

# Practice Final

Math 3150 001 Summer 2015  
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Name and Unid:

*Solution S*

by writing my name I swear by the honor code

**Read all of the following information before starting the exam:**

**Note this is the practice exam ignore point values on problems there are extras but the exam will look similar**

- Show all work, clearly and in order, if you want to get full credit. I reserve the right to take off points if I cannot see how you arrived at your answer (even if your final answer is correct).
- Justify your answers algebraically whenever possible to ensure full credit. Sketch all relevant graphs and explain all relevant mathematics.
- Circle or otherwise indicate your final answers.
- You MAY NOT use a calculator or notes. You may use the formulas provided and your 3X5 card.
- The actual test has 8 problems and is worth 150 points It is your responsibility to make sure that you have all of the pages!
- Good luck!

Good Luck!

Here are some Formulas and rules for you.

once I write the final  
I will post the formulas.

1. (10 points) Find the equilibrium temperature distribution of a rod of length  $L = 1$  with no internal heat source and insulated boundaries. That is, subject to the boundary conditions  $\frac{\partial u}{\partial x}(0) = 0$  and  $\frac{\partial u}{\partial x}(1) = 0$ , with an initial temperature distribution of  $u(x, 0) = 3x^2$

In equilibrium  $\frac{\partial u}{\partial t} = 0$  so the 1-d heat equation becomes

$$0 = k \frac{\partial^2 u}{\partial x^2} \Rightarrow \frac{\partial^2 u}{\partial x^2} = 0 \Rightarrow u = C_1 x + C_2.$$

Applying the boundary conditions gives:

$$\frac{\partial u}{\partial x}(0) = C_1 = 0 \quad \frac{\partial u}{\partial x}(1) = C_1 = 0 \Rightarrow \boxed{C_1 = 0}$$

$$\text{So } u(x, t) = C_2$$

Then  $u(x, 0) = 3x^2$  with insulated boundaries the total thermal Energy must remain constant.

$$\text{Thus: } \int_0^1 u(x, 0) dx = \int_0^1 u(x, t) dx \quad \text{for all } t \text{ including at equilibrium}$$

$$\text{so } \int_0^1 3x^2 dx = \int_0^1 C_2 dx$$

$$\Rightarrow \left. \frac{3x^3}{3} \right|_0^1 = C_2 x \Big|_0^1$$

$$\Rightarrow C_2 = \frac{3(1)^3}{3} - 0$$

$$\Rightarrow C_2 = 1$$

$$\Rightarrow \boxed{u(x, t) = 1}$$

2. (10 points) Prove that the total thermal energy is constant for a 1-d rod of length  $L$  when it has perfectly insulated boundaries and no internal sources.

Diagram of a rod of length  $L$  with insulated boundaries (no flux) at  $x=0$  and  $x=L$ . The heat equation is  $\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}$ . The total energy is  $E = \int_0^L A \rho c u dx$ , where  $A$  is cross-sectional area,  $\rho$  is density, and  $c$  is heat capacity. The energy is constant, so  $\frac{dE}{dt} = 0$ .

$$\frac{\partial E}{\partial t} = \frac{\partial}{\partial t} \int_0^L A \rho c u dx = A \rho c \int_0^L \frac{\partial u}{\partial t} dx = A \rho c \int_0^L k \frac{\partial^2 u}{\partial x^2} dx$$

$$= A \rho c k \int_0^L \frac{\partial^2 u}{\partial x^2} dx = A \rho c k \left[ \frac{\partial u}{\partial x} \right]_0^L = A \rho c k \left( \frac{\partial u}{\partial x}(L) - \frac{\partial u}{\partial x}(0) \right) = 0 \quad \checkmark$$

3. (10 points) For Laplace's equation in 2-d cartesian coordinates,  $\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$  apply the assumption of separation of variables  $u(x,y) = h(x)\phi(y)$  and derive the resulting ODE equations.

Assume  $u(x,y) = h(x)\phi(y)$  plug in to PDE

$$\frac{\partial^2}{\partial x^2} (h(x)\phi(y)) + \frac{\partial^2}{\partial y^2} (h(x)\phi(y)) = 0$$

$$\Rightarrow \phi(y) \frac{d^2 h}{dx^2} + h(x) \frac{d^2 \phi}{dy^2} = 0$$

$$\Rightarrow \frac{\phi(y) \frac{d^2 h}{dx^2}}{\phi(y) h(x)} = - \frac{h(x) \frac{d^2 \phi}{dy^2}}{\phi(y) h(x)} \quad \text{divide by } \phi(y)h(x)$$

$$\Rightarrow \frac{1}{h(x)} \frac{d^2 h}{dx^2} = - \frac{1}{\phi(y)} \frac{d^2 \phi}{dy^2} = \lambda$$

since each side only depends on one variable they must be constant call it  $\lambda$

$$\Rightarrow \boxed{\frac{1}{h(x)} \frac{d^2 h}{dx^2} = \lambda} \quad \boxed{- \frac{1}{\phi(y)} \frac{d^2 \phi}{dy^2} = \lambda}$$

4. (10 points) For Laplace's equation in polar coordinates,  $\nabla^2 u = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = 0$  apply the assumption of separation of variables  $u(r, \theta) = G(r)\phi(\theta)$  and derive the resulting ODE equations.

$$u(r, \theta) = G(r)\phi(\theta) \quad \text{plus into PDE}$$

$$\Rightarrow \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial}{\partial r} G(r)\phi(\theta) \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} (G(r)\phi(\theta)) = 0$$

$$\Rightarrow \phi(\theta) \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial}{\partial r} G \right) + \frac{1}{r^2} G(r) \frac{\partial^2}{\partial \theta^2} \phi(\theta) = 0$$

$$\Rightarrow \cancel{\phi(\theta)} \frac{1}{r} \frac{d}{dr} \left( r \frac{dG}{dr} \right) = -G(r) \frac{1}{r^2} \frac{d^2 \phi}{d\theta^2} \quad \begin{array}{l} \text{divide} \\ \text{by } \phi(\theta)G(r) \end{array}$$

$$\Rightarrow r^2 \left( \frac{1}{G(r)} \frac{1}{r} \frac{d}{dr} \left( r \frac{dG}{dr} \right) \right) = \left( -\frac{1}{\phi(\theta)} \frac{1}{r^2} \frac{d^2 \phi}{d\theta^2} \right) \quad \begin{array}{l} \text{multiply by } r^2 \\ \text{multiply by } r^2 \end{array}$$

$$\Rightarrow \frac{r}{G} \frac{d}{dr} \left( r \frac{dG}{dr} \right) = -\frac{1}{\phi(\theta)} \frac{d^2 \phi}{d\theta^2} = 1$$

$$\boxed{\frac{r}{G} \frac{d}{dr} \left( r \frac{dG}{dr} \right) = 1}$$

$$\boxed{-\frac{1}{\phi(\theta)} \frac{d^2 \phi}{d\theta^2} = 1}$$

5. (10 points) For the 1-d wave equation  $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$  apply the assumption of separation of variables,  $u(x, t) = \phi(x)h(t)$  and derive the resulting ODE equations.

$$\frac{\partial^2}{\partial t^2} \phi(x)h(t) = c^2 \frac{\partial^2}{\partial x^2} \phi(x)h(t)$$

$$\Rightarrow \cancel{\phi(x)} \frac{d^2 h}{dt^2} = h(t) c^2 \frac{d^2 \phi}{dx^2} \quad \begin{array}{l} \text{divide by} \\ \phi(x)h(t) \end{array}$$

$$\Rightarrow \frac{1}{h(t)} \frac{d^2 h}{dt^2} = \frac{c^2}{\phi(x)} \frac{d^2 \phi}{dx^2} \quad \begin{array}{l} \text{often we want the constants} \\ \text{away from the ODE which} \\ \text{gives our eigen value problem} \\ \text{so I'll move it over} \end{array}$$

$$= \frac{1}{c^2 h(t)} \frac{d^2 h}{dt^2} = \frac{1}{\phi(x)} \frac{d^2 \phi}{dx^2} = 1$$

$$\Rightarrow \boxed{\frac{1}{c^2 h(t)} \frac{d^2 h}{dt^2} = 1} \quad \boxed{\frac{1}{\phi(x)} \frac{d^2 \phi}{dx^2} = 1}$$

6. (10 points) Solve Laplace's equation inside a rectangle  $0 \leq x \leq L$ ,  $0 \leq y \leq H$  with the following boundary conditions.  $\frac{\partial u}{\partial y}(x, 0) = 0$ ,  $\frac{\partial u}{\partial y}(x, H) = 0$ ,  $u(0, y) = 0$ ,  $u(L, y) = f(y)$

The ODE's which come from separation of variables for

Laplace's Equation are  $\frac{d^2 h}{dx^2} = \lambda h$   $\frac{d^2 \phi}{dy^2} = -\lambda h$

The Homogeneous Boundary conditions become:

$$\frac{\partial u}{\partial y}(x, 0) \rightarrow \frac{d\phi}{dy}(0) = 0$$

$$\frac{\partial u}{\partial y}(x, H) = 0 \rightarrow \frac{d\phi}{dy}(H) = 0$$

$$u(0, y) = 0 \rightarrow h(0) = 0$$

we choose  $\phi$  to be a function of  $y$  since our two homogeneous boundary conditions are in  $y$

The General Solutions are:

$$\phi(y) = c_1 y + c_2$$

$$\lambda = 0$$

$$h(x) = c_3 x + c_4$$

applying the BC's

$$\frac{d\phi}{dy} = c_1, \quad \frac{\partial \phi}{\partial y}(0) = \frac{\partial \phi}{\partial y}(H) = c_1 = 0$$

$$\Rightarrow c_1 = 0 \Rightarrow \phi(y) = c_2$$

$$h(0) = 0 \Rightarrow c_3(0) + c_4 = 0$$

$$\Rightarrow c_4 = 0 \Rightarrow h(x) = c_3 x$$

$$\Rightarrow u(x, y) = A_0 x \quad \lambda = 0$$

$$\phi(y) = c_1 \cos(\sqrt{\lambda} y) + c_2 \sin(\sqrt{\lambda} y)$$

$$\lambda > 0$$

$$h(y) = c_3 \cosh(\sqrt{\lambda} y) + c_4 \sinh(\sqrt{\lambda} y)$$

Applying BC's

$$\frac{d\phi}{dy} = -c_1 \sqrt{\lambda} \sin(\sqrt{\lambda} y) + c_2 \sqrt{\lambda} \cos(\sqrt{\lambda} y)$$

$$\frac{d\phi}{dy}(0) = 0 = c_2 \sqrt{\lambda} \Rightarrow c_2 = 0$$

$$\frac{d\phi}{dy}(H) = 0 = -c_1 \sqrt{\lambda} \sin(\sqrt{\lambda} H) \Rightarrow \sqrt{\lambda} H = n\pi$$

$$\Rightarrow \lambda = \left(\frac{n\pi}{H}\right)^2 \Rightarrow \phi(y) = c_1 \cos\left(\frac{n\pi}{H} y\right)$$

$$h(0) = 0 = c_3 \cosh(0) + c_4 \sinh(0) \Rightarrow c_3 = 0$$

$$\Rightarrow h(x) = c_2 \sinh(\sqrt{\lambda} x) = c_2 \sinh\left(\frac{n\pi}{H} x\right)$$

$$\Rightarrow u(x, y) = A_0 \sinh\left(\frac{n\pi}{H} x\right) \cos\left(\frac{n\pi}{H} y\right)$$

Applying superposition

$$u(x, y) = A_0 x + \sum_{n=1}^{\infty} A_n \sinh\left(\frac{n\pi}{H} x\right) \cos\left(\frac{n\pi}{H} y\right)$$

Applying  $u(L, y) = f(y)$

$$u(L, y) = f(y) = A_0 L + \sum_{n=1}^{\infty} A_n \sinh\left(\frac{n\pi}{H} L\right) \cos\left(\frac{n\pi}{H} y\right)$$

compare to cosine series  $f(y) = \tilde{A}_0 + \sum_{n=1}^{\infty} \tilde{A}_n \cos\left(\frac{n\pi}{H} y\right)$

$$\tilde{A}_0 = \frac{1}{H} \int_0^H f(y) dy \quad \tilde{A}_n = \frac{2}{H} \int_0^H f(y) \cos\left(\frac{n\pi}{H} y\right) dy$$

$$A_0 L = \frac{1}{H} \int_0^H f(y) dy$$

$$\Rightarrow A_0 = \frac{1}{HL} \int_0^H f(y) dy$$

$$A_n \sinh\left(\frac{n\pi}{H} L\right) = \frac{2}{H} \int_0^H f(y) \cos\left(\frac{n\pi}{H} y\right) dy$$

$$\Rightarrow A_n = \frac{2}{H \sinh\left(\frac{n\pi}{H} L\right)} \int_0^H f(y) \cos\left(\frac{n\pi}{H} y\right) dy$$

7. (20 points) Consider the general solution Laplace's equation inside a circle of radius  $r = a$  given by

$$u(r, \theta) = A_0 + \sum_{n=1}^{\infty} A_n r^n \cos(n\theta) + B_n r^n \sin(n\theta)$$

a. (10 pts) Find the full solution if on the boundary,  $u(a, \theta) = 1 + 2 \sin(3\theta) + 12 \sin(\theta) + 3 \cos(4\theta)$  Here  $u(a, \theta)$  is made of eigen functions so we match coeff's

$$u(a, \theta) = A_0 + \sum_{n=1}^{\infty} A_n a^n \cos(n\theta) + B_n a^n \sin(n\theta) = 1 + 2 \sin(3\theta) + 12 \sin(\theta) + 3 \cos(4\theta)$$

$\downarrow \quad \downarrow \quad \underbrace{\quad}_{n=3} \quad \downarrow \quad \underbrace{\quad}_{n=1} \quad \downarrow \quad \underbrace{\quad}_{n=4}$   
 $A_0 \quad B_3 \quad \quad B_1 \quad \quad A_4$

$$\Rightarrow A_0 = 1 \quad B_1 a^1 = 12 \quad B_3 a^3 = 2 \quad A_4 a^4 = 3 \quad \text{all other } B_n's \text{ } A_n's = 0$$

$$\Rightarrow A_0 = 1 \quad B_1 = \frac{12}{a} \quad B_3 = \frac{2}{a^3} \quad A_4 = \frac{3}{a^4}$$

$$\Rightarrow u(r, \theta) = 1 + \frac{3}{a^4} r^4 \cos(4\theta) + \frac{12}{a} r \sin(\theta) + \frac{2}{a^3} r^3 \sin(3\theta)$$

b. (10 pts) Find the full solution if on the boundary,  $u(a, \theta) = 1$

$$A_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} 1 d\theta = \frac{1}{2\pi} \theta \Big|_{-\pi}^{\pi} = \frac{1}{2\pi} [\pi + \pi] = 1$$

$$A_n a^n = \frac{1}{\pi} \int_{-\pi}^{\pi} (1) \cos(n\theta) d\theta = \frac{1}{\pi} \frac{1}{n} \sin(n\theta) \Big|_{-\pi}^{\pi} = \frac{1}{n\pi} [\sin(n\pi) - \sin(-n\pi)] = 0$$

$$B_n a^n = \frac{1}{\pi} \int_{-\pi}^{\pi} (1) \sin(n\theta) d\theta = \frac{1}{\pi} \frac{1}{n} \cos(n\theta) \Big|_{-\pi}^{\pi} = \frac{1}{n\pi} [\cos(n\pi) - \cos(-n\pi)] = 0$$

$$\text{So } A_0 = 1 \quad A_n = 0 \quad B_n = 0$$

$$\Rightarrow u(r, \theta) = 1$$

8. (10 points) Consider the two ODE's coming from Laplace's equation in polar coordinates,  
 $\frac{d^2\phi}{d\theta^2} = -\lambda\phi$  and  $\frac{r}{G} \frac{d}{dr} \left( r \frac{dG}{dr} \right) = \lambda$

a. (5 pts) Find the general solution to both if  $\lambda = 0$

$$\int \frac{d^2\phi}{d\theta^2} = 0$$

$$\int \frac{d\phi}{d\theta} = C_1$$

$$\boxed{\phi = C_1\theta + C_2}$$

$$\frac{r}{G} \frac{d}{dr} \left( r \frac{dG}{dr} \right) = 0$$

$$\int \frac{d}{dr} \left( r \frac{dG}{dr} \right) = 0$$

$$r \frac{dG}{dr} = C_1$$

$$\int \frac{dG}{dr} = \frac{C_1}{r}$$

$$\boxed{G = C_1 \ln(r) + C_2}$$

b. (5 pts) Find the general solution to both if  $\lambda > 0$

$$\frac{d^2\phi}{d\theta^2} + \lambda\phi = 0$$

$$r^2 + \lambda = 0$$

$$r^2 = -\lambda$$

$$r = \pm\sqrt{-\lambda} = \pm\sqrt{\lambda}i$$

$$\Rightarrow \boxed{\phi(\theta) = C_1 \cos(\sqrt{\lambda}\theta) + C_2 \sin(\sqrt{\lambda}\theta)}$$

$$\frac{r}{G} \frac{d}{dr} \left( r \frac{dG}{dr} \right) = \lambda$$

$$r \frac{d}{dr} \left( r \frac{dG}{dr} \right) = \lambda G$$

product rule

$$r \left[ \frac{dG}{dr} + r \frac{d^2G}{dr^2} \right] = \lambda G$$

$$r^2 \frac{d^2G}{dr^2} + r \frac{dG}{dr} - \lambda G = 0$$

Guess  $r^p$

$$r^2 \frac{d^2}{dr^2} (r^p) + r \frac{d}{dr} r^p - \lambda r^p = 0$$

$$r^2 p(p-1)r^{p-2} + r p r^{p-1} - \lambda r^p = 0$$

$$p(p-1)r^p + p r^p - \lambda r^p = 0$$

$$(p(p-1) + p - \lambda)r^p = 0$$

$$p^2 - p + p - \lambda = 0$$

$$\Rightarrow p^2 - \lambda = 0$$

$$p^2 = \lambda$$

$$p = \pm\sqrt{\lambda}$$

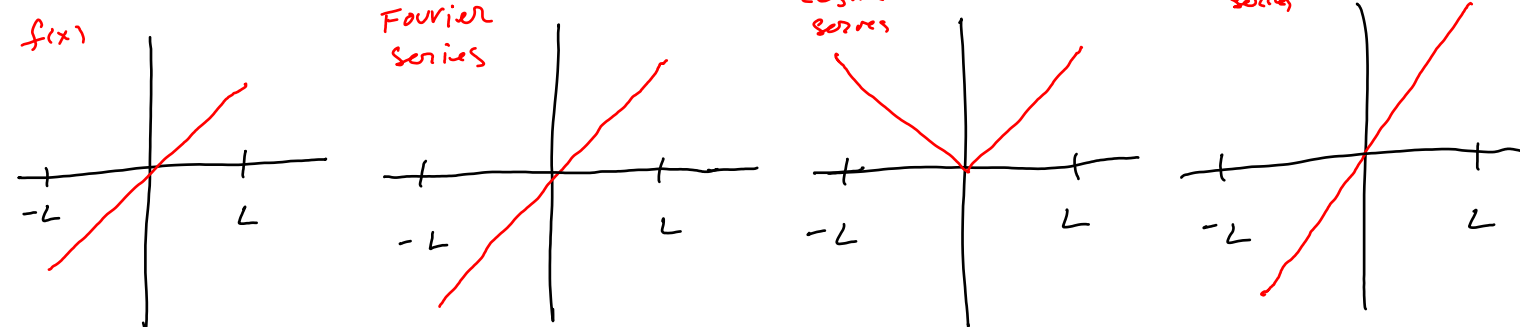
$$\Rightarrow \boxed{G(r) = C_1 r^{\sqrt{\lambda}} + C_2 r^{-\sqrt{\lambda}}}$$

9. (15 points) For the following functions sketch  $f(x)$ , the Fourier series for  $f(x)$ , the Fourier Sine series of  $f(x)$ , and the Fourier Cosine series of  $f(x)$  on the interval  $-L < x < L$ .

a. (5 pts)  $f(x) = x$

even extension

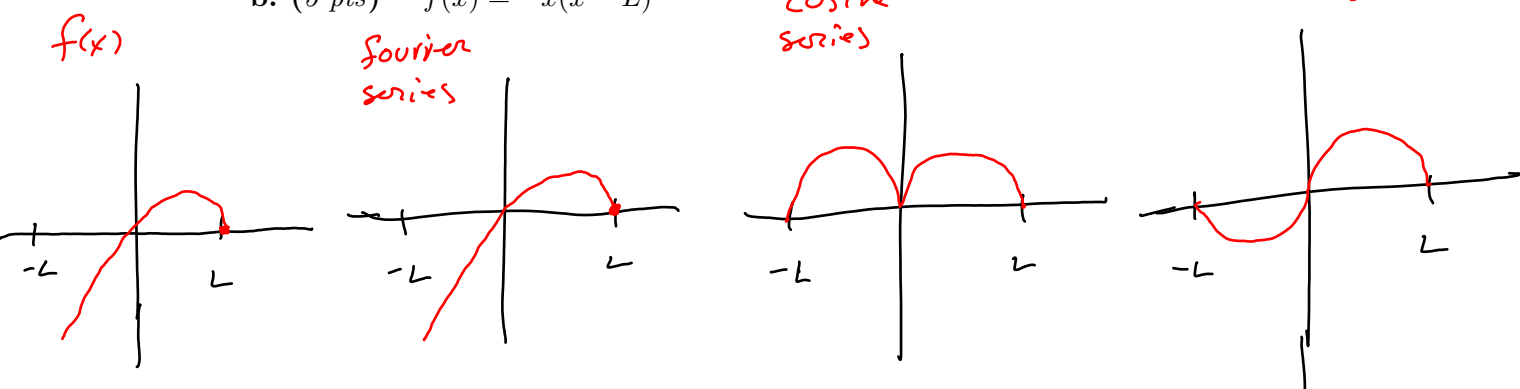
Sine series



b. (5 pts)  $f(x) = -x(x - L)$

cosine series

sine series

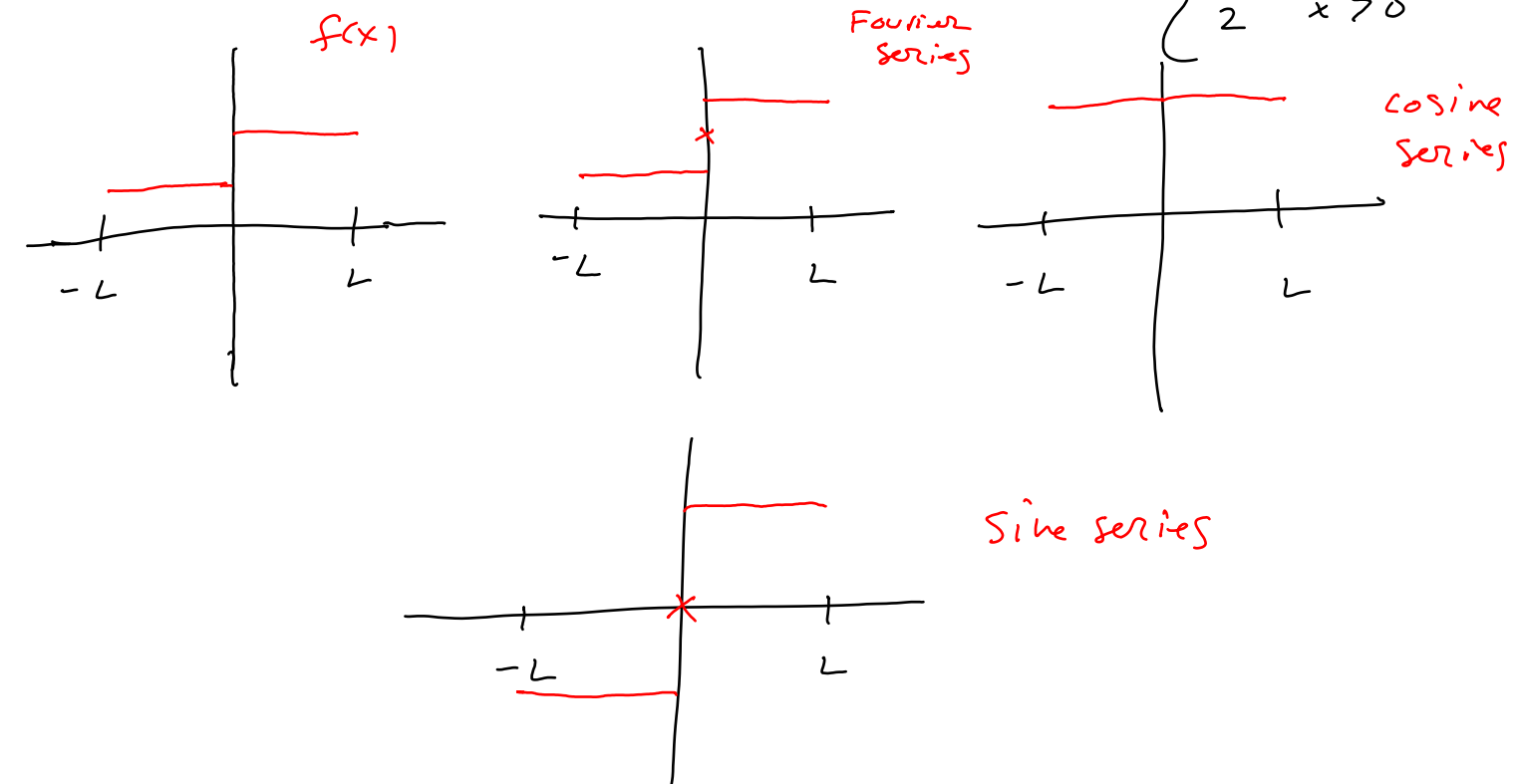


c. (5 pts)  $f(x) = 1$  for  $x > 0$  and  $f(x) = 2$  for  $x < 0$

$$f(x) = \begin{cases} 1 & x < 0 \\ 2 & x > 0 \end{cases}$$

Fourier series

cosine series



10. (5 points)

Find the Fourier Series coefficients for  $f(x) = x$  on the interval  $-L < x < L$

$$A_0 = \frac{1}{2L} \int_{-L}^L f(x) dx = \frac{1}{2L} \int_{-L}^L x dx = \frac{1}{2L} \left[ \frac{x^2}{2} \right]_{-L}^L = 0$$

$$A_n = \frac{1}{2L} \int_{-L}^L f(x) \cos\left(\frac{n\pi}{L}x\right) dx = \frac{1}{2L} \int_{-L}^L x \cos\left(\frac{n\pi}{L}x\right) dx = 0$$

can also be done by parts

↓ odd · even = odd  $\Rightarrow A_n = 0$

$$B_n = \frac{1}{2L} \int_{-L}^L f(x) \sin\left(\frac{n\pi}{L}x\right) dx = \frac{1}{2L} \int_{-L}^L x \sin\left(\frac{n\pi}{L}x\right) dx$$

$u = x \quad dv = \sin\left(\frac{n\pi}{L}x\right)$   
 $du = dx \quad v = -\frac{L}{n\pi} \cos\left(\frac{n\pi}{L}x\right)$   
 $\int u dv = uv - \int v du$   
 integration by parts

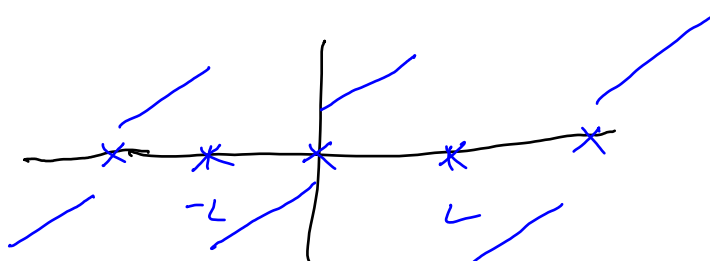
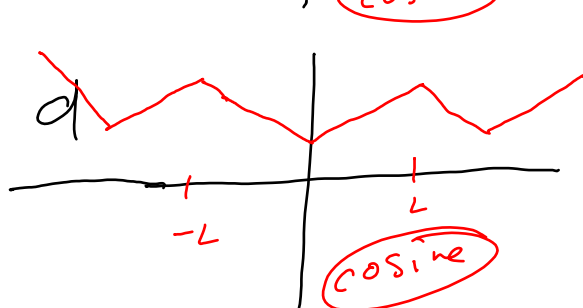
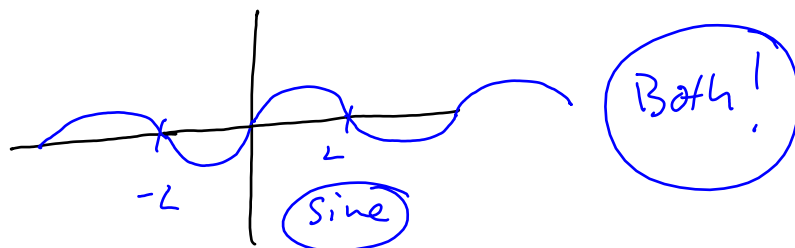
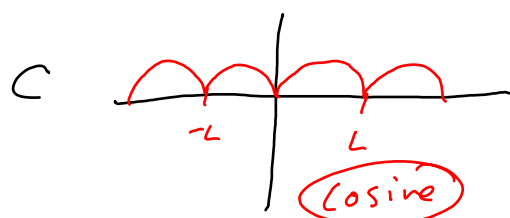
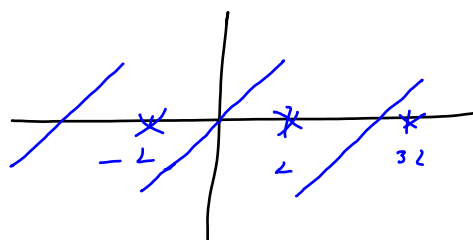
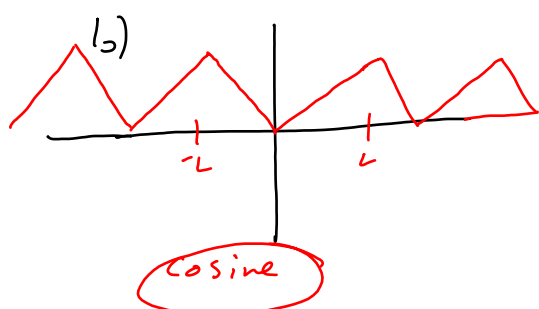
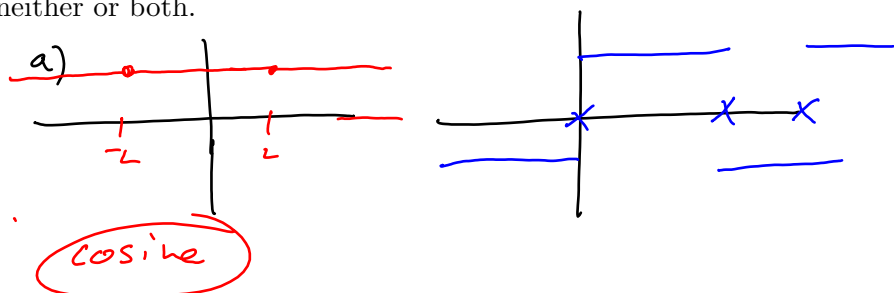
$$= \frac{1}{2L} \left[ -x \frac{L}{n\pi} \cos\left(\frac{n\pi}{L}x\right) - \int_{-L}^L -\frac{L}{n\pi} \cos\left(\frac{n\pi}{L}x\right) dx \right] = \frac{1}{2L} \left[ -\frac{xL}{n\pi} \cos\left(\frac{n\pi}{L}x\right) + \frac{L^2}{n^2\pi^2} \sin\left(\frac{n\pi}{L}x\right) \right]_{-L}^L$$

$$= \frac{1}{2L} \left[ -\frac{L^2}{n\pi} \cos(n\pi) - \frac{L^2}{n\pi} \cos(-n\pi) + \frac{L^2}{n^2\pi^2} \sin(n\pi) - \sin(-n\pi) \right] = \frac{-L}{n\pi} \cos(n\pi) = \frac{-L}{n\pi} (-1)^n = \frac{L}{n\pi} (-1)^{n+1} = B_n$$

$\cos(n\pi)$   
 even

11. (5 points) For the following functions state whether it has a continuous Fourier cosine series or continuous Fourier sine series neither or both.

- (1 pt)  $f(x) = 1$
- (1 pt)  $f(x) = x$
- (1 pt)  $f(x) = -(x)(x-L)$
- (1 pt)  $f(x) = 1 + x$



Both!

12. (10 points) Verify that  $u(x, t) = \int_0^\infty e^{-k\lambda t} (c_1 \cos(\sqrt{\lambda}x) + c_2 \sin(\sqrt{\lambda}x)) d\lambda$  is a solution to the 1-d heat equation  $\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}$  plus it in to the ODE.

$$\begin{aligned} \frac{\partial}{\partial t} \int_0^\infty e^{-k\lambda t} (c_1 \cos(\sqrt{\lambda}x) + c_2 \sin(\sqrt{\lambda}x)) d\lambda &\stackrel{?}{=} k \frac{\partial^2}{\partial x^2} \int_0^\infty e^{-k\lambda t} (c_1 \cos(\sqrt{\lambda}x) + c_2 \sin(\sqrt{\lambda}x)) d\lambda \\ \downarrow & \qquad \qquad \qquad \downarrow \\ \int_0^\infty \frac{\partial}{\partial t} e^{-k\lambda t} (c_1 \cos(\sqrt{\lambda}x) + c_2 \sin(\sqrt{\lambda}x)) d\lambda &= k \int_0^\infty e^{-k\lambda t} \frac{\partial^2}{\partial x^2} (c_1 \cos(\sqrt{\lambda}x) + c_2 \sin(\sqrt{\lambda}x)) d\lambda \\ \downarrow \text{Take time derivative} & \qquad \qquad \qquad \downarrow \text{Take two derivatives} \\ \int_0^\infty -k\lambda e^{-k\lambda t} (c_1 \cos(\sqrt{\lambda}x) + c_2 \sin(\sqrt{\lambda}x)) d\lambda &= k \int_0^\infty e^{-k\lambda t} (-c_1 \lambda \cos(\sqrt{\lambda}x) - c_2 \lambda \sin(\sqrt{\lambda}x)) d\lambda \\ \downarrow \text{factor out } (-k\lambda) & \qquad \qquad \qquad \downarrow \text{factor out } (-\lambda) \\ -k\lambda \int_0^\infty e^{-k\lambda t} (c_1 \cos(\sqrt{\lambda}x) + c_2 \sin(\sqrt{\lambda}x)) d\lambda &= -k\lambda \int_0^\infty e^{-k\lambda t} (c_1 \cos(\sqrt{\lambda}x) + c_2 \sin(\sqrt{\lambda}x)) d\lambda \end{aligned}$$

□

13. (10 points) Solve the 1-d wave equation  $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$  on the whole real line subject to the initial conditions  $u(x, 0) = f(x)$  and  $\frac{\partial u}{\partial t}(x, 0) = 0$

To do this we take the Fourier transform of the PDE

$$\mathcal{F}\left[\frac{\partial^2 u}{\partial t^2}\right] = \mathcal{F}\left[c^2 \frac{\partial^2 u}{\partial x^2}\right]$$

$$\Rightarrow \frac{\partial^2}{\partial t^2} \mathcal{F}[u] = c^2 (i\omega)^2 \mathcal{F}[u] \quad \text{let } \mathcal{F}[u] = \bar{u}(\omega, t)$$

$$\Rightarrow \frac{\partial^2}{\partial t^2} \bar{u} = -c^2 \omega^2 \bar{u} \quad r^2 = -c^2 \omega^2 \quad r = \pm c\omega i$$

$$\Rightarrow \bar{u} = c_1(\omega) \cos(c\omega t) + c_2(\omega) \sin(c\omega t)$$

then

$$\mathcal{F}\left[\frac{\partial u}{\partial t}(x, 0)\right] = \frac{d}{dt} \mathcal{F}[u(x, 0)] = \frac{d}{dt} \bar{u}(x, 0) = 0$$

$$\Rightarrow \frac{d\bar{u}}{dt} = -c_1 c\omega \sin(c\omega t) + c_2 c\omega \cos(c\omega t) = 0 \Rightarrow c_2 = 0$$

$$\Rightarrow \bar{u}(\omega, t) = c_1(\omega) \cos(c\omega t) \quad \text{then take inverse Fourier transform}$$

$$u(x, t) = \mathcal{F}^{-1}[\bar{u}(\omega, t)] = \int_{-\infty}^{\infty} c_1(\omega) \cos(c\omega t) e^{i\omega x} d\omega$$

This is as far as you need to go.

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