

Fourier Transform and PDE's (Chapter 16)

We have spent a great deal of time studying the 1-d heat equation with various boundary conditions. However, in all cases we used a rod which had finite length. Being good math students we might ask, what about an infinite rod?

Let's Examine the heat equation defined on the interval $(-\infty, \infty)$

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}$$

Let's proceed in the usual way by separating variables. With our usual assumption $u(x, t) = h(t)\phi(x)$ we obtain our old friends:

$$\frac{dh}{dt} = -\lambda kh$$

$$\frac{d^2\phi}{dx^2} = -\lambda\phi$$

With our usual solutions:

$$h(t) = c_1 e^{-\lambda kt}$$

$$\phi(x) = c_1 \cos(\sqrt{\lambda}x) + c_2 \sin(\sqrt{\lambda}x)$$

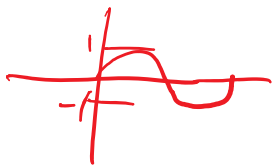
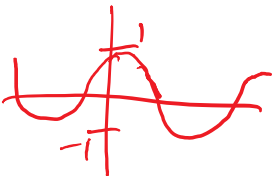
Normally at this point we would apply our boundary conditions to determine our separation constant λ alas we have no boundaries as this is an infinite interval. So instead we will make the requirement that:

$$u(\pm\infty, t) < \infty \quad \Leftarrow \quad \underline{|\phi(\pm\infty)| < \infty}$$

The question then becomes, for what values of λ do we satisfy $|\phi(\pm\infty)| < \infty$

$$\phi(x) = |c_1 \cos \sqrt{\lambda}x + c_2 \sin \sqrt{\lambda}x| \leq |c_1 \cos \sqrt{\lambda}x| + |c_2 \sin \sqrt{\lambda}x|$$

$$\begin{aligned} &= |c_1| |\cos(\sqrt{\lambda}x)| + |c_2| |\sin \sqrt{\lambda}x| \leq |c_1| (1) + |c_2| (1) \\ &\leq |c_1| + |c_2| < \infty \end{aligned}$$



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So as long as $\lambda \geq 0$ our condition of $|\phi(\pm\infty)| < \infty$ is satisfied, this is very different from the case we had before where $\lambda = \left(\frac{n\pi}{L}\right)^2$. We often refer to the set of eigen values as the spectra of a problem. When we had a finite rod we had a discrete spectra for this problem we have a continuous spectra.

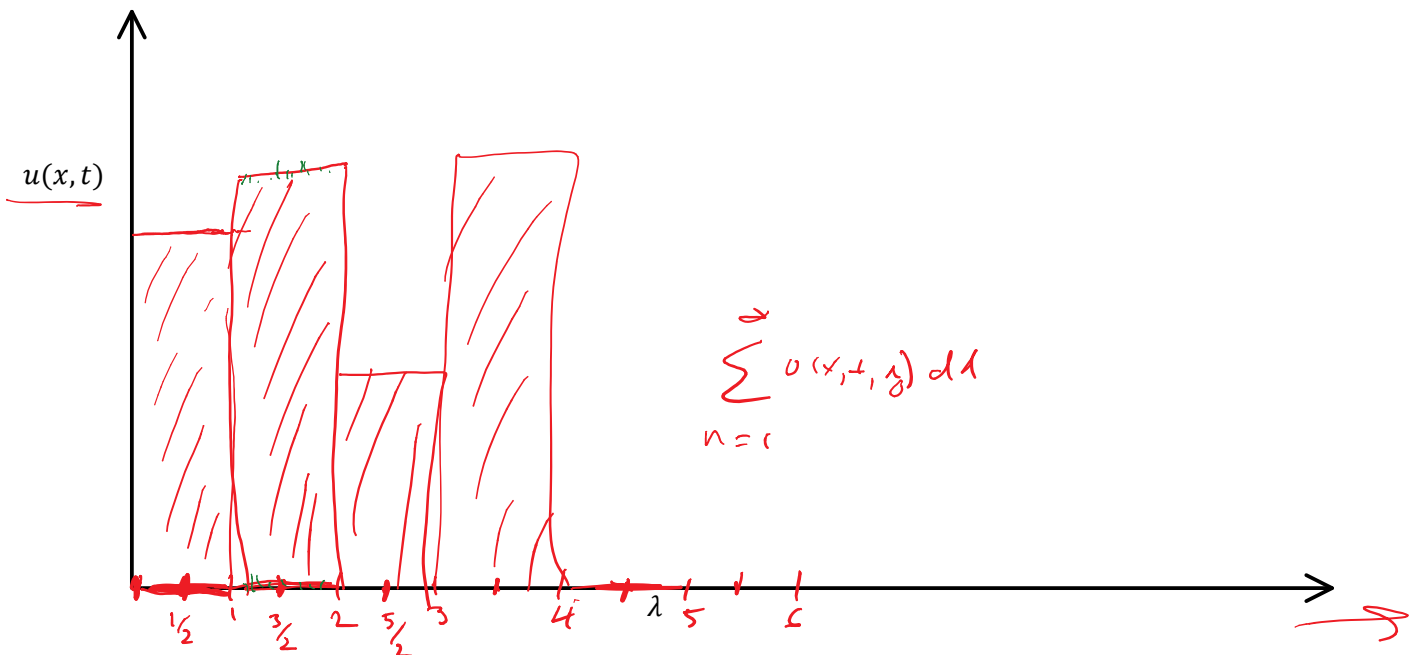
Discrete Spectra $\lambda = \left(\frac{n\pi}{L}\right)^2$



Continuous Spectra $\lambda \geq 0$



So now we have a solution $u(x, t) = h(t)\phi(x) = e^{-k\lambda t}(c_1 \cos(\sqrt{\lambda}x) + c_2 \sin(\sqrt{\lambda}x))$ for each value of $\lambda \geq 0$, so again we have infinitely many solutions but with a continuous spectra, the question now is how do we apply the principle of super position.



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With these considerations instead of summing over λ we integrate over λ

$$u(x, t) = \int_0^{\infty} e^{-k\lambda t} (c_1(\lambda) \cos(\sqrt{\lambda}x) + c_2(\lambda) \sin(\sqrt{\lambda}x)) d\lambda$$

Possible Final Problem:

Verify that the solution $u(x, t) = \int_0^{\infty} e^{-k\lambda t} (c_1(\lambda) \cos(\sqrt{\lambda}x) + c_2(\lambda) \sin(\sqrt{\lambda}x)) d\lambda$ satisfies the 1-d heat equation:

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}$$

$$\frac{\partial}{\partial t} \int_0^{\infty} e^{-k\lambda t} (c_1(\lambda) \cos(\sqrt{\lambda}x) + c_2(\lambda) \sin(\sqrt{\lambda}x)) d\lambda = k \frac{\partial^2}{\partial x^2} \int_0^{\infty} e^{-k\lambda t} (c_1(\lambda) \cos(\sqrt{\lambda}x) + c_2(\lambda) \sin(\sqrt{\lambda}x)) d\lambda$$

$$\int_0^{\infty} -k\lambda e^{-k\lambda t} (c_1(\lambda) \cos(\sqrt{\lambda}x) + c_2(\lambda) \sin(\sqrt{\lambda}x)) d\lambda = k \int_0^{\infty} e^{-k\lambda t} (-c_1(\lambda)\lambda \cos(\sqrt{\lambda}x) - c_2(\lambda)\lambda \sin(\sqrt{\lambda}x)) d\lambda$$

↓

$$-k \int_0^{\infty} \lambda e^{-k\lambda t} (c_1(\lambda) \cos(\sqrt{\lambda}x) + c_2(\lambda) \sin(\sqrt{\lambda}x)) d\lambda$$

$$= -k \int_0^{\infty} \lambda e^{-k\lambda t} (c_1(\lambda) \cos(\sqrt{\lambda}x) + c_2(\lambda) \sin(\sqrt{\lambda}x)) d\lambda$$

✓

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So now that we have our solution we need to find the coefficients. We assume we have some initial condition $u(x, 0) = f(x)$ and proceed in the usual way.

$$u(x, t) = \int_0^\infty e^{-k\lambda t} (c_1(\lambda) \cos(\sqrt{\lambda}x) + c_2(\lambda) \sin(\sqrt{\lambda}x)) d\lambda$$

Real quickly, since lambda is positive anyway we will define $\lambda = \omega^2$ and write our solution in the form.

$$u(x, t) = \int_0^\infty e^{-k\omega^2 t} (A(\omega) \cos(\omega x) + B(\omega) \sin(\omega x)) d\omega$$

Let's now apply our initial condition:

$$u(x, 0) = f(x) = \int_0^\infty (A(\omega) \cos(\omega x) + B(\omega) \sin(\omega x)) d\omega$$

The question then becomes how do we find $A(\omega)$ and $B(\omega)$, this is where the Fourier transform will come in.

Before we do that though we need to re-write our initial condition equation and solution using complex exponentials

It can be shown that:

$$u(x, t) = \int_0^\infty e^{-k\omega^2 t} (A(\omega) \cos(\omega x) + B(\omega) \sin(\omega x)) d\omega = \int_{-\infty}^\infty c(\omega) e^{-i\omega x} e^{-k\omega^2 t} d\omega$$

$\cos(-\omega x) + i \sin(-\omega x)$

$$f(x) = \int_0^\infty (A(\omega) \cos(\omega x) + B(\omega) \sin(\omega x)) d\omega = \int_{-\infty}^\infty c(\omega) e^{-i\omega x} d\omega$$

$\int_{-\infty}^0 + \int_0^\infty$

with $A(\omega), B(\omega) \in \mathbb{R}$ and $c(\omega) \in \mathbb{C}$

→ In fact a great possible final question would be to determine $c(\omega)$ in terms of $A(\omega)$ and $B(\omega)$ using the equation for $u(x, t)$ above!

For now let's just use the fact that we could write:

$$u(x, t) = \int_{-\infty}^\infty c(\omega) e^{-i\omega x} e^{-k\omega^2 t} d\omega$$

$$f(x) = \int_{-\infty}^\infty c(\omega) e^{-i\omega x} d\omega$$

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In order to find the coefficient $c(\omega)$ we now introduce the Fourier Transform and the Inverse Fourier Transform

The Fourier transformation of a function $f(x)$ is defined as:

$$\mathcal{F}[f(x)] = \underline{F(\omega)} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \underline{f(x)} \underline{e^{i\omega x}} \underline{dx}$$

The inverse Fourier transform of a function $F(\omega)$ is defined as:

$$\mathcal{F}^{-1}[\underline{F(\omega)}] = \underline{f(x)} = \int_{-\infty}^{\infty} \underline{F(\omega)} \underline{e^{-i\omega x}} d\omega$$

Let's examine our initial condition equation:

$$f(x) = \int_{-\infty}^{\infty} \underline{c(\omega)} e^{-i\omega x} d\omega \quad \underline{c(\omega) = \mathcal{F}[f(x)]}$$

Looking at this we notice that this is the inverse Fourier Transform of $c(\omega)$! What does this say about $c(\omega)$?

$$c(\omega) = \mathcal{F}[f(x)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{i\omega x} dx$$

Fourier Transform and PDE's (Chapter 16)

So we have a solution to our infinite domain heat equation:

$$u(x, t) = \int_{-\infty}^{\infty} c(\omega) e^{-i\omega x} e^{-k\omega^2 t} d\omega$$

where:

$$c(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{i\omega x} dx$$

$$u(x, t) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi k t}} e^{-\frac{\omega^2}{4k} t} e^{-i\omega x} e^{-k\omega^2 t} d\omega$$

It might be useful to see an example of the calculation of a Fourier transform. Suppose $f(x) = e^{-\alpha x^2}$ is our initial condition.

Let's find our $c(\omega)$

$$c(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \overbrace{e^{-\alpha x^2}}^{f(x)} e^{i\omega x} dx$$

$$c(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\alpha x^2 + i\omega x} dx$$

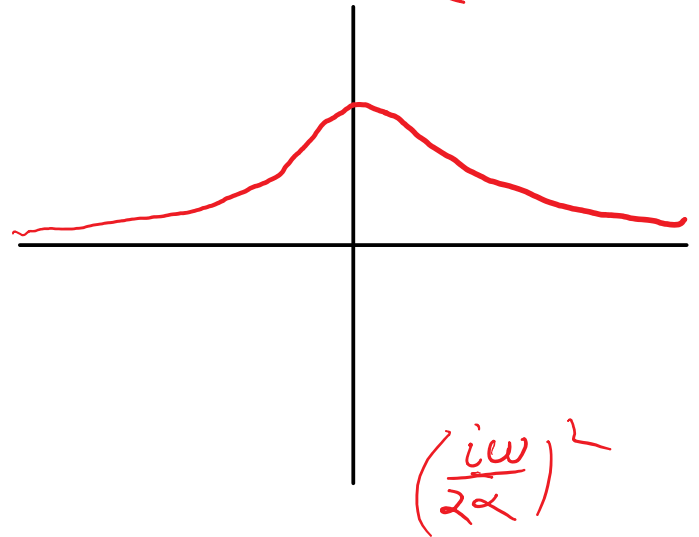
$$c(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\alpha \left(x - \frac{i\omega}{2\alpha}\right)^2} dx$$

$$c(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\alpha \left(x^2 - \frac{i\omega}{\alpha} x + \left(\frac{i\omega}{2\alpha}\right)^2 - \left(\frac{i\omega}{2\alpha}\right)^2\right)} dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\alpha \left(x^2 - \frac{i\omega}{\alpha} x + \left(\frac{i\omega}{2\alpha}\right)^2\right)} dx$$

$$c(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\alpha \left(x - \frac{i\omega}{2\alpha}\right)^2 - \left(\frac{i\omega}{2\alpha}\right)^2} dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\alpha \left(x - \frac{i\omega}{2\alpha}\right)^2} e^{-\frac{\omega^2}{4\alpha}} dx$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\alpha \left(x - \frac{i\omega}{2\alpha}\right)^2} e^{-\frac{\omega^2}{4\alpha}} dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\left(\sqrt{\alpha} x - \frac{i\omega}{2\sqrt{\alpha}}\right)^2} e^{-\frac{\omega^2}{4\alpha}} dx$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\left(\sqrt{\alpha} x - \frac{i\omega}{2\sqrt{\alpha}}\right)^2} e^{-\frac{\omega^2}{4\alpha}} dx = \frac{e^{-\frac{\omega^2}{4\alpha}}}{2\pi} \int_{-\infty}^{\infty} e^{-\left(\sqrt{\alpha} x - \frac{i\omega}{2\sqrt{\alpha}}\right)^2} dx$$



$$\frac{e^{-\frac{\omega^2}{4\alpha}}}{2\pi} \int_{-\infty}^{\infty} e^{-\left(\sqrt{\alpha}x - \frac{i\omega}{2\sqrt{\alpha}}\right)^2} dx$$

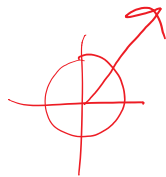
$$u = \sqrt{\alpha}x - \frac{i\omega}{2\sqrt{\alpha}}$$

$$du = \sqrt{\alpha} dx$$

$$\frac{e^{-\frac{\omega^2}{4\alpha}}}{2\pi\sqrt{\alpha}} \int_{-\infty}^{\infty} e^{-u^2} du = \sqrt{\pi}$$

$$I = \int_{-\infty}^{\infty} e^{-u^2} du \quad I^2 = \int_{-\infty}^{\infty} e^{-u^2} du \int_{-\infty}^{\infty} e^{-w^2} dw$$

$$I^2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-u^2-w^2} du dw = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(u^2+w^2)} du dw$$



$$I^2 = \int_0^{2\pi} \int_0^{\infty} e^{-r^2} r dr d\theta \quad u = r^2 \quad r^2 = x^2 + y^2$$

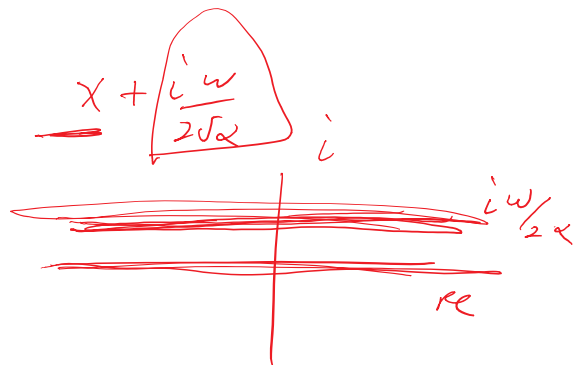
$$du = 2r dr \quad r^2 = u^2 + w^2$$

$$I^2 = \int_0^{2\pi} \frac{1}{2} \int_0^{\infty} e^{-u} du d\theta = \frac{1}{2} \int_0^{2\pi} -e^{-u} \Big|_0^{\infty} d\theta = \frac{1}{2} \int_0^{2\pi} 0 - (-e^0) d\theta$$

$$= \frac{1}{2} \int_0^{2\pi} d\theta = \frac{1}{2} \theta \Big|_0^{2\pi} = \frac{2\pi}{2} = \pi$$

$$I^2 = \pi \rightarrow I = \sqrt{\pi}$$

$$\mathcal{F}[e^{-\alpha x^2}] = \frac{e^{-\frac{\omega^2}{4\alpha}}}{2\pi\sqrt{\alpha}} \sqrt{\pi} = \frac{e^{-\frac{\omega^2}{4\alpha}}}{\sqrt{4\pi\alpha}} = F(\omega)$$



Fourier Transform and PDE's (Chapter 16)

It turns out we use Fourier Transforms to turn PDE's into ODE's! Let's do this with our heat equation.

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}$$

$$\mathcal{F}\left[\frac{\partial u}{\partial t}\right] = \mathcal{F}\left[k \frac{\partial^2 u}{\partial x^2}\right]$$

The trick will be to take the Fourier Transform of both sides of this equation. Before we do that let's just name the Fourier Transform of $u(x, t)$ to be $\bar{U}(\omega, t)$ that is:

$$\mathcal{F}[u(x, t)] = \bar{U}(\omega, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} u(x, t) e^{i\omega x} dx$$

Now Let's see what happens when we take the Fourier Transform of both sides of our PDE.

The time derivative side is easy

$$\begin{aligned} \mathcal{F}\left[\frac{\partial u}{\partial t}\right] &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\partial u}{\partial t} e^{i\omega x} dx = \frac{\partial}{\partial t} \frac{1}{2\pi} \int_{-\infty}^{\infty} u(x, t) e^{i\omega x} dx \\ &= \frac{\partial}{\partial t} \bar{U}(\omega, t) \end{aligned}$$

For the Spatial derivative side Let's first examine what happens when we take the Fourier Transform of $\frac{\partial u}{\partial x}$

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$$\mathcal{F}\left[\frac{\partial u}{\partial x}\right] = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\partial u}{\partial x} e^{i\omega x} dx$$

Let's throw integration by parts at it.

$$u = e^{i\omega x} \quad dv = \frac{\partial u}{\partial x}$$

$$du = i\omega e^{i\omega x} dx \quad v = u$$

$$\int u dv = u \cdot v - \int v du$$

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\partial u}{\partial x} e^{i\omega x} dx = \frac{1}{2\pi} \left[e^{i\omega x} u \Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} u i\omega e^{i\omega x} dx \right]$$

$$u(\pm\infty, t) = 0$$

$$\lim_{x \rightarrow \pm\infty} \left| e^{i\omega x} \frac{\partial u}{\partial x} \right| = 0$$

$$-\frac{i\omega}{2\pi} \int_{-\infty}^{\infty} u(x, t) e^{i\omega x} dx$$

$$\lim_{x \rightarrow \pm\infty} |u(x)| < \infty$$

$$\Rightarrow \lim_{x \rightarrow \pm\infty} |u(x, t)| = 0$$

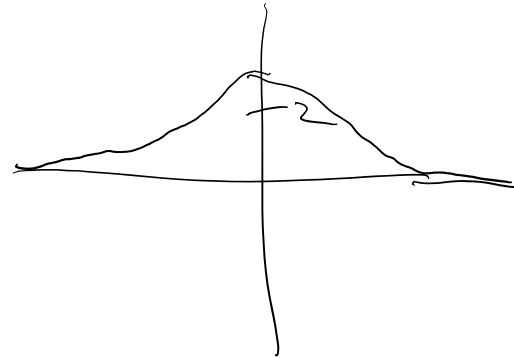
$$-i\omega \frac{1}{2\pi} \int_{-\infty}^{\infty} u(x, t) e^{i\omega x} dx = -i\omega \bar{U}(\omega, t)$$

$$\text{So } \mathcal{F}\left[\frac{\partial u}{\partial x}\right] = -i\omega \mathcal{F}[u]$$

Any guesses for $\mathcal{F}\left[\frac{\partial^2 u}{\partial x^2}\right]$?

$$\mathcal{F}\left[\frac{\partial^2 u}{\partial x^2}\right] = (i\omega)^2 \mathcal{F}[u] = -\omega^2 \mathcal{F}[u]$$

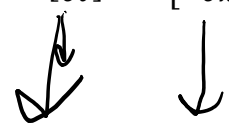
$$= -\omega^2 \bar{U}(\omega, t)$$



Fourier Transform and PDE's (Chapter 16)

Then if we take the Fourier transform of our heat equation on both sides we get:

$$\mathcal{F}\left[\frac{\partial u}{\partial t}\right] = \mathcal{F}\left[k \frac{\partial^2 u}{\partial x^2}\right]$$



$$\frac{\partial}{\partial t} \bar{U}(\omega, t) = k(-i\omega)^2 \bar{U}(\omega, t)$$

An ODE! Let's solve this thing.

$$\frac{d\bar{U}}{dt} = k(i\omega)^2 \bar{U} =$$
$$\int \frac{d\bar{U}}{\bar{U}} = \int k(i\omega)^2 dt$$

$-k\omega^2 t$
 Ce
 $\uparrow$

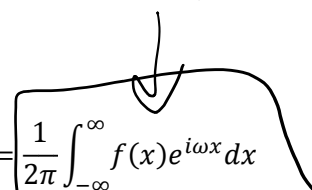
$$e^{\ln \bar{U}} = \frac{k(i\omega)^2 t + C}{e} \quad \bar{U} = Ce^{k(i\omega)^2 t}$$

So we have $\bar{U}(\omega, t) = c(\omega)e^{-k\omega^2 t}$ we can then simply apply the inverse Fourier Transform to obtain $u(x, t)$ That is $\uparrow$

$$u(x, t) = \int_{-\infty}^{\infty} \bar{U}(\omega, t) e^{-i\omega t} d\omega = \int_{-\infty}^{\infty} c(\omega) e^{-k\omega^2 t} e^{-i\omega t} d\omega$$

Just as we had from separation of variables!

We could find $c(\omega)$ again using:



$$c(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{i\omega x} dx$$

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Another representation for our solution using the Convolution Theorem. First Let's define the convolution of two functions, $f * g$

$$f * g = \int_{-\infty}^{\infty} f(\bar{x})g(x - \bar{x})d\bar{x}$$

Let's ask the following question, What is the inverse Fourier transform $h(x)$ of a function $H(\omega) = F(\omega)G(\omega)$

Where:

$$F(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x)e^{-i\omega x}dx$$

$$f(x) = \int_{-\infty}^{\infty} F(\omega)e^{i\omega x}d\omega$$

$$G(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} g(x)e^{-i\omega x}dx$$

$$g(x) = \int_{-\infty}^{\infty} G(\omega)e^{i\omega x}d\omega$$

We start by just applying the inverse Fourier transform to $H(\omega)$

$$h(x) = \int_{-\infty}^{\infty} H(\omega)e^{i\omega x}d\omega = \int_{-\infty}^{\infty} F(\omega)G(\omega)e^{i\omega x}d\omega$$

$$\begin{aligned} & \int_{-\infty}^{\infty} F(\omega)G(\omega)e^{i\omega x}d\omega \\ & \int_{-\infty}^{\infty} F(\omega) \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} g(\bar{x})e^{-i\omega \bar{x}}d\bar{x} \right) e^{i\omega x}d\omega \\ & \int_{-\infty}^{\infty} F(\omega) \frac{1}{2\pi} \int_{-\infty}^{\infty} g(\bar{x})e^{-i\omega \bar{x} + i\omega x}d\bar{x}d\omega \\ & \int_{-\infty}^{\infty} F(\omega) \frac{1}{2\pi} \int_{-\infty}^{\infty} g(\bar{x})e^{-i\omega(\bar{x}-x)}d\bar{x}d\omega \\ & \frac{1}{2\pi} \int_{-\infty}^{\infty} g(\bar{x}) \left(\int_{-\infty}^{\infty} F(\omega)e^{i\omega(x-\bar{x})}d\omega \right) d\bar{x} \\ & \frac{1}{2\pi} \int_{-\infty}^{\infty} g(\bar{x}) f(x-\bar{x})d\bar{x} \end{aligned}$$

$$\begin{aligned} & i\omega(-1)(\bar{x}-x) \\ & i\omega(-\bar{x}+x) \\ & i\omega(x-\bar{x}) \end{aligned}$$

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So we have that if $H(\omega) = F(\omega)G(\omega)$ then $h(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} g(\bar{x})f(x-\bar{x})d\bar{x} = (g * f) \frac{1}{2\pi}$

In words we are just saying that inverse Fourier transform of a product of two Fourier transforms is $\frac{1}{2\pi}$ times the convolution of the product of the original functions.

What does this do for us with regard to the heat equation? Well recall that the solution was:

$$u(x, t) = \int_{-\infty}^{\infty} \underbrace{c(\omega)}_{\text{initial condition}} \underbrace{e^{-k\omega^2 t}}_{\text{diffusion}} e^{i\omega x} d\omega = \mathcal{F}^{-1}[c(\omega) e^{-k\omega^2 t}]$$

Let's name $G(\omega) = c(\omega)$ and $F(\omega) = e^{-k\omega^2 t}$

Then by the convolution theorem $u(x, t)$ will be equal to the $\frac{1}{2\pi}$ times convolution of the inverse Fourier transforms of $G(\omega) = c(\omega)$ and $F(\omega) = e^{-k\omega^2 t}$

$$\mathcal{F}^{-1}(c(\omega)) = f(x)$$

We know that the inverse Fourier transform of $c(\omega)$ is just of $f(x)$ and we found out earlier that:

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\alpha x^2} e^{i\omega x} dx = \frac{1}{\sqrt{4\pi\alpha}} e^{-\frac{\omega^2}{4\alpha}}$$

If we take the Inverse Fourier transform of $F(\omega) = e^{-k\omega^2 t}$ it would read: $\int_{-\infty}^{\infty} e^{-k\omega^2 t} e^{-i\omega x} d\omega$ which we can figure out easily using the known integral above. Let $\alpha = kt, x = -\omega$

$$\int_{-\infty}^{\infty} e^{-k\omega^2 t} e^{-i\omega x} d\omega \quad ?$$

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\alpha x^2} e^{i\omega x} dx = \frac{1}{\sqrt{4\pi\alpha}} e^{-\frac{\omega^2}{4\alpha}}$$

$\omega = -x$
 $x = -\omega$
 $dx = -d\omega$

$$-\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-kt\omega^2} e^{-i\omega^2} d\omega = \frac{1}{\sqrt{4\pi kt}} e^{-\frac{x^2}{4kt}}$$

$$-\int_{-\infty}^{\infty} e^{-k\omega^2 t} e^{i\omega x} d\omega = \frac{2\pi}{\sqrt{4\pi kt}} e^{-\frac{x^2}{4kt}}$$

$$\begin{aligned}
 & \int_{-\infty}^{\infty} e^{-k\omega^2 t} e^{i\omega x} d\omega \\
 &= \int_{-\infty}^{\infty} e^{-k\omega^2 t - i\omega x} d\omega = \frac{\sqrt{\pi}}{\sqrt{k t}} e^{-x^2/4kt}
 \end{aligned}$$

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so we have that $\mathcal{F}^{-1}[c(\omega)] = f(x)$ and $\mathcal{F}^{-1}[e^{-k\omega^2 t}] = \sqrt{\frac{\pi}{kt}} e^{-\frac{x^2}{4kt}}$

What we want is $\mathcal{F}^{-1}[c(\omega)e^{-k\omega^2 t}] = \mathcal{F}^{-1}(\bar{U})$

The convolution Theorem States that:

$$\mathcal{F}^{-1}[H(\omega)] = \mathcal{F}^{-1}[F(\omega)G(\omega)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} g(\bar{x})f(x - \bar{x})d\bar{x}$$

where

$$g(x) = \mathcal{F}^{-1}[F(\omega)] \quad \text{and} \quad f(x) = \mathcal{F}^{-1}[G(\omega)]$$

If we apply this to our solution

$$F(\omega) \subseteq c(\omega)$$

$$\bar{U}(\omega, t) = c(\omega)e^{-k\omega^2 t}$$

$$\mathcal{F}^{-1}(\bar{U}) = \mathcal{F}^{-1}(c(\omega)e^{-k\omega^2 t}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \boxed{f(\bar{x})} \sqrt{\frac{\pi}{kt}} e^{-\frac{(x-\bar{x})^2}{4kt}} d\bar{x}$$

$$U(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \underline{f(\bar{x})} \sqrt{\frac{\pi}{kt}} e^{-\frac{(x-\bar{x})^2}{4kt}} d\bar{x}$$