

Sec 13.1-13.2

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Sec 13.1

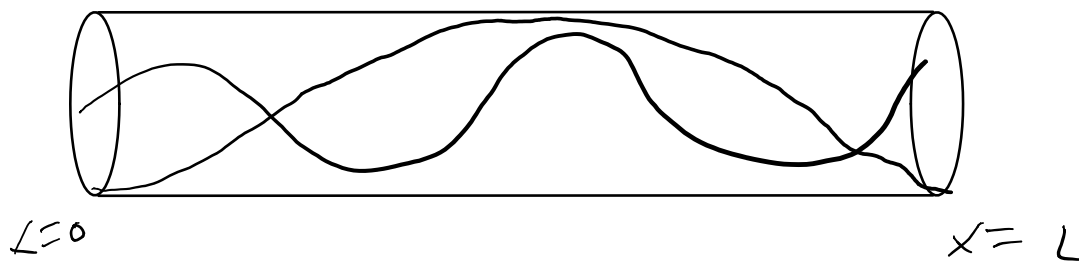
Let's Get ready to solve our First PDE! For this week we will focus on the 1-d heat equation and apply the method of separation of variables. We will focus on the heat equation with constant density ρ and heat capacity c .

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} + \frac{Q(x, t)}{c\rho}$$

$k \leftarrow k_0/c\rho$

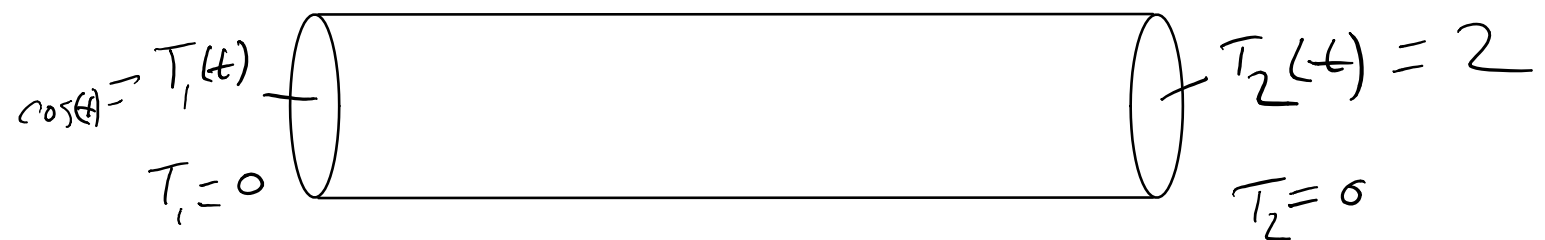
Subject to some initial condition:

$$u(x, 0) = f(x) \quad 0 < x < L$$



With Dirichlet boundary conditions at first:

$$u(0, t) = T_1(t) \quad u(L, t) = T_2(t) \quad t > 0$$



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Sec 13.2

However before we get into the thick of it we will need to talk about some important concepts and properties which will allow us to use the method of separation of variables.

Linear Operator: A linear operator L is an operator which satisfies the linear property.

$$\underline{L(c_1 u_1 + c_2 u_2) = c_1 L(u_1) + c_2 L(u_2)}$$

$$L(u_1 + u_2) = L(u_1) + L(u_2)$$
$$L(cu) = cL(u)$$

In words, the operator L applied to a sum of more than one term is the same as the sum of those terms with L applied to them individually and L applied to the product of a constant and a function is the same as the constant multiplied by L applied to the function.

One the first linear operators you were introduced to in calculus class was that of differentiation.

$$L(u) = \frac{d}{dx} u$$

Let's check its properties of linearity to do so we simply apply it to a linear combination of two functions:

$$L = \frac{d}{dx}$$

$$L(u) = \frac{d}{dx} u$$

$$\underline{L(c_1 u_1 + c_2 u_2) = \frac{d}{dx} (c_1 u_1 + c_2 u_2) = \frac{d}{dx} c_1 u_1 + \frac{d}{dx} c_2 u_2}$$

$$= c_1 \frac{d}{dx} u_1 + c_2 \frac{d}{dx} u_2 = \underline{c_1 L(u_1) + c_2 L(u_2)}$$

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As it turns out, because differentiation is linear, many combinations of differentiations are also linear.
For example if we take the heat equation:

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} + \frac{Q(x,t)}{c\rho}$$

And move every derivative term to one side

$$\frac{\partial u}{\partial t} - k \frac{\partial^2 u}{\partial x^2} = \frac{Q(x,t)}{c\rho}$$

Then let's just rename the right hand side to be $f(x,t)$ to add some generality.

$$\frac{\partial u}{\partial t} - k \frac{\partial^2 u}{\partial x^2} = f(x,t) \quad L(u) = f$$

Then we can define the Linear operator L to be:

$$L(u) = \frac{\partial u}{\partial t} - k \frac{\partial^2 u}{\partial x^2}$$

That is, L applied to any u means take its time derivative and subtract from that k times its second spatial derivative. We often then solve a linear equation involving L that is

$$L(u) = f$$

Let's show that L for the 1-d heat equation is indeed linear:

$$\begin{aligned} L(c_1 u_1 + c_2 u_2) &= c_1 L u_1 + c_2 L u_2 \\ \frac{\partial}{\partial t}(c_1 u_1 + c_2 u_2) - k \frac{\partial^2}{\partial x^2}(c_1 u_1 + c_2 u_2) \\ &\downarrow \\ c_1 \frac{\partial}{\partial t} u_1 + c_2 \frac{\partial}{\partial t} u_2 - \left(c_1 k \frac{\partial^2 u_1}{\partial x^2} + c_2 k \frac{\partial^2 u_2}{\partial x^2} \right) \\ &= c_1 \frac{\partial u_1}{\partial t} - c_1 k \frac{\partial^2 u_1}{\partial x^2} + c_2 \frac{\partial u_2}{\partial t} - c_2 k \frac{\partial^2 u_2}{\partial x^2} \\ &= c_1 \left(\frac{\partial u_1}{\partial t} - k \frac{\partial^2 u_1}{\partial x^2} \right) + c_2 \left(\frac{\partial u_2}{\partial t} - k \frac{\partial^2 u_2}{\partial x^2} \right) = c_1 L(u_1) + c_2 L(u_2) \end{aligned}$$

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When is an operator not Linear? Well, when it does not satisfy the linearity property.

Homework Example sec 13.2 #1(b)

Show that usually $L(u) = \frac{\partial}{\partial x} \left[\overset{\downarrow \downarrow}{k_0(x, u)} \frac{\partial u}{\partial x} \right]$ is not linear: Let $k_0 = \sin(u)$

$$L(u) = \frac{\partial}{\partial x} \left(\sin(u) \frac{\partial u}{\partial x} \right)$$

$$L(c_1 u_1 + c_2 u_2) = \underline{c_1 L(u_1) + c_2 L(u_2)} \leftarrow$$

$$L(c_1 u_1 + c_2 u_2) = \frac{\partial}{\partial x} \left(\sin(c_1 u_1 + c_2 u_2) \frac{\partial}{\partial x} (c_1 u_1 + c_2 u_2) \right)$$

$$\stackrel{?}{\neq} \frac{c_1 \frac{\partial}{\partial x} \left(\sin(u_1) \frac{\partial u_1}{\partial x} \right) + c_2 \left(\sin(u_2) \frac{\partial u_2}{\partial x} \right)}{}$$

$$\sin(\underbrace{c_1 u_1 + c_2 u_2}_{\beta}) \neq c_1 \sin u_1 + c_2 \sin u_2$$

$$\underline{\sin(\alpha + \beta) = \sin \alpha \cos \beta + \sin \beta \cos \alpha}$$

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One very important linear equation is the homogeneous equation (when $f = 0$) which reads:

$$L(u) = 0$$

The importance of this equation comes from the fact that its solutions imply the principle of superposition:

Principle of Superposition

If u_1 and u_2 satisfy a linear homogeneous equation ($L(u) = 0$) then so does any ^{arbitrary} linear combination of those solutions: $c_1 u_1 + c_2 u_2$

Proof:

$$\underline{L(u_1) = 0} \quad \underline{L(u_2) = 0}$$

$$L(c_1 u_1 + c_2 u_2) = c_1 L(u_1) + c_2 L(u_2)$$

$$u = c_1 u_1 + c_2 u_2$$

$$L(u) = 0$$

$$L(u) = L(c_1 u_1 + c_2 u_2) = c_1 \overset{0}{L(u_1)} + c_2 \overset{0}{L(u_2)} = 0$$
$$L(u) = 0$$

Why is this important?

Homework Example: sec 13.2 3(a)

Consider $L(u) = f$. If u_p is a particular solution, ($L(u_p) = f$), and if u_1 and u_2 are homogeneous solutions, ($L(u_1) = 0$ and $L(u_2) = 0$), show that $u = u_p + c_1 u_1 + c_2 u_2$ is also a particular solution.

That is $L(u_p + c_1 u_1 + c_2 u_2) = f$

$$\underline{u = u_p + c_1 u_1 + c_2 u_2}$$

$$L(u_p) = f \quad L(u_1) = 0 \quad L(u_2) = 0$$

$$L(u) = f$$

$$L(u) = L(u_p + c_1 u_1 + c_2 u_2) = \overset{f}{L(u_p)} + c_1 \overset{0}{L(u_1)} + c_2 \overset{0}{L(u_2)} = f$$

$$L(u) = f$$

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A specific example from ODE's

$$y(x)$$

Consider the ODE: $y'' + y = x$

We could find using the method of undetermined coefficients that $y = x$ solves the above equation:

However recall the general solution to the homogeneous equation:

$y'' + y = 0$ solved using the characteristic equation $r^2 + 1 = 0 \Rightarrow r = \pm i$

$$r^2 + 1 = 0$$

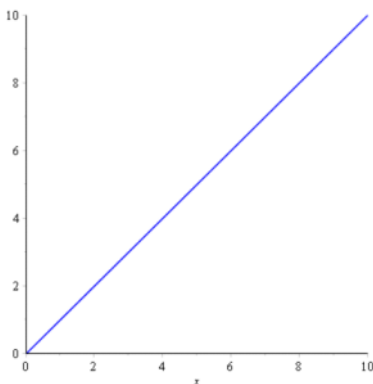
And thus $y = c_1 \cos(x) + c_2 \sin(x)$

$$y'' + y = x \quad y_p = x$$

Let's just show that $y = x + c_1 \cos(x) + c_2 \sin(x)$ also solves the particular ode:

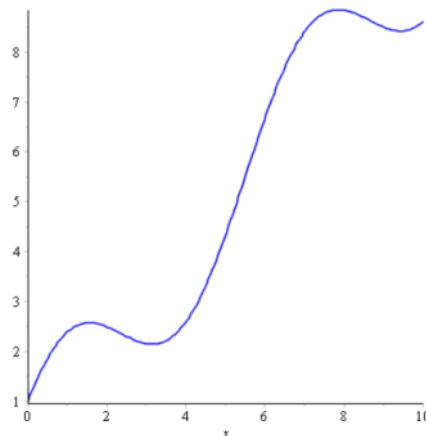
$$\begin{aligned} y' &= 1 - c_1 \sin(x) + c_2 \cos(x) \\ y'' &= -c_1 \cos(x) - c_2 \sin(x) \\ y'' + y &= x \\ -c_1 \cos(x) - c_2 \sin(x) + x + c_1 \cos(x) + c_2 \sin(x) &= x \checkmark \end{aligned}$$

Making sure to include the full solution is important, especially for physical problems we want to make sure to capture the correct behavior which can be very different if the homogeneous solutions are not included.



Solution without homogeneous solutions
Added in

$$y=x$$



Full true solution to the ode
 $y = x + \cos(x) + \sin(x)$