

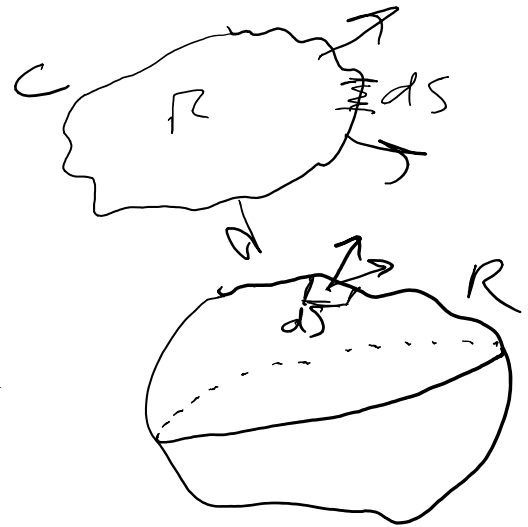
Sec 12.5 Heat Equation in 2-d or 3-d

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We will start this derivation with the idea of conservation of energy for the total thermal energy inside an arbitrary region R. The total thermal energy in R is

$$E = \int_S e(x, y, t) dS = \int_S \overbrace{c(x, y) \rho(x, y) u(x, y, t)}^{e(x, y, t)} dS \text{ In 2d}$$

$$E = \int_V e(x, y, z, t) dV = \int_V c(x, y, z) \rho(x, y, z) u(x, y, z, t) dV \text{ in 3d}$$



Conservation of energy then states:

Time rate of change
of heat energy

= Heat energy flowing across
The boundaries per unit time

+

Heat energy generated or lost
Inside per unit time



$$\frac{d}{dt} \int_S c \rho u(x, y, t) dS = - \int_C \vec{\Phi} \cdot \vec{n} dS + \int_S Q dS \text{ In 2d}$$

$$\frac{d}{dt} \int_V c \rho u(x, y, z, t) dV = - \int_S \vec{\Phi} \cdot \vec{n} dS + \int_V Q dV \text{ In 3d}$$

We would like everything in terms of a volume integral if we can. To that end recall our favorite divergence theorem.

$$\int_A \vec{F} \cdot \vec{n} dS = \int_V \nabla \cdot \vec{F} dV \text{ in 3d}$$

or

$$\int_C \vec{F} \cdot \vec{n} ds = \int_S \nabla \cdot \vec{F} dS \text{ in 2d}$$

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We will do this in 3d as the derivation is the same in 2d.

Our conservation of thermal energy equation states:

$$\frac{d}{dt} \int_V c\rho u dV = - \int_S \vec{\phi} \cdot \vec{n} dS + \int_V Q dV$$

Applying the divergence theorem ($\int_A \vec{F} \cdot \vec{n} dS = \int_V \nabla \cdot \vec{F} dV$) to our surface integral gives us:

$$- \int_S \vec{\phi} \cdot \vec{n} dS = - \int_V \nabla \cdot \vec{\phi} dV$$

$$\frac{d}{dt} \int_V c\rho u dV = - \int_V \nabla \cdot \vec{\phi} dV + \int_V Q dV$$

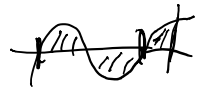
Passing the time derivative into the volume integral gives us:

$$\int_V c\rho \frac{\partial u}{\partial t} dV = - \int_V \nabla \cdot \vec{\phi} dV + \int_V Q dV$$

Moving all the integrals to one side we obtain:

$$\Rightarrow \int_V c\rho \frac{\partial u}{\partial t} dV + \int_V \nabla \cdot \vec{\phi} dV - \int_V Q dV = 0$$

$$\int_V \left(c\rho \frac{\partial u}{\partial t} + \nabla \cdot \vec{\phi} - Q \right) dV = 0$$



Because this is an integral over an arbitrary volume we may say:

$$c\rho \frac{\partial u}{\partial t} + \nabla \cdot \vec{\phi} - Q = 0$$

$$\boxed{c\rho \frac{\partial u}{\partial t} = - \nabla \cdot \vec{\phi} + Q}$$

1-d

$$c\rho \frac{\partial u}{\partial t} = - \frac{\partial}{\partial x} \phi + Q$$

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Now we have a continuity equation

$$c\rho \frac{\partial u}{\partial t} = -\nabla \cdot \vec{\phi} + Q$$

but just like in the 1d case we have two different functions namely u and $\vec{\phi}$.

To obtain a PDE in terms of just the temperature u we again appeal to Fourier and his law of heat conduction.

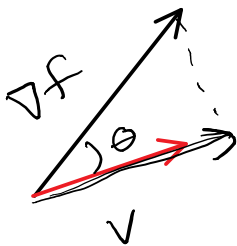
Fourier's Law of heat conduction in 2d and 3d

$$\vec{\phi} = -K_0 \nabla u$$

Let's pause to examine this law physically and mathematically for a moment.

You might recall the definition of the directional derivative from Calc III

$D_{\vec{v}}f = \nabla f \cdot \vec{v}$ where $\vec{v}$ is a unit vector. This measures how much f is changing in the direction of $\vec{v}$



We can write $\nabla f \cdot \vec{v} = \|\nabla f\| \|\vec{v}\| \cos(\theta)$

Thus we can see if $\theta = 0$ we have the greatest rate of increase of f and if $\theta = 180$ We have the greatest rate of decrease.

Note that when $\theta = 0$ $\nabla f \cdot \vec{v} = \|\nabla f\| \|\vec{v}\| \cos(0) = \|\nabla f\| (1)(1) = \|\nabla f\|$

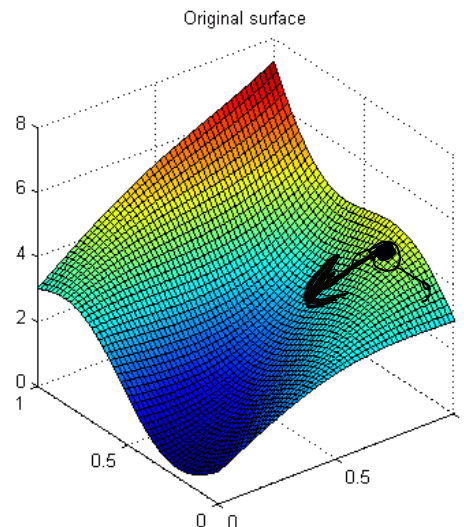
When $\theta = 180$ $\nabla f \cdot \vec{v} = \|\nabla f\| \|\vec{v}\| \cos(180) = \|\nabla f\| (1)(-1) = -\|\nabla f\|$

So the here the direction of greatest decrease of the scalar function f is the opposite of the gradient or $-\nabla f$

If f happens to be a temperature field and if heat flows from hot to cold, we would expect the flux to follow the negative gradient! In math speak:

$$\vec{\phi} = -K_0 \nabla u$$

↑ ↑
negative gradient



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Back to the Derivation.

We have: $c\rho \frac{\partial u}{\partial t} = -\nabla \cdot \vec{\phi} + Q$ and $\vec{\phi} = -K_0 \nabla u$

To finish off our derivation we only need substitute Fourier's law into our continuity equation

$$c\rho \frac{\partial u}{\partial t} = -\nabla \cdot (-\underline{k_0} \nabla u) + Q$$

Again if ρ , c and K_0 are constant we can divide by ρ and c and pull K_0 out of the derivative for a simpler heat equation.

$$c\rho \frac{\partial u}{\partial t} = k_0 (\nabla \cdot \nabla u) + Q$$

$$c\rho \frac{\partial u}{\partial t} = k_0 \nabla^2 u + Q$$

$$\frac{\partial u}{\partial t} = k \nabla^2 u + \tilde{Q}$$

Behold! The Heat Equation in 2-d or 3d!!

$$\frac{\partial u}{\partial t} = k \nabla^2 u + Q$$

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} + Q$$

Recall that: $\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$

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Initial and Boundary conditions in 2-d and 3-d

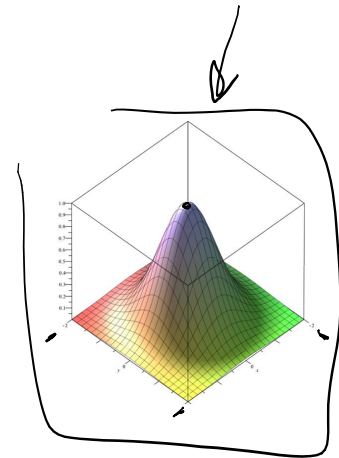
We will still have some kind of initial condition at time $t = 0$ which will represent the initial temperature distribution.

In 3d: $u(x, y, z, 0) = f(x, y, z)$ in and on the boundary of the region of interest

I can't draw a 4 dimensional surface :(

In 2d: $u(x, y, 0) = f(x, y)$ in and on the boundary of the region of interest

As an example suppose the initial temperature distribution looked like $f(x, y) = e^{-x^2 - y^2}$

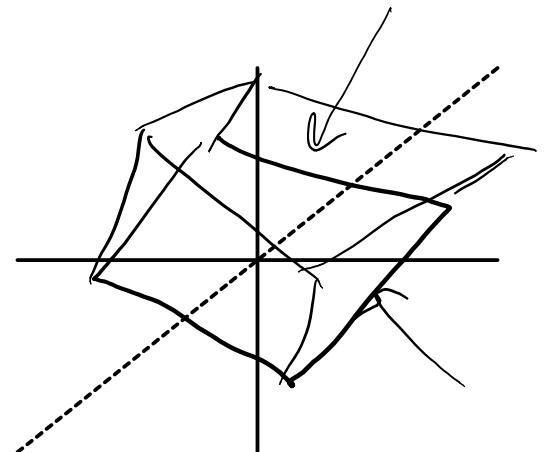
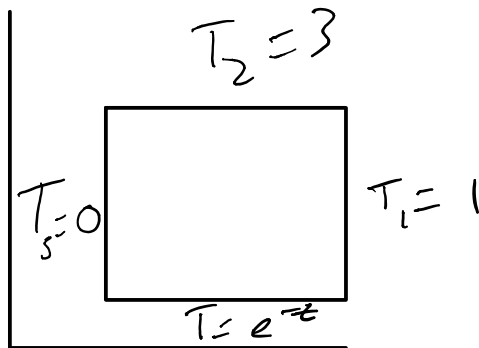
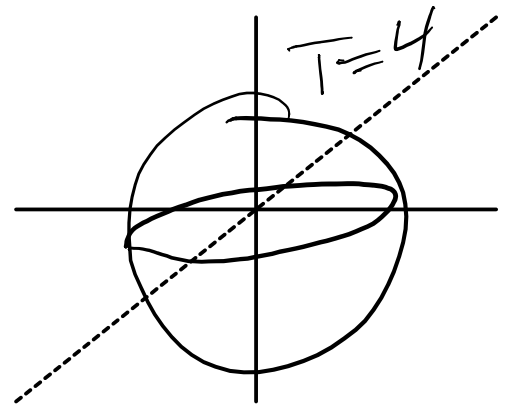
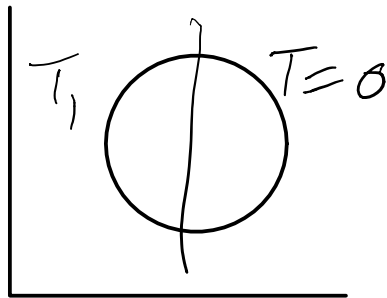


Boundary Conditions:

Dirichlet (specified temperature on the boundaries)

belongs to boundary

$$u(x, y, z, t) = T(x, y, z, t) \text{ for } (x, y, z) \in \partial R$$



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Neumann Boundary condition: Specifying the flux at the boundary. $\vec{\phi} \cdot \vec{n} = f(x, y, z, t)$ for $(x, y, z) \in \partial R$

Recall from Fourier that $\vec{\phi} = -K_0 \nabla u$ thus $\vec{\phi} \cdot \vec{n} = -K_0 \nabla u \cdot \vec{n} = f(x, y, z, t)$

We usually just divide by $-K_0$ and absorb it into the left hand side for what ever function we have. In this way flux boundary conditions are often written in terms of just the gradient and the normal vector.

$$\nabla u \cdot \vec{n} = f(x, y, z, t) \quad \text{or in another common notation} \quad \frac{\partial u}{\partial \vec{n}} = f(x, y, z, t)$$

Keep in mind this is only on the boundary of the region of interest often denoted ∂R

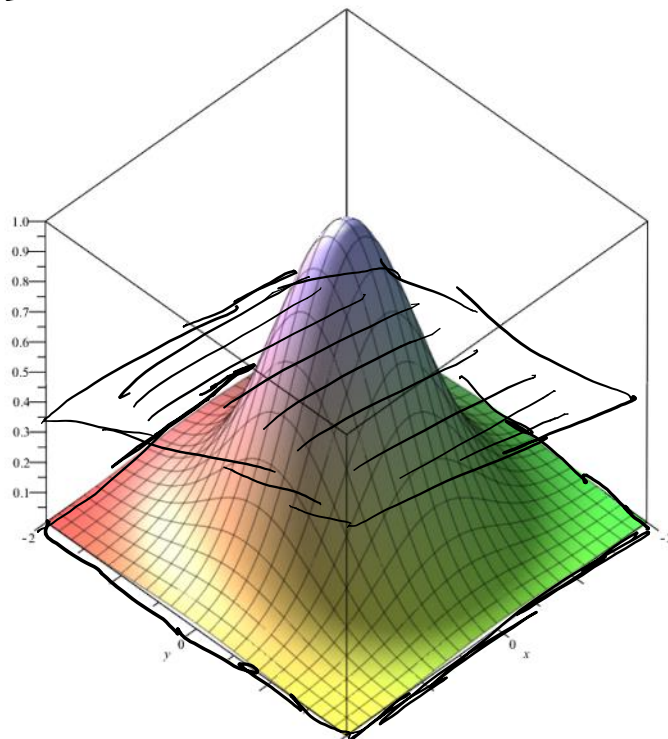
$$\frac{\partial u}{\partial \vec{n}} = \nabla u \cdot \vec{n}$$

Let's think about what an insulated boundary means. $\nabla u \cdot \vec{n} = 0$ for $(x, y, z) \in \partial R$

Let's suppose we in 2d and are interested in the region defined by: $[-2, 2] \times [-2, 2]$ and have the initial temperature distribution $u(x, y, 0) = f(x, y) = e^{-x^2 - y^2}$



If there is no flux ($\nabla u \cdot \vec{n} = 0$) at the boundary what will the solution at equilibrium look like?



Let's watch it happen!

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Equilibrium solutions to the heat equation.

Let's first look at the case where we have no heat sources ($Q = 0$) in the volume on interest and let's assume everything is in equilibrium $\frac{\partial u}{\partial t} = 0$

The heat equation ($\frac{\partial u}{\partial t} = k\nabla^2 u + Q$) then becomes: $k\nabla^2 u = 0$

Or: $\nabla^2 u = 0$

This is what we call Laplace's equation which we will learn to solve later.

If there is an internal source of heat $Q \neq 0$ but we assume equilibrium $\frac{\partial u}{\partial t} = 0$ then we obtain what is called Poisson's equation

$$\nabla^2 u = -\frac{Q}{K_0}$$

Which we will also learn to solve later.

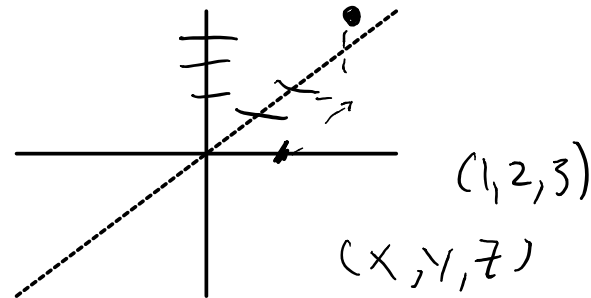
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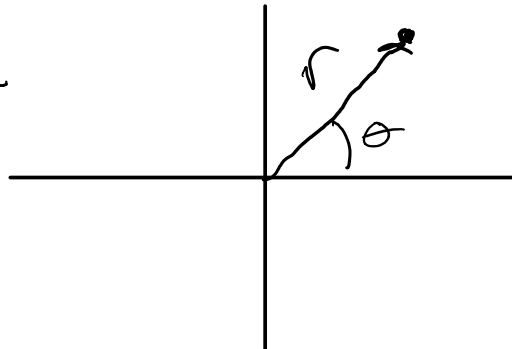
A look at the Laplacian in other coordinate systems.

Depending on the geometry of our problem we may want to use a specific coordinate system. Thus we will need to know that $\nabla^2 u$ looks like in what ever coordinate system we choose.

In Good 'ol Cartesian: $\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$

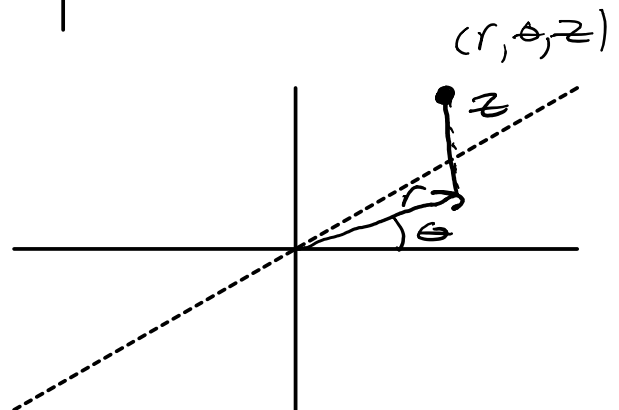


In polar(2d): $\nabla^2 u = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2}$



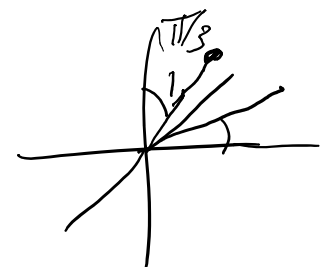
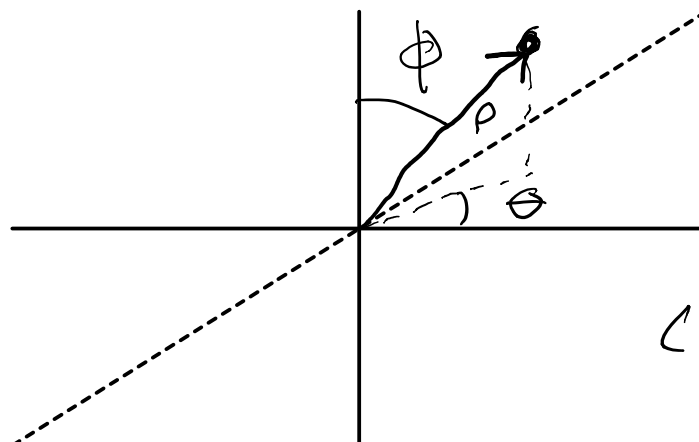
In Cylindrical: $\nabla^2 u = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2}$

(r, θ, z)



In Spherical: $\nabla^2 u(\rho, \theta, \phi) = \frac{1}{\rho^2} \frac{\partial}{\partial \rho} \left(\rho^2 \frac{\partial u}{\partial \rho} \right) + \frac{1}{\rho^2 \sin(\phi)} \frac{\partial}{\partial \phi} \left(\sin(\phi) \frac{\partial u}{\partial \phi} \right) + \frac{1}{\rho^2 \sin^2 \phi} \frac{\partial^2 u}{\partial \theta^2}$

(ρ, θ, ϕ)



$(1, \pi/4, \pi/3)$

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Homework Examples:

Let $u(r, \theta, z) = z + r \sin(\theta)$ calculate $\nabla^2 u$ $\frac{\partial u}{\partial r}$ In Cylindrical: $\nabla^2 u = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2}$

$$\frac{\partial u}{\partial r} = \sin \theta \quad \frac{1}{r} \frac{\partial}{\partial r} (r \sin \theta) = \frac{1}{r} \sin \theta$$

$$\frac{\partial^2 u}{\partial \theta^2} = \frac{\partial}{\partial \theta} \frac{\partial u}{\partial \theta} = \frac{\partial}{\partial \theta} (r \cos \theta) = -r \sin \theta$$

$$\frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = \frac{1}{r^2} (-r \sin \theta) = -\frac{1}{r} \sin \theta$$

$$\frac{\partial^2 u}{\partial z^2} = \frac{\partial}{\partial z} \frac{\partial u}{\partial z} = \frac{\partial}{\partial z} (1) = 0$$

$$\nabla^2 u = \frac{1}{r} \sin \theta - \frac{1}{r} \sin \theta + 0 = 0$$

Let $u(\rho, \theta, \phi) = \rho + \sin(\theta) + \cos(\phi)$ calculate $\nabla^2 u$

In Spherical: $\nabla^2 u(\rho, \theta, \phi) = \frac{1}{\rho^2} \frac{\partial}{\partial \rho} \left(\rho^2 \frac{\partial u}{\partial \rho} \right) + \frac{1}{\rho^2 \sin(\phi)} \frac{\partial}{\partial \theta} \left(\sin(\phi) \frac{\partial u}{\partial \theta} \right) + \frac{1}{\rho^2 \sin^2 \phi} \frac{\partial^2 u}{\partial \phi^2}$

$$\frac{1}{\rho^2} \frac{\partial}{\partial \rho} \rho^2 \frac{\partial u}{\partial \rho} = \frac{1}{\rho^2} \frac{\partial}{\partial \rho} (\rho^2 (1)) = \frac{1}{\rho^2} 2\rho = \frac{2}{\rho}$$

$$\begin{aligned} \frac{1}{\rho^2 \sin \phi} \frac{\partial}{\partial \theta} \sin \phi \frac{\partial u}{\partial \theta} &= \frac{1}{\rho^2 \sin \phi} \frac{\partial}{\partial \theta} \sin \phi (-\sin \phi) = \frac{1}{\rho^2 \sin \phi} \frac{\partial}{\partial \theta} \sin^2 \phi \\ &= \frac{1}{\rho^2 \sin \phi} - 2 \sin \phi \cos \phi = \frac{-2}{\rho^2} \cos \phi \end{aligned}$$

$$\frac{1}{\rho^2 \sin^2 \phi} \frac{\partial}{\partial \phi} \left(\frac{\partial u}{\partial \phi} \right) = \frac{1}{\rho^2 \sin^2 \phi} \frac{\partial}{\partial \phi} \cos \phi = \frac{1}{\rho^2 \sin^2 \phi} (-\sin \phi)$$

$$= -\frac{\sin \phi}{\rho^2 \sin^2 \phi}$$

$$\boxed{\nabla^2 u = \frac{2}{\rho} - \frac{2}{\rho^2} \cos \phi - \frac{\sin \phi}{\rho^2 \sin^2 \phi}}$$