

Sec 13.4

Wednesday, May 27, 2015 5:09 PM

Let's now turn our attention to some different boundary conditions. We suppose that we have perfectly insulated ends which means there is no flux of heat at either boundary. The Problem we want to solve is then:

PDE: $\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}$



BCS: $\frac{\partial u}{\partial x}(0, t) = 0$
 $\frac{\partial u}{\partial x}(L, t) = 0$

IC: $u(x, 0) = f(x)$

To solve this problem we again employ the assumption of separation of variables $u(x, t) = \phi(x)G(t)$

Which again gives us two ODE's to solve:

$$\frac{dG}{dt} = -\lambda k G$$

$$\frac{d^2 \phi}{dx^2} = -\lambda \phi$$

For the time dependent ODE, we obtain the same general solution as before:

$$G(t) = ce^{-\lambda kt}$$

However for the spatial ODE we will again have to go through the cases.....

Sec 13.4

Thursday, May 28, 2015 6:23 PM

We wish to solve:

$$\frac{d^2\phi}{dx^2} = -\lambda\phi \quad \text{Subject to the boundary conditions:}$$
$$\frac{d\phi}{dx}(0) = 0 \quad \text{and} \quad \frac{d\phi}{dx}(L) = 0$$

Case 1 ($\lambda < 0$) a positive right hand side

Case 2 ($\lambda = 0$)

Sec 13.4

Thursday, May 28, 2015 6:27 PM

We wish to solve:

$$\frac{d^2\phi}{dx^2} = -\lambda\phi$$

Subject to the boundary conditions:

$$\frac{d\phi}{dx}(0) = 0 \quad \text{and} \quad \frac{d\phi}{dx}(L) = 0$$

Case 1 ($\lambda > 0$) a negative right hand side

Sec 13.4

Thursday, May 28, 2015 7:00 PM

Case 1 cont.

Sec 13.4

Thursday, May 28, 2015 6:28 PM

So we have now that:

$$G(t)=c$$

$$\phi(x) = c_2$$

$$\lambda = 0$$

$$G(t) = ce^{-\left(\frac{n\pi}{L}\right)^2 kt}$$

$$\phi(x) = A \cos\left(\frac{n\pi}{L}x\right)$$

$$\lambda = \frac{n\pi}{L} \quad n = 1, 2, 3, \dots$$

And $u(x, t) = G(t)\phi(t)$

$$\text{Thus } u(x, t) = A_0 \quad \lambda = 0 \quad \text{and } u_n(x, t) = A_n \cos\left(\frac{n\pi}{L}x\right) e^{-\left(\frac{n\pi}{L}\right)^2 kt} \quad \lambda = \frac{n\pi}{L} \quad n = 1, 2, 3, \dots$$

This with the principle of superposition implies

$$u(x, t) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi}{L}x\right) e^{-\left(\frac{n\pi}{L}\right)^2 kt}$$

We are again faced with the problem of determining the constants, in this case A_0 and A_n for $n = 1, 2, 3, \dots$. To determine the constants we use the initial condition again and if we are lucky it is made of our eigen functions.

HW Ex. 13.3 #1b Suppose $u(x, 0) = f(x) = 3 + 4 \cos\left(\frac{3\pi}{L}x\right)$ for the heat equation of a 1-d rod with insulated boundaries. Determine the coefficients for the solution $u(x, t)$

Sec 13.4

Thursday, May 28, 2015 7:04 PM

What if our initial condition is not made of eigenfunctions? :(

We play the same game as before only this time we will multiply both sides of our initial condition equation:

$$f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi}{L}x\right)$$

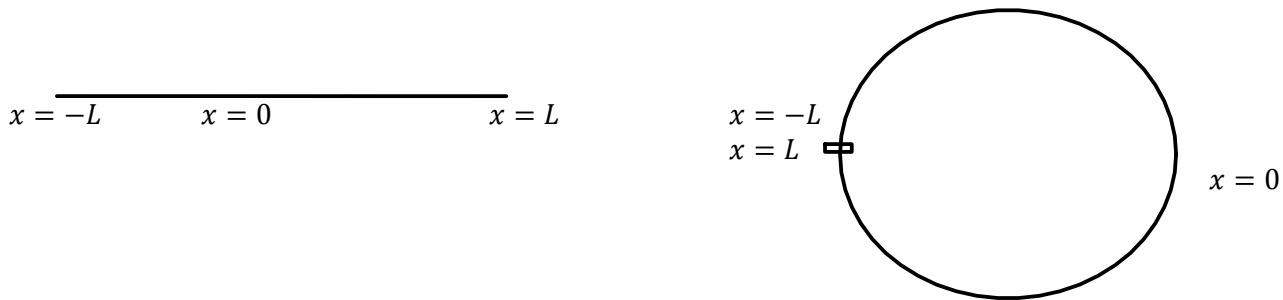
By $\cos\left(\frac{m\pi}{L}x\right)$ and use the fact that

$$\int_0^L \cos\left(\frac{n\pi}{L}x\right) \cos\left(\frac{m\pi}{L}x\right) dx = \begin{cases} 0 & \text{if } n \neq m \\ \frac{L}{2} & \text{if } n = m \neq 0 \\ L & \text{if } n = m = 0 \end{cases}$$

Sec 13.4

Thursday, May 28, 2015 7:11 PM

Yet another set of boundary conditions! Let's take our 1-d rod and make a circle.



As you might expect if we glue the ends together we will require both the temperature and fluxes at that point to be the same. This leads to the following boundary conditions:

BCS:

$$u(-L, t) = u(L, t)$$

$$\frac{\partial u}{\partial x}(-L, t) = \frac{\partial u}{\partial x}(L, t)$$

IC: $u(x, 0) = f(x)$

Let the fun begin! As you might guess $\lambda < 0$ Leads to a trivial solution so we will focus on $\lambda < 0$ and $\lambda = 0$

As before our time dependent solution is not as fussy about the value of lambda solving it's ODE: $\frac{dG}{dt} = -\lambda kG$ gives us our old dying exponential friend: $G(t) = ce^{-\lambda kt}$ Thus we again must fight with the spatial ODE

$$\frac{d^2\phi}{dx^2} = -\lambda\phi$$

Case 1 $\lambda = 0$

Sec 13.4

Thursday, May 28, 2015 7:39 PM

Case 2 $\lambda > 0$ Again in this case $\frac{d^2\phi}{dx^2} = -\lambda\phi$ has the general solution $\phi(x) = c_1 \cos(\sqrt{\lambda}x) + c_2 \sin(\sqrt{\lambda}x)$

Let's apply our BCS

$$u(-L, t) = u(L, t) \Rightarrow \phi(-L) = \phi(L)$$

$$\frac{\partial u}{\partial x}(-L, t) = \frac{\partial u}{\partial x}(L, t) \Rightarrow \frac{d\phi}{dx}(-L) = \frac{d\phi}{dx}(L)$$

Sec 13.4

Thursday, May 28, 2015 7:48 PM

Case 2 cont.

Sec 13.4

Thursday, May 28, 2015 7:44 PM

So for this case we obtain the family of solutions

$$u_0(x, t) = A_0$$

$$u_n(x, t) = A_n \cos\left(\frac{n\pi}{L}x\right)$$

$$u_n(x, t) = B_n \sin\left(\frac{n\pi}{L}x\right)$$

Again by super position this means:

$$u(x, t) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi}{L}x\right) e^{-\left(\frac{n\pi}{L}\right)^2 kt} + \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}x\right) e^{-\left(\frac{n\pi}{L}\right)^2 kt}$$

With initial condition equation:

$$u(x, 0) = f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi}{L}x\right) + \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}x\right)$$

We are then again tasked with finding those pesky coefficients! We might again get lucky and have our initial condition in terms of our eigenfunctions, in this case sines and cosines. If so we play the same game and simply match coefficients again. If not we will need some formula as before.

The trick is to multiply both sides by $\cos\left(\frac{m\pi}{L}x\right)\sin\left(\frac{m\pi}{L}x\right)$ and integrate as before but from $-L$ to L . To help out we can use the integral identities:

$$\int_{-L}^L \cos\left(\frac{n\pi}{L}x\right) \cos\left(\frac{m\pi}{L}x\right) dx = \begin{cases} 0 & \text{if } n \neq m \\ L & \text{if } n = m \neq 0 \\ 2L & \text{if } n = m = 0 \end{cases}$$

$$\int_{-L}^L \sin\left(\frac{n\pi}{L}x\right) \sin\left(\frac{m\pi}{L}x\right) dx = \begin{cases} 0 & \text{when } m \neq n \\ L & \text{when } m = n \end{cases}$$

$$\int_{-L}^L \sin\left(\frac{n\pi}{L}x\right) \cos\left(\frac{m\pi}{L}x\right) dx = 0$$

Sec 13.4

Thursday, May 28, 2015 8:06 PM

If we do this we obtain the following formulas for our coefficients

$$A_0 = \frac{1}{2L} \int_{-L}^L f(x) dx$$

$$A_n = \frac{1}{L} \int_{-L}^L f(x) \cos\left(\frac{n\pi}{L}x\right) dx$$

$$B_n = \frac{1}{L} \int_{-L}^L f(x) \sin\left(\frac{n\pi}{L}x\right) dx$$

To go with our solution:

$$u(x, t) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi}{L}x\right) e^{-\left(\frac{n\pi}{L}\right)^2 kt} + \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}x\right) e^{-\left(\frac{n\pi}{L}\right)^2 kt}$$

With all of this work it is nice to have a summary: pg 774 of your book

Boundary conditions	$\phi(0) = 0$ $\phi(L) = 0$	$\frac{d\phi}{dx}(0) = 0$ $\frac{d\phi}{dx}(L) = 0$	$\phi(-L) = \phi(L)$ $\frac{d\phi}{dx}(-L) = \frac{d\phi}{dx}(L)$
Eigenvalues λ_n	$\left(\frac{n\pi}{L}\right)^2$ $n = 1, 2, 3, \dots$	$\left(\frac{n\pi}{L}\right)^2$ $n = 0, 1, 2, 3, \dots$	$\left(\frac{n\pi}{L}\right)^2$ $n = 0, 1, 2, 3, \dots$
Eigenfunctions	$\sin \frac{n\pi x}{L}$	$\cos \frac{n\pi x}{L}$	$\sin \frac{n\pi x}{L}$ and $\cos \frac{n\pi x}{L}$
Series	$f(x) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L}$	$f(x) = \sum_{n=0}^{\infty} A_n \cos \frac{n\pi x}{L}$	$f(x) = \sum_{n=0}^{\infty} a_n \cos \frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}$
Coefficients	$B_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx$	$A_0 = \frac{1}{L} \int_0^L f(x) dx$ $A_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L} dx$	$a_0 = \frac{1}{2L} \int_{-L}^L f(x) dx$ $a_n = \frac{1}{L} \int_{-L}^L f(x) \cos \frac{n\pi x}{L} dx$ $b_n = \frac{1}{L} \int_{-L}^L f(x) \sin \frac{n\pi x}{L} dx$