

# Sec 12.1 and 12.2 and 12.3

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## Section 12.1

What is a PDE?

Strictly speaking, a PDE is an equation that relates the partial derivatives of one or more functions of several variables.

Example:

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} \text{ or in another notation } u_t = k u_{xx}$$

This PDE relates the time rate of change of a quantity  $u$  to its second spatial derivative. This is the 1-d heat equation which we will derive today.

Just as with ODE's we need some initial conditions to be able to find a unique solution. For a PDE rather than just an initial value we have an initial function and usually some boundary conditions.

ODE:

$$\frac{du}{dt} = au$$

Where  $u = u(t)$

Initial Condition  
 $u_0 = u(0) = 2$

PDE

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}$$

Where  $u = u(x, t)$

Initial Condition  
 $u(x, 0) = f(x) = -x^2 + x$

Boundary Conditions  
 $u(0, t) = 0 \quad u(1, t) = 0$



What does the solution to the 1-d heat equation look like?



$t = 0$



$\rightarrow$  increasing  $t$

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## Section 12.2

Derivation of the 1-d Heat equation:

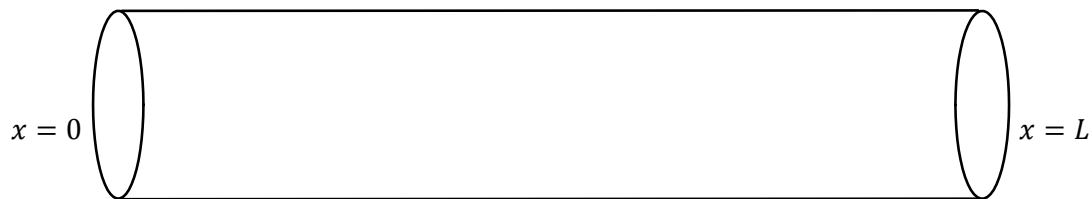
For this derivation we employ physical principles and arguments to give you a sense of what the heat equation actually says:

Thermal Energy Density

Let's define the thermal energy density as the amount of thermal energy  $e$  in joules per unit volume  $m^3$

Let  $e(x, t) = \text{thermal energy density}$ .

Let's first think of a metal rod and derive an equation for heat flow with in the rod.



We will assume that the rod is perfectly insulated except at the ends so that heat can only flow in 1 direction in this case we will call it the x direction.

Let's define the heat flux field

$\phi(x, t) = \text{amount thermal energy flowing per unit time per unit surface area}$

We know from physics that energy is conserved. For this system, this means that whatever total energy we started remains the same, it can leave the rods through the ends but does not disappear from existence.

Let  $E$  be the total thermal energy of the rod. Conservation of energy in in this case means that

$\frac{dE}{dt} = \text{sum of the thermal energy coming in or leaving through the ends of the rod}$

How can we calculate that? Recall that if we integrate the flux field over the surface of the rod this tells us how much of a quantity is leaving through the surface. In our case heat can only leave through the ends. We then obtain

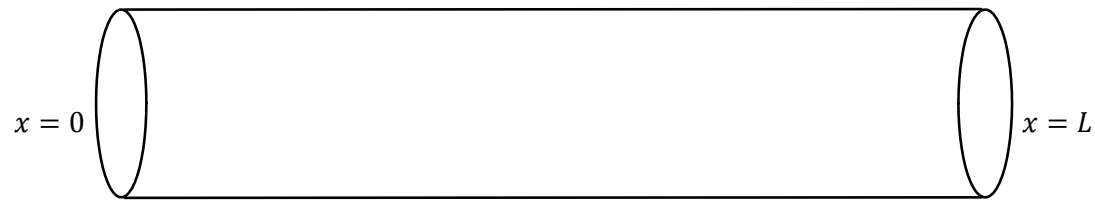
$$\frac{dE}{dt} = \int_S \vec{\phi} \cdot \vec{n} ds = \vec{\phi}(L, t) A - \vec{\phi}(0, t) A$$

where  $A$  is the surface area of the ends.

Let's derive the last part of that equality.

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$$\frac{dE}{dt} = \int_S \vec{\phi} \cdot \vec{n} ds$$

Now that we have this we can employ the fundamental theorem of calculus which yields

$$A(\phi(L, t) - \phi(0, t)) = A \int_0^L \frac{\partial \phi}{\partial x} dx$$

We now have the statement that

$$\frac{dE}{dt} = A \int_0^L \frac{\partial \phi}{\partial x} dx$$

This is nice and all but we have no idea what the flux actually is, we need an equation we can actually solve, we are headed for a PDE.

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Recall that we defined  $e(x, t) = \text{thermal energy density}$ . The thermal energy per unit volume

$$\text{With this } E = A \int_0^L e(x, t) dx$$

Let's Examine what this says about  $\frac{dE}{dt}$

So we now have that:

$$A \int_0^L \frac{\partial e}{\partial t} dx = A \int_0^L \frac{\partial \phi}{\partial x} dx$$

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So now we have a PDE namely that:  $\frac{\partial e}{\partial t} = -\frac{\partial \phi}{\partial x}$

However, we are still not quite at something we can solve.. Here we have an equation relating the partials of two different functions what we want is an equation relating the partials of the *same* function

Temperature and specific heat:

Define:  $u(x, t) = \text{temperature}$ ,  $c = \text{specific heat}$ , and  $\rho(x) = \text{mass density}$

Specific heat is the amount of energy required to raise a unit of mass of a material one degree and has units of  $\frac{\text{energy}}{\text{mass temperature}}$

Thus we can say:  $e(x, t) = c(x)\rho(x)u(x, t) = \frac{\text{energy}}{\text{mass temp}} \times \frac{\text{mass}}{\text{vol}} \times \text{temp} = \frac{\text{energy}}{\text{vol}}$

So far we have:

$$\frac{\partial e}{\partial t} = c(x)\rho(x) \left( \frac{\partial u}{\partial t} \right) = -\frac{\partial \phi}{\partial x}$$

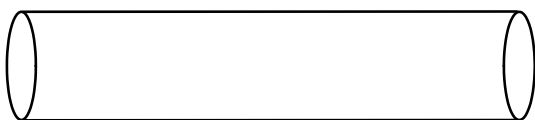
If we can write  $\phi$  in terms of the temperature then we will finally have a PDE in terms of one function's partials.

Fourier's Law: Heat flows from hot to cold, and the greater the difference the faster the flow

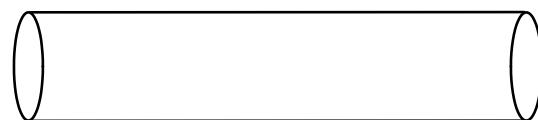
$$\phi = -K_0 \frac{\partial u}{\partial x}$$

Recall that a gradient points in the direction of greatest change  
This is the 1-d gradient of temperature.

What's with the minus sign? This ensures hot to cold flow.



$$\frac{\partial u}{\partial x} > 0$$



$$\frac{\partial u}{\partial x} < 0$$

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Now we have:

$$c(x)\rho(x)\frac{\partial u}{\partial t} = -\frac{\partial \phi}{\partial x} \text{ and } \frac{\partial \phi}{\partial x} = -K_o \frac{\partial u}{\partial x}$$

Combining these we obtain the heat equation in 1-d!

$$c(x)\rho(x)\frac{\partial u}{\partial t} = \frac{\partial}{\partial x} \left( -K_o \frac{\partial u}{\partial x} \right) \text{ and if } K_o \text{ happens to be constant in } x$$

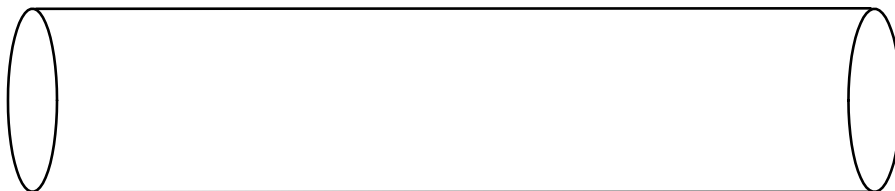
$$c(x)\rho(x)\frac{\partial u}{\partial t} = K_o \frac{\partial^2 u}{\partial x^2}$$

Often we will deal with the case where both  $c(x)$  and  $\rho(x)$  also constant allowing us to divide both sides by them and obtain a new constant  $k$

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}$$

We have derived the heat equation for the case where there were no internal sources or sinks. If there is an internal source of heat or heat loss in the rod we can simply add it to the left hand side of the equation.

$$c(x)\rho(x)\frac{\partial u}{\partial t} = K_o \frac{\partial^2 u}{\partial x^2} + Q$$



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## Section 12.3

Boundary Conditions describe what is happening on the boundary! For the 1-d heat equation we typically have three types of Boundary conditions.

Prescribed temperature at the ends:



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Prescribed flux at the ends:



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Newton's Law of Cooling  $\phi(t) = -H(T_{object} - T_{air})$

