

Sec 15.1-15.4

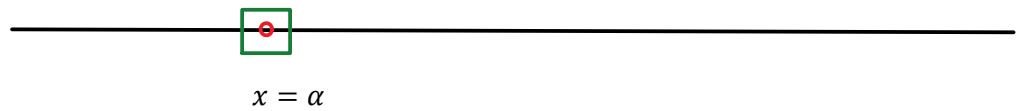
A vertically vibrating string.

Let's derive the PDE for a vertically vibrating string using Newton's second law, $\sum \vec{F} = m\vec{a}$

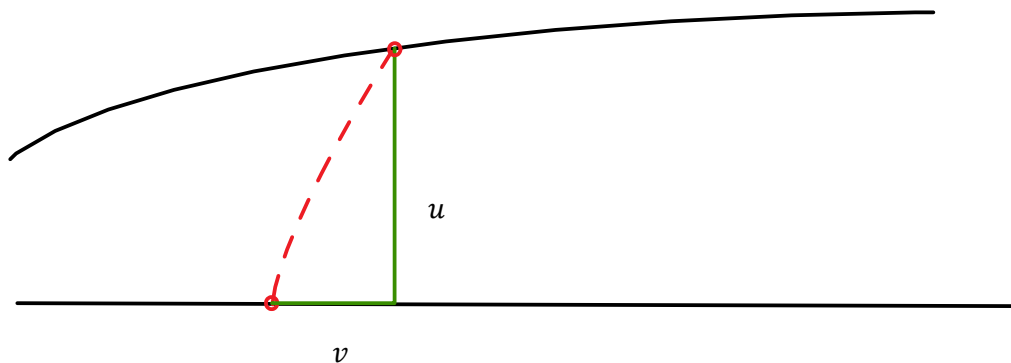
Suppose we have a string in its equilibrium position.

Let's single out a small particle on the string say at $x = \alpha$ and track the motion of that particle.

Equilibrium position.



Let's compare the equilibrium position to a perturbed position and name the Vertical displacement u and horizontal displacement v



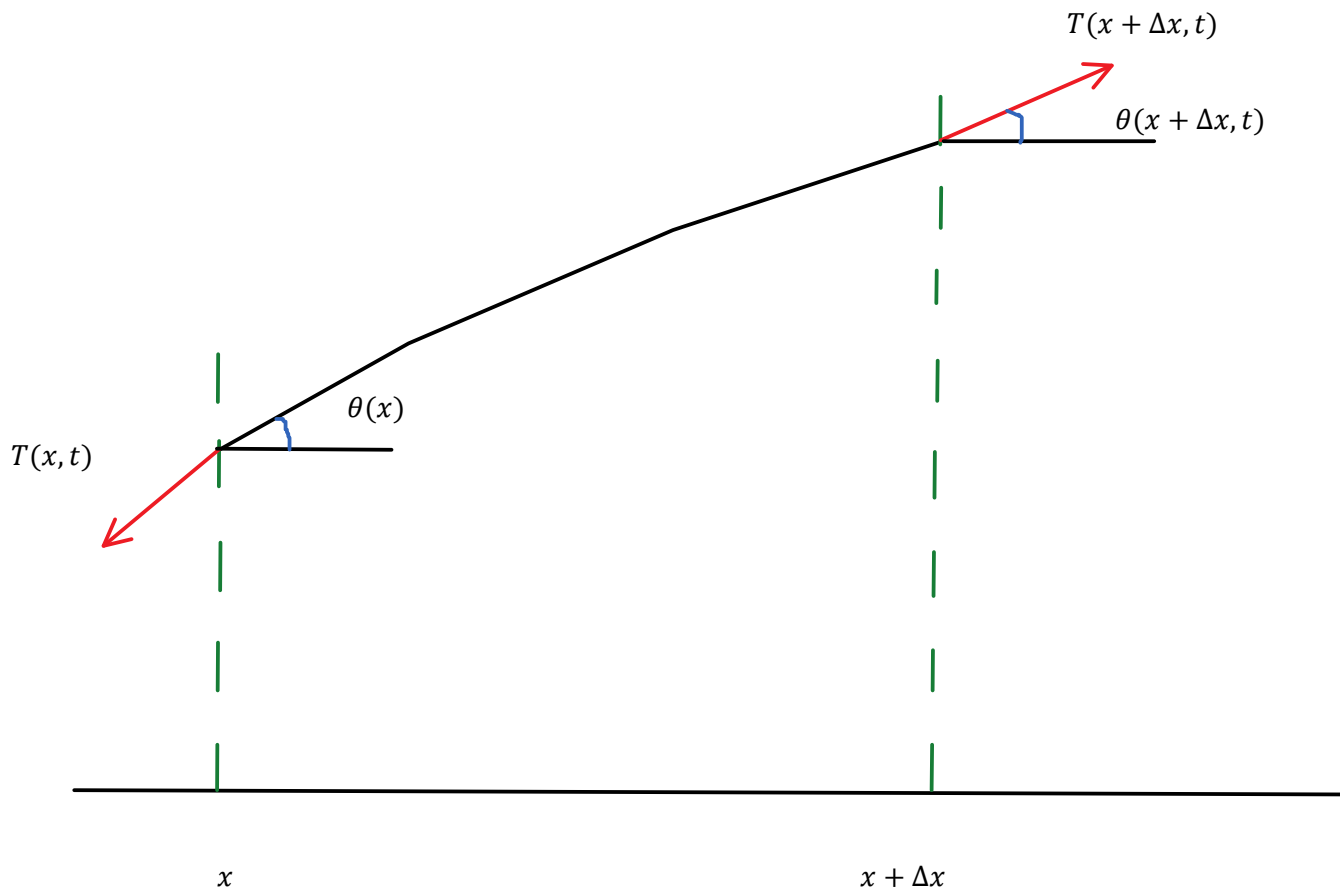
We notice that the vertical displacement of the point is much larger than the horizontal displacement for small enough perturbations. We will assume this is the case this will mean that our vertical displacement u will be a function of x and time only.

$$u = u(x, t)$$

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Now let's apply Newton's laws to a small section of the string of density $\rho_0(x)$ between x and $x + \Delta x$. If Δx is small enough we can simply use Newton's law for a point mass like the one we were tracking previously.

Let's consider the forces... We will have a tension T at both ends which will depend on both position x and time t . We will also have body forces such as gravity which we will wrap up into a term $Q(x, t)$ which will be the *body force per unit mass*



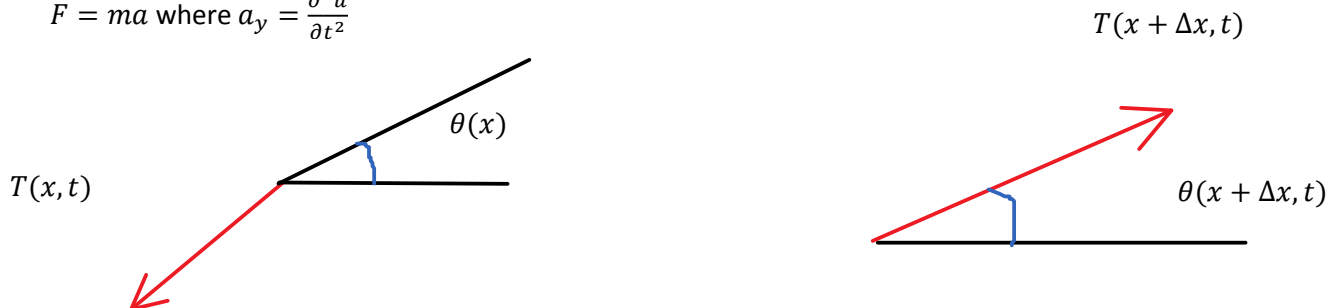
The mass of this section of string is the density times the length so: $m = \rho_0(x)\Delta x$

If we choose x to the right as positive then the sum of the forces is: $\sum \vec{F} = T(x + \Delta x, t) - T(x, t) + \rho_0(x)\Delta x Q(x, t)$

We claimed that the horizontal displacement was negligible thus we are only interested in finding an equation involving the y component of the displacement which we said was a function of x and time.

$u(x, t)$ Thus we will need only the vertical components of our forces to use with Newton's second law

$F = ma$ where $a_y = \frac{\partial^2 u}{\partial t^2}$



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So we have:

$$m = \rho_0(x)\Delta x \quad a_y = \frac{\partial^2 u}{\partial t^2} \quad \sum \vec{F}_y = T(x + \Delta x, t)\sin(\theta(x + \Delta x, t)) - T(x, t)\sin(\theta(x, t)) + \rho_0(x)\Delta x Q(x, t)$$

Thus:

$$\rho_0(x)\Delta x \frac{\partial^2 u}{\partial t^2} = T(x + \Delta x, t)\sin(\theta(x + \Delta x, t)) - T(x, t)\sin(\theta(x, t)) + \rho_0(x)\Delta x Q(x, t)$$

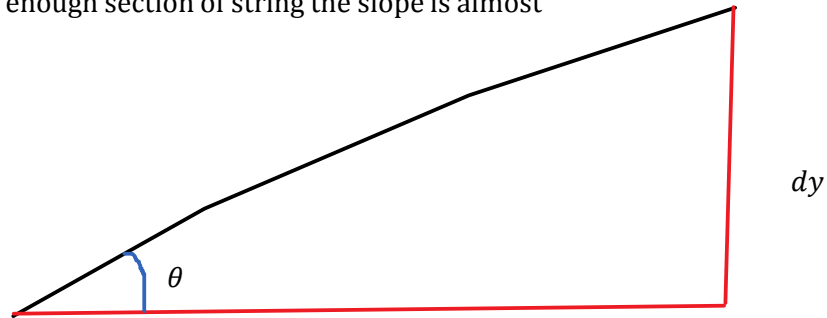
Now we want to get rid of Δx so we will divide by it and take the limit as $\Delta x \rightarrow 0$

So now we have

$$\rho_0(x) \frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x} (T(x, t) \sin(\theta(x, t))) + \rho_0(x) Q(x, t)$$

Now let's think about that $\sin(\theta(x, t))$ term.. For a small enough section of string the slope is almost constant

$$\text{slope} = \frac{dy}{dx} = \tan(\theta(x, t)) = \frac{\partial u}{\partial x}$$



and for small enough vertical displacements we have angles close to zero

thus $\cos(\theta) \approx 1$ and $\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} \approx \sin(\theta)$

If we put this all together we can get an equation for u in terms of just t and x !

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Behold! The 1-d wave equation!

$$\rho_0(x) \frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x} \left(T(x, t) \frac{\partial u}{\partial x} \right) + \rho_0(x) Q(x, t)$$

We get some variations of this equations making a few assumptions, if the tension does not change much in the string then $T(x, t) = \text{constant}$ so we could have:

$$\rho_0(x) \frac{\partial^2 u}{\partial t^2} = \left(T_0 \frac{\partial^2 u}{\partial x^2} \right) + \rho_0(x) Q(x, t)$$

If there are no body forces or we let a sagged state of the string be it's equilibrium position we can ditch $Q(x, t)$ and obtain:

$$\rho_0(x) \frac{\partial^2 u}{\partial t^2} = T_0 \frac{\partial^2 u}{\partial x^2}$$

If we divide by $\rho_0(x)$ we obtain the most commnly written form of this PDE

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \text{where } c^2 = \frac{T_0}{\rho_0(x)}$$

What kinds of boundary and initial conditions can we have?

Common Boundary Conditions

Fixed ends: $u(0, t) = 0$ or $u(L, t) = 0$

Forced ends: $u(0, t) = f(t)$ or $u(L, t) = f(t)$

Common Initial Conditions

Initial shape of string: $u(x, 0) = f(x)$

Initial velocity of string: $\frac{\partial u}{\partial t}(x, 0) = g(x)$

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Solving the wave equation

We are going to solve the wave equation assuming a uniform density $\rho_0(x) = \rho_0 = \text{constant}$ and constant tension T_0 and with two fixed ends a general initial shape and general initial velocity.

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \text{where } c^2 = \frac{T_0}{\rho_0(x)}$$

$$u(0, t) = 0 \text{ and } u(L, t) = 0$$

$$\text{Initial shape of string: } u(x, 0) = f(x)$$

$$\text{Initial velocity of string: } \frac{\partial u}{\partial t}(x, 0) = g(x)$$

We begin the usual way by separating variables we assume: $u(x, t) = \phi(x)h(t)$



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Think about what we might want our solution to be, we choose $\lambda > 0$ and attach a minus sign, that is we expect to have two oscillatory solutions here. It does turn out that for $\lambda = 0$ and $\lambda < 0$ we get only trivial solutions.

$$\frac{1}{c^2} \frac{1}{h} \frac{d^2 h}{dt^2} = \frac{1}{\phi} \frac{d^2 \phi}{dx^2} = -\lambda$$

Let's take a look at our equation for ϕ

$$\frac{1}{\phi} \frac{d^2 \phi}{dx^2} = -\lambda$$

$$\phi(0) = 0 \quad \text{and} \quad \phi(L) = 0$$

Any guesses for how this will play out?

Now let's look at our time dependent ODE for which we have no homogeneous boundary conditions.

$$\frac{1}{c^2} \frac{1}{h} \frac{d^2 h}{dt^2} = -\lambda$$

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So we have $\phi_n(x) = c_2 \sin\left(\frac{n\pi x}{L}\right)$ and $h_n(t) = c_3 \cos\left(\frac{cn\pi}{L}t\right) + c_4 \sin\left(\frac{cn\pi}{L}t\right)$

Thus taking the product of the solutions

$$u_n(x, t) = c_2 \sin\left(\frac{n\pi x}{L}\right) \left(c_3 \cos\left(\frac{cn\pi}{L}t\right) + c_4 \sin\left(\frac{cn\pi}{L}t\right) \right) = A_n \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{cn\pi}{L}t\right) + B_n \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{cn\pi}{L}t\right)$$

Applying super position

$$u(x, t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{cn\pi}{L}t\right) + B_n \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{cn\pi}{L}t\right)$$

Now we can apply our initial conditions. $u(x, 0) = f(x)$ $\frac{\partial u}{\partial t}(x, 0) = g(x)$

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So we have initial condition equations:

$$f(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right)$$

We then find our A_n and B_n in any of the usual ways.....

$$g(x) = \sum_{n=1}^{\infty} B_n \frac{n\pi c}{L} \sin\left(\frac{n\pi x}{L}\right)$$

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Homework Example: 15.4 #7

Suppose we have a vibrating string satisfying the following PDE problem.

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \quad u(0, t) = 0 \text{ and } u(L, t) = 0 \quad \text{Initial shape of string: } u(x, 0) = f(x)$$

$$\text{Initial velocity of string: } \frac{\partial u}{\partial t}(x, 0) = 0$$

Show that: $u(x, t) = \frac{1}{2} [F(x - ct) + F(x + ct)]$