

Of interest in many physical problems is what we call the equilibrium state or steady state. This is the state where a system stops undergoing change.

$$\frac{\partial u}{\partial t} = 0$$

Let's apply this to the 1-d heat equation. $\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} + Q$ with $Q = 0$

If we are in equilibrium this means that the temperature is no longer changing or that $\frac{\partial u}{\partial t} = 0$

Thus we have the now ODE

$$\frac{d^2 u}{dx^2} = 0$$

$$0 = k \frac{\partial^2 u}{\partial x^2} \quad \frac{\partial^2 u}{\partial x^2} = 0$$

We will still have an initial condition

$$u(x, 0) = f(x)$$

$$e^{-x^2} \quad \cos(x)$$

Whether or not we will need the initial condition will depend on our boundary conditions.

Let's first consider Dirichlet boundary conditions (prescribed temperature at the ends)

$$u(0, t) = T_1 \text{ and } u(L, t) = T_2$$

We then want to solve the ODE: $\frac{\partial^2 u}{\partial x^2} = 0$ with boundary conditions $u(0, t) = T_1$ and $u(L, t) = T_2$

$$\int \frac{\partial^2 u}{\partial x^2} dx = \int 0 dx$$

$$\text{APPLY BC'S} \\ u(0, t) = T_1$$

$$\int \frac{\partial u}{\partial x} = \int C_1$$

$$u(0) = C_1(0) + C_2 = T_1$$

$$C_2 = T_1$$

$$u(x) = C_1 x + C_2$$

$$u(L) = C_1(L) + T_1 = T_2$$

$$C_1(L) = T_2 - T_1$$

$$C_1 = \frac{T_2 - T_1}{L}$$

$$u(x) = \frac{T_2 - T_1}{L} x + T_1$$



$$u(x) = \frac{T_2 - T_1}{L} x + T_1$$

$$u(x) = \frac{0}{L} x + T_1$$

$$u(x) = T_1$$

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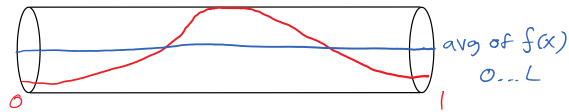
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Insulated boundaries:

Let's solve the same problem but with insulated end boundary conditions. BCs: $\frac{\partial u}{\partial x}(0) = 0$ and $\frac{\partial u}{\partial x}(L) = 0$

With an initial condition of $u(x, 0) = f(x) = e^{-(x-\frac{L}{2})^2}$ $\int_0^L e(x, t) dx = C$

First let's think about it...



$$e(x, t) = \rho c u$$

$$\int_0^L \rho c u = \text{const}$$

$$\frac{\partial^2 u}{\partial x^2} = 0 \Rightarrow c_1 x + c_2 = u(x)$$

$$\frac{\partial u}{\partial x} = c_1 \quad \frac{\partial u}{\partial x} \Big|_{x=0} = 0 = c_1$$

$$u = c_2$$

$$\int_0^L \rho c c_2 dx = \int_0^L \rho c f(x) dx$$

$$\cancel{\rho c} \frac{c_2 L}{L} = \cancel{\rho c} \frac{\int_0^L f(x) dx}{L} \Rightarrow c_2 = \frac{1}{L} \int_0^L f(x) dx$$

$$\text{avg } f(x) \Big|_a^b = \frac{\int_a^b f(x) dx}{b-a}$$

$$u = \frac{1}{L} \int_0^L f(x) dx = \text{avg}(f(x)) \Big|_0^L$$

$$\int_0^L \rho c u dx = \text{Total energy}$$

$$\int_0^L \rho c \underbrace{f(x)}_{t=0} dx = \text{Total energy}$$

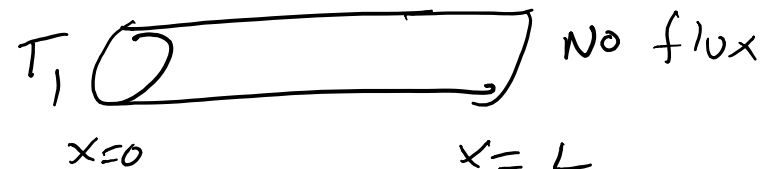
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Homework example.

From sec 12.4 Problem 1 part (f)

Determine the equilibrium temperature distribution for a one dimensional rod with the following sources and boundary conditions.

$$\frac{Q}{K_0} = x^2, \quad u(0) = T_1, \quad \frac{\partial u}{\partial x}(L) = 0$$


$x=0$ $x=L$

$$\rho C \frac{\partial u}{\partial t} = -K_0 \frac{\partial^2 u}{\partial x^2} + Q$$

$$0 = -K_0 \frac{\partial^2 u}{\partial x^2} + Q$$

$$\frac{\partial}{\partial x} \left(-K_0 \frac{\partial u}{\partial x} \right) = 0$$

$$-\frac{\partial^2 u}{\partial x^2} + \frac{Q}{K_0} = 0$$

$$\int \frac{\partial^2 u}{\partial x^2} = \int x^2$$

$$u(0) = T_1, \quad \frac{\partial u}{\partial x} \Big|_{x=L} = 0$$

$$\int \frac{\partial u}{\partial x} = \int \frac{x^3}{3} + C_1$$

$$u(0) = \frac{0^4}{12} + C_1(0) + C_2 = T_1$$

$$T_1 = C_2$$

$$\frac{\partial u}{\partial x} = \frac{x^3}{3} + C_1 \Big|_{x=L} = 0$$

$$\frac{L^3}{3} + C_1 = 0 \Rightarrow C_1 = -\frac{L^3}{3}$$

$$u(x) = \frac{x^4}{12} + C_1 x + C_2$$

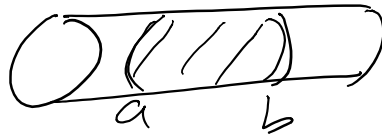
$$u(x) = \frac{x^4}{12} - \frac{L^3}{3} x + T_1$$


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Home Work Example

From Sec 12.4 Problem 4



If both ends of a rod are insulated show, using the partial differential equation, that the total thermal energy in the rod is constant.

$$\int e = \int \rho c u$$

$$E = \int_0^L \rho c u dx$$

$$\frac{\partial u}{\partial t} = -k \frac{\partial^2 u}{\partial x^2}$$

$$\frac{dE}{dt} = 0$$

$$\frac{d}{dt} \int_0^L \rho c u dx \rightarrow \int_0^L \rho c \frac{\partial u}{\partial t} dx$$

$$= \int_0^L \rho c (-k) \frac{\partial^2 u}{\partial x^2} dx = \int_0^L -\rho c k \frac{\partial}{\partial x} \left(\frac{\partial u}{\partial x} \right) dx$$

$$\left(\text{rod} \right) \frac{\partial u}{\partial x} = 0 \quad = \int_0^L 0 dx = 0$$

$$\frac{dE}{dt} = 0$$