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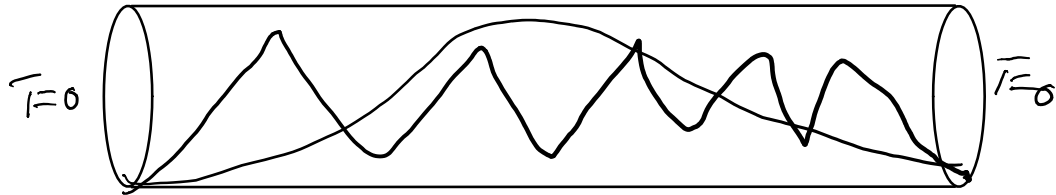
Let's now start solving some PDEs! To start let's deal with the 1-d heat equation with no sources and a temperature of zero at both ends. That is we will solve:

~~$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} \text{ for } 0 < x < L \text{ and } t > 0$$~~

BCs: $u(0, t) = 0$

$u(L, t) = 0$

IC: $u(x, 0) = \underline{f(x)}$



To find the solution we will apply *separation of variables* a trick which will turn our PDE into two ODE's which we will be able to solve.

The idea behind separation of variables for PDEs is that our solution $u(x, t)$ is simply a product of a function of x and t , that is $u(x, t) = \phi(x)G(t)$. Let's see what this would say about our PDE:

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} \quad u = \phi(x)G(t)$$

$$\frac{\partial}{\partial t} (\phi(x)G(t)) = k \frac{\partial^2}{\partial x^2} (\phi(x)G(t))$$

$$= \phi(x) \frac{dG}{dt} = k G(t) \frac{d^2 \phi}{dx^2} \Rightarrow \cancel{\phi} \frac{dG}{dt} = k \cancel{\phi} \frac{d^2 \phi}{dx^2}$$

$$\frac{1}{G} \frac{dG}{dt} = \frac{k}{\phi} \frac{d^2 \phi}{dx^2}$$

$$f(t) = g(x) = c$$

$$2^2 = 3^2 = 4^2 = 9$$

$$\frac{1}{kG} \frac{dG}{dt} = \frac{1}{\phi} \frac{d^2 \phi}{dx^2} = \text{constant}$$

only t

only x

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So now we have a statement that says two functions of two different variables are always equal. The only way that this can be true for all x, t is if both are constant:

$$\frac{1}{kG} \frac{dG}{dt} = \frac{1}{\phi} \frac{d^2\phi}{dx^2} = \text{constant}$$

We then have 3 possibilities for that constant, either it is positive, negative or zero. One very important question we will have to answer and which will depend on the PDE and boundary conditions themselves is which type of constant we have. We will typically choose the sign of the constant based on physical considerations as well as to avoid a trivial solution.

→ What is the trivial solution? The trivial solution is the solution that is always the solution, the zero solution ($u(x, t) = 0$) it tells us nothing about the system and by the superposition principle is always part of any solution to any homogeneous PDE.

Let's go through the cases before us and decide on what type of constant we should have for the heat equation with zero temperature at the ends. Before we do so I am going to confuse you with a minus sign. We will choose our constant to be λ and write our problem in the following form:

$$\frac{1}{kG} \frac{dG}{dt} = \frac{1}{\phi} \frac{d^2\phi}{dx^2} = \underline{-\lambda}$$

Thus if $\lambda > 0$ the constant on the right hand side comes out negative, if $\lambda < 0$ the right hand side is positive.

Why would we do this????

For this particular problem, it is convenient. It turns out, as we will see, that we will want the right hand side of our equation to be negative both for physical reasons and to avoid a trivial solution.

Further λ will actually take on specific values which are positive, also in order to avoid a trivial solution.

In this way it is convenient to use the negative of a positive number on the right hand side.

Let's run through the cases that lead to trivial solutions then finally find the true solution.

Case 1 ($\lambda < 0$) (a positive right hand side)

$$\sinh(x) = \frac{e^x - e^{-x}}{2} \quad \cosh(x) = \frac{e^x + e^{-x}}{2}$$

We have that: $\frac{1}{kG} \frac{dG}{dt} = \frac{1}{\phi} \frac{d^2\phi}{dx^2} = -\lambda$

Which we can separate this in to two equations each of which gives us an ODE to solve:

$$\rightarrow \frac{1}{kG} \frac{dG}{dt} = -\lambda \text{ and } \frac{1}{\phi} \frac{d^2\phi}{dx^2} = -\lambda \rightarrow \frac{d^2\phi}{dx^2} = -\lambda\phi = \frac{d^2\phi}{dx^2} + \lambda\phi = 0$$

Let's focus in on the ODE in x:

That is we will solve $\frac{d^2\phi}{dx^2} + \lambda\phi = 0$ keeping in mind that $u(x, t) = G(t)\phi(x)$ and $u(0, t) = u(L, t) = 0$

$$\frac{d^2\phi}{dx^2} + \lambda\phi = 0 \rightarrow \phi'' + \lambda\phi = 0 \quad r^2 + \lambda = 0 \quad \uparrow$$

$$r^2 = -\lambda \Rightarrow r = \pm\sqrt{\lambda}$$

$$\phi(x) = c_1 e^{\sqrt{\lambda}x} + c_2 e^{-\sqrt{\lambda}x} = c_1 \sinh(\sqrt{\lambda}x) + c_2 \cosh(\sqrt{\lambda}x)$$

$$\Rightarrow \phi(x) = 0 + 0 = 0$$

$$u(x, t) = G(t) (c_1 e^{\sqrt{\lambda}x} + c_2 e^{-\sqrt{\lambda}x})$$

$$u(0, t) = 0 = G(t) (c_1 + c_2) \quad c_1 + c_2 = 0$$

$$u(L, t) = 0 = G(t) (c_1 e^{\sqrt{\lambda}L} + c_2 e^{-\sqrt{\lambda}L}) \Rightarrow c_1 = -c_2$$

$$c_1 e^{\sqrt{\lambda}L} + c_2 e^{-\sqrt{\lambda}L} = 0$$

$$c_1 e^{\sqrt{\lambda}L} - c_1 e^{-\sqrt{\lambda}L} = 0 \Rightarrow c_1 (e^{\sqrt{\lambda}L} - e^{-\sqrt{\lambda}L}) = 0$$

$$\Rightarrow c_1 = 0 = 0$$

$$c_1 e^x + c_2 e^{-x}$$

$$\tilde{c}_1 \sinh x + \tilde{c}_2 \cosh(x) = \tilde{c}_1 \left(\frac{e^x - e^{-x}}{2} \right) + \tilde{c}_2 \left(\frac{e^x + e^{-x}}{2} \right)$$

$$\frac{\tilde{c}_1}{2} e^x + \frac{\tilde{c}_2}{2} e^x - \frac{\tilde{c}_1}{2} e^{-x} + \frac{\tilde{c}_2}{2} e^{-x} = \left(\frac{\tilde{c}_1 + \tilde{c}_2}{2} \right) e^x + \left(\frac{\tilde{c}_2 - \tilde{c}_1}{2} \right) e^{-x}$$

$$y'' + y' + y = 0$$

$$y = e^{rx}$$

$$y' = c_1 r e^{rx}$$

$$y'' = c_1 r^2 e^{rx}$$

$$c_1 r^2 e^{rx} + c_1 r e^{rx} + c_1 e^{rx} = 0$$

$$r^2 + r + 1 = 0$$

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Case 2 ($\lambda = 0$)

We have that: $\frac{1}{kG} \frac{dG}{dt} = \frac{1}{\phi} \frac{d^2\phi}{dx^2} = 0$

Which we can separate this in to two equations each of which gives us an ODE to solve:

$$\frac{1}{kG} \frac{dG}{dt} = 0 \quad \text{and} \quad \frac{1}{\phi} \frac{d^2\phi}{dx^2} = 0$$

Let's focus in on the ODE in x:

That is we will solve $\frac{d^2\phi}{dx^2} = 0$ keeping in mind that $u(x, t) = G(t)\phi(x)$ and $u(0, t) = u(0, L) = 0$

$$\phi(0) = 0 \quad \rightarrow \quad \phi(L) = 0$$

$$\frac{1}{\phi} \frac{d^2\phi}{dx^2} = 0 \Rightarrow \int \frac{d^2\phi}{dx^2} = \int 0 \Rightarrow \int \frac{d\phi}{dx} = \int c_1$$

$$\phi(x) = c_1 x + c_2$$

$$u = G(t) \phi(x)$$

$$u(0, t) = G(t) \phi(0) = 0$$

$$\phi(0) = c_1(0) + c_2 = 0$$

$$\Rightarrow c_2 = 0$$

$$\phi(L) = c_1(L) + c_2 = 0 \Rightarrow c_1(L) = 0$$

$$\Rightarrow c_1 = 0$$

$$\boxed{\phi = 0}$$

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Case 3 ($\lambda > 0$) (a negative right hand side)

We have that: $\frac{1}{kG} \frac{dG}{dt} = \frac{1}{\phi} \frac{d^2\phi}{dx^2} = -\lambda$

Which we can separate this in to two equations each of which gives us an ODE to solve:

$$\frac{1}{kG} \frac{dG}{dt} = -\lambda \quad \text{and} \quad \frac{1}{\phi} \frac{d^2\phi}{dx^2} = -\lambda \quad \rightarrow \quad \frac{d^2\phi}{dx^2} = -\lambda\phi \Rightarrow \frac{d^2\phi}{dx^2} + \lambda\phi = 0$$

Let's focus in on the ODE in x:

That is we will solve $\frac{d^2\phi}{dx^2} + \lambda\phi = 0$ keeping in mind that $u(x, t) = G(t)\phi(x)$ and $u(0, t) = \cancel{u(0, t)} = 0$ $u(L, t) = 0$

$$r^2 + \lambda = 0 \Rightarrow r^2 = -\lambda \quad r = \pm\sqrt{-\lambda} = \pm\sqrt{\lambda}i$$

$$\boxed{e^{i\sqrt{\lambda}x} = \cos(\sqrt{\lambda}x) + i\sin(\sqrt{\lambda}x)} \quad \phi(x) = C_1 \cos(\sqrt{\lambda}x) + C_2 \sin(\sqrt{\lambda}x)$$

$$\phi(0) = 0 \quad \phi(L) = 0$$

$$\phi(0) = C_1 \cos(0) + C_2 \sin(0) = 0 \Rightarrow C_1 = 0$$

$$\phi(L) = C_2 \sin(\sqrt{\lambda}L) = 0$$

$$C_2 \sin(n\pi) = 0$$

$$\sqrt{\lambda}L = n\pi$$

$$n = 1, 2, 3, \dots$$

$$\Rightarrow \sqrt{\lambda}L = n\pi \Rightarrow \sqrt{\lambda} = \frac{n\pi}{L} \Rightarrow \lambda = \left(\frac{n\pi}{L}\right)^2$$

$$\phi(x) = C_2 \sin\left(\frac{n\pi}{L}x\right)$$

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Case 3 cont.

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$$A\vec{x} = \lambda\vec{x}$$

So we only have a non-trivial solution to our PDE when $\lambda = \left(\frac{n\pi}{L}\right)^2$ where $n=1,2,3,\dots$ indeed λ is a special number not only because it determines when we have a solution but because it is an eigenvalue of our ODE.

Recall that an eigenvalue is a number such that when an operator L is applied to a function it returns a multiple of that function, that multiple is our eigen value.

$$\tilde{L}\phi = \lambda\phi$$

This whole time we were solving an eigenvalue problem with $L = \frac{d^2}{dx^2}$

$$\frac{d^2\phi}{dx^2} = \lambda\phi$$

$$\phi = c_2 \sin \frac{n\pi}{L} x$$

$$\frac{d\phi}{dx} = c_2 \frac{n\pi}{L} \cos \frac{n\pi}{L} x$$

$$\frac{d^2\phi}{dx^2} = -c_2 \left(\frac{n^2\pi^2}{L^2}\right) \sin\left(\frac{n\pi}{L} x\right)$$

$$\tilde{L}\phi = \frac{d^2}{dx^2} \phi = \lambda\phi$$

$$\tilde{L}\phi = \lambda\phi$$

↑
operator

For our boundary conditions we found $\lambda = \left(\frac{n\pi}{L}\right)^2$ and $\phi_n = \sin\left(\frac{n\pi}{L} x\right)$, $n=1,2,3,\dots$. The functions we get are called eigenfunctions.

↑
length of rod

The solution to our ODE which was the sum of all the eigenfunctions is called the eigenfunction expansion

It is important to be aware of this terminology as this type of problem will come up again.

Due to the frequency of the eigenproblem lots and lots of mathematical theory has been developed around eigenvalue/eigenfunction problems knowing when you have one allows you to apply the many useful theorems that those that came before have developed.

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So we have half of our problem solved, we know $\phi(x) = c_2 \sin\left(\frac{n\pi}{L}x\right)$, but $u = G(t)\phi(x)$ so we still need to solve our time dependent ODE:

$$\frac{1}{kG} \frac{dG}{dt} = -\lambda$$

And we know that $\lambda = \left(\frac{n\pi}{L}\right)^2$

$$\frac{1}{kG} \frac{dG}{dt} = -\left(\frac{n\pi}{L}\right)^2$$

$$\frac{dG}{dt} = -k\left(\frac{n\pi}{L}\right)^2 G$$

$$\int \frac{dG}{G} = \int -k\left(\frac{n\pi}{L}\right)^2 dt$$

$$\ln(G) = -k\left(\frac{n\pi}{L}\right)^2 t + C_1$$

$$G = e^{-k\left(\frac{n\pi}{L}\right)^2 t + C_1}$$

$$= e^{-k\left(\frac{n\pi}{L}\right)^2 t} e^{C_1}$$

$$G = Ce^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

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$$u(x,t) = G(t) \phi(x) =$$

So we only have a non-trivial solution to our PDE when $\lambda = \left(\frac{n\pi}{L}\right)^2$ where $n=1,2,3,\dots$

$$G(t) = \tilde{c}_n e^{-k\left(\frac{n\pi}{L}\right)^2 t} \quad \phi(x) = c_n \sin\left(\frac{n\pi}{L}x\right) \Rightarrow u_n(x,t) = \tilde{c}_n e^{-k\left(\frac{n\pi}{L}\right)^2 t} c_n \sin\left(\frac{n\pi}{L}x\right)$$

All of a sudden this means we have lots and lots, infinitely many in fact, of solutions to our PDE.

$$u = u_1 + u_2 + u_3 + \dots$$

$$u_n = B_n e^{-k\left(\frac{n\pi}{L}\right)^2 t} \sin\left(\frac{n\pi}{L}x\right) \text{ for } n = 1, 2, 3, \dots$$

To that end we will have to apply the principle of superposition in order to define the one true solution that is we will just create a solution which is a sum of all of the solutions.

$$u(x,t) = \sum_{n=1}^{\infty} B_n e^{-k\left(\frac{n\pi}{L}\right)^2 t} \sin\left(\frac{n\pi}{L}x\right)$$

How do we then determine the coefficients B_n ? Let's first consider the easy case when our initial condition is made up of our eigenfunctions.

$$\text{Suppose } u(x,0) = 4 \sin\left(\frac{\pi x}{L}\right) + 6 \sin\left(\frac{3\pi x}{L}\right) + 3 \sin\left(\frac{100\pi x}{L}\right)$$



$$u(x,0) = \sum_{n=1}^{\infty} B_n e^{-k\left(\frac{n\pi}{L}\right)^2 \cdot 0} \sin\left(\frac{n\pi}{L}x\right) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}x\right) = f(x)$$

$$\sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}x\right) = 4 \sin\frac{\pi x}{L} + 6 \sin\frac{3\pi x}{L} + 3 \sin\left(\frac{100\pi x}{L}\right)$$

$$B_1 \sin\left(\frac{\pi x}{L}\right) + B_2 \sin\left(\frac{2\pi x}{L}\right) + B_3 \sin\left(\frac{3\pi x}{L}\right) + \dots + B_{100} \sin\left(\frac{100\pi x}{L}\right) + \dots$$

$$= 4 \sin\frac{\pi x}{L} + 6 \sin\frac{3\pi x}{L} + 3 \sin\frac{100\pi x}{L}$$

$$u(x,t) = 4 e^{-k\left(\frac{\pi}{L}\right)^2 t} \sin\frac{\pi x}{L} + 6 e^{-k\left(\frac{3\pi}{L}\right)^2 t} \sin\frac{3\pi x}{L} + 3 e^{-k\left(\frac{100\pi}{L}\right)^2 t} \sin\frac{100\pi x}{L}$$

$$3x^3 + 2x^2 + 1 = c_n x^n + c_{n-1} x^{n-1} + \dots + c_3 x^3 + c_2 x^2 + c_1 x + c_0$$

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What if our initial condition is not made up of eigenfunctions?

Suppose our initial condition is $u(x, 0) = f(x)$ for some function of x then we can write

$$u(x, 0) = \sum_{n=1}^{\infty} B_n e^{\left(\frac{n\pi}{L}\right)^2 0} \sin\left(\frac{n\pi}{L}x\right) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}x\right) = f(x)$$

Or just that:

$$f(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}x\right)$$

The trick we will use to find B_n will be to multiply both sides by $\sin\left(\frac{m\pi}{L}x\right)$ and make use of the fact that:

$$\int_0^L \sin\left(\frac{n\pi}{L}x\right) \sin\left(\frac{m\pi}{L}x\right) dx = \begin{cases} 0 & \text{when } m \neq n \\ \frac{L}{2} & \text{when } m = n \end{cases}$$

Which you will prove in your homework.:

For now let's find B_n !

$$\int_0^L f(x) \sin\left(\frac{m\pi}{L}x\right) dx = \int_0^L \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}x\right) \sin\left(\frac{m\pi}{L}x\right) dx$$

$$\int_0^L f(x) \sin\left(\frac{m\pi}{L}x\right) dx = \sum_{n=1}^{\infty} B_n \int_0^L \sin\left(\frac{n\pi}{L}x\right) \sin\left(\frac{m\pi}{L}x\right) dx$$

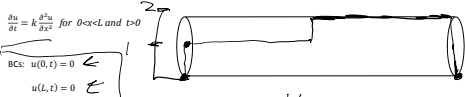
$$\int_0^L f(x) \sin\left(\frac{m\pi}{L}x\right) dx = B_m \int_0^L \sin\left(\frac{m\pi}{L}x\right) \sin\left(\frac{m\pi}{L}x\right) dx$$

$$\int_0^L f(x) \sin\left(\frac{m\pi}{L}x\right) dx = B_m \frac{L}{2}$$

$$B_m = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{m\pi}{L}x\right) dx$$

Homework Example: #3 (d)

Solve the heat equation



$$IC: u(x,0) = \begin{cases} 1 & \text{for } 0 < x < \frac{L}{2} \\ 2 & \text{for } \frac{L}{2} < x < L \end{cases} = f(x)$$

$$u(x,t) = \sum_{n=1}^{\infty} B_n e^{-k \left(\frac{n\pi}{L}\right)^2 t} \sin\left(\frac{n\pi}{L} x\right)$$

$$u(x,t) = \sum_{n \text{ odd}} \left(\frac{L}{n\pi}\right) e^{-k \left(\frac{n\pi}{L}\right)^2 t} \sin\left(\frac{n\pi}{L} x\right) + \sum_{n \text{ even}} \left(\frac{2}{n\pi}(-1)^{n-1} - \frac{2}{n\pi}\right) e^{-k \left(\frac{n\pi}{L}\right)^2 t} \sin\left(\frac{n\pi}{L} x\right)$$

$$B_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx = \frac{2}{L} \int_0^{L/2} 1 \sin\left(\frac{n\pi x}{L}\right) dx + \frac{2}{L} \int_{L/2}^L 2 \sin\left(\frac{n\pi x}{L}\right) dx$$

$$= \frac{2}{L} \int_0^{L/2} \sin\left(\frac{n\pi x}{L}\right) dx + \frac{4}{L} \int_{L/2}^L \sin\left(\frac{n\pi x}{L}\right) dx = -\frac{2}{L} \frac{L}{n\pi} \cos\left(\frac{n\pi x}{L}\right) \Big|_0^{L/2} - \frac{4}{L} \frac{L}{n\pi} \cos\left(\frac{n\pi x}{L}\right) \Big|_{L/2}^L$$

$$= -\frac{2}{L} \frac{L}{n\pi} \cos\left(\frac{n\pi}{2}\right) + \frac{4}{L} \frac{L}{n\pi} (1) - \left(-\frac{4}{L} \frac{L}{n\pi} \cos\left(\frac{n\pi}{2}\right) - \frac{4}{L} \frac{L}{n\pi} \cos\left(\frac{n\pi}{2}\right) \right)$$

$$= -\frac{2}{n\pi} \cos\left(\frac{n\pi}{2}\right) + \frac{4}{n\pi} - \frac{4}{n\pi} \cos(n\pi) + \frac{4}{n\pi} \cos\left(\frac{n\pi}{2}\right) = B_n \quad n \text{ odd}$$

$$B_n = \frac{2}{n\pi} \cos\left(\frac{n\pi}{2}\right) - \frac{4}{n\pi} \cos(n\pi) + \frac{4}{n\pi}$$

$$B_n = \frac{-4}{n\pi} (-1) + \frac{2}{n\pi}$$

$$B_n = \frac{4}{n\pi} + \frac{2}{n\pi} = \frac{6}{n\pi}$$

even

$$\frac{2}{n\pi} (-1)^{n-1} - \frac{4}{n\pi} + \frac{2}{n\pi} = \frac{2}{n\pi} (-1)^{n-1} - \frac{2}{n\pi} = B_n$$