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From our study of the 2-d and 3-d heat equation in thermal equilibrium ($\frac{\partial u}{\partial t} = 0$) we arrive at another PDE which we will learn to solve. Namely Laplace's Equation

$$\nabla^2 u = 0$$

$$\text{In 3-d } \nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

$$\text{In 2-d } \nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

We will begin by first solving this equation on a rectangle in 2 dimensions with prescribed boundary data at each edge.

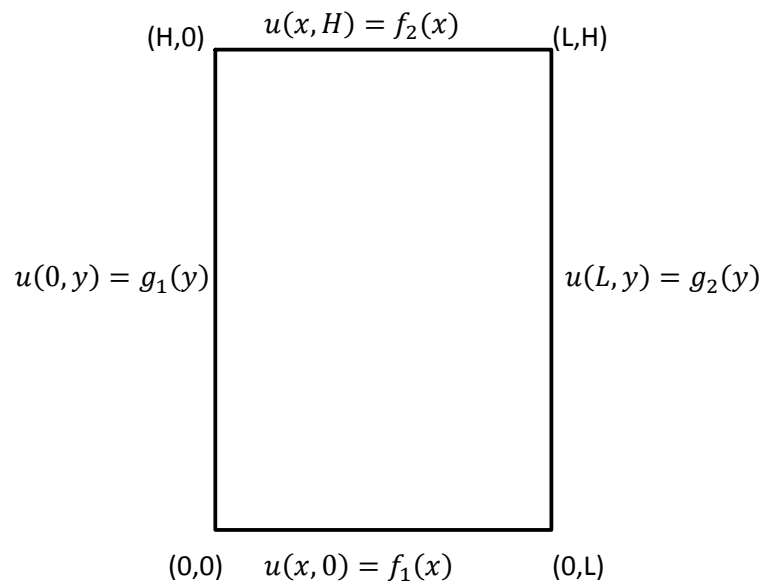
$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$u(0, y) = g_1(y)$$

$$u(L, y) = g_2(y)$$

$$u(x, 0) = f_1(x)$$

$$u(x, H) = f_2(x)$$



We might be tempted to employ separation of variables to this problem right away however this will leave us unable to determine our eigenvalue λ , some times also called the separation constant. Let's give it a try to see why it fails and then I'll show you a trick to fix it.

We are going to assume that: $u(x, y) = h(x)\phi(y)$

And substitute it into

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

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So we have that:

$$u(x, y) = h(x)\phi(y) \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

So we now have: $\frac{1}{h} \frac{d^2 h}{dx^2} = -\frac{1}{\phi} \frac{d^2 \phi}{dy^2} = \lambda$

In contrast to the heat equation both ODE's are essentially the same and only differ by a sign. Because of this we won't assume anything about lambda to start and we could work with either ODE to try and find out what lambda would need to be. As before we will have three cases to consider, $\lambda > 0$ $\lambda < 0$ $\lambda = 0$

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Let's just work with $-\frac{1}{\phi} \frac{d^2\phi}{dy^2} = \lambda$ first. Perhaps one way to see that separation of variables will not quite work right away is through the boundary condition equations. We are trying to solve:

$$\frac{d^2\phi}{dy^2} + \lambda\phi = 0$$

$$u(x, 0) = f_1(x)$$

$$u(x, H) = f_2(x)$$

Let's just check the $\lambda > 0$ case and check the first boundary condition

So we end up with a statement that:

$$\phi(0) = c_1 h(x) = f_1(x)$$

This means that $h(x) = \frac{1}{c_1} f_1(x)$. While this may seem fine or even nice as it relates an unknown function $h(x)$ to our known boundary condition, however this could be disastrous. If $f_1(x)$ does not satisfy the ODE for $h(x)$ we don't have a solution to that ODE:

$$\frac{d^2h}{dx^2} - h\lambda = 0 \Rightarrow \frac{d^2f_1}{dx^2} - f_1\lambda = 0$$

Our boundary condition is supposed to be arbitrary and if this is true it is not. While it may be that we can only solve the PDE if we have boundary conditions that satisfy the separation ODE's being good math students we will try to find another way if possible.

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Recall from our study of the heat equation with homogeneous boundary conditions that we were able to solve the problem

$$\frac{d^2\phi}{dx^2} + \lambda\phi = 0$$

$$\phi(0) = 0$$

$$\phi(L) = 0$$

And in fact it was because both boundary conditions were zero that we found $\lambda = \left(\frac{n\pi}{L}\right)^2$

Motivated by our fond memories of this problem we might try to re-write our current problem in some way to make it look like our old heat equation problem. In fact thanks to the superposition principle we can!

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

All we really need is some function $u(x, y)$ such that:

$$u(0, y) = g_1(y)$$

$$u(L, y) = g_2(y)$$

$$u(x, 0) = f_1(x)$$

$$u(x, H) = f_2(x)$$

What we will do here is split our solution into 4 solutions: $u(x, y) = u_1(x, y) + u_2(x, y) + u_3(x, y) + u_4(x, y)$

Where u_1 solves
 $\nabla^2 u_1 = 0$

$$u_1(0, y) = g_1(y)$$

$$u_1(L, y) = 0$$

$$u_1(x, 0) = 0$$

$$u_1(x, H) = 0$$

Where u_2 solves
 $\nabla^2 u_2 = 0$

$$u_2(0, y) = 0$$

$$u_2(L, y) = g_2(y)$$

$$u_2(x, 0) = 0$$

$$u_2(x, H) = 0$$

Where u_3 solves
 $\nabla^2 u_3 = 0$

$$u_3(0, y) = 0$$

$$u_3(L, y) = 0$$

$$u_3(x, 0) = f_1(x)$$

$$u_3(x, H) = 0$$

Where u_4 solves
 $\nabla^2 u_4 = 0$

$$u_4(0, y) = 0$$

$$u_4(L, y) = 0$$

$$u_4(x, 0) = 0$$

$$u_4(x, H) = f_2(x)$$

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Before we proceed let's verify that our sum of solutions indeed satisfies our original PDE:

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$u(0, y) = g_1(y)$$

$$u(L, y) = g_2(y)$$

$$u(x, 0) = f_1(x)$$

$$u(x, H) = f_2(x)$$

For reference $u(x, y) = u_1(x, y) + u_2(x, y) + u_3(x, y) + u_4(x, y)$ Where:

Where u_1 solves

$$\nabla^2 u_1 = 0$$

$$u_1(0, y) = g_1(y)$$

$$u_1(L, y) = 0$$

$$u_1(x, 0) = 0$$

$$u_1(x, H) = 0$$

Where u_2 solves

$$\nabla^2 u_2 = 0$$

$$u_2(0, y) = 0$$

$$u_2(L, y) = g_1(y)$$

$$u_2(x, 0) = 0$$

$$u_2(x, H) = 0$$

Where u_3 solves

$$\nabla^2 u_3 = 0$$

$$u_3(0, y) = 0$$

$$u_3(L, y) = 0$$

$$u_3(x, 0) = f_1(x)$$

$$u_3(x, H) = 0$$

Where u_4 solves

$$\nabla^2 u_4 = 0$$

$$u_4(0, y) = 0$$

$$u_4(L, y) = 0$$

$$u_4(x, 0) = 0$$

$$u_4(x, H) = f_2(x)$$

Let's check it out.

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While this may seem like a lot more work 4 PDE's instead of 1 we at least can now have arbitrary boundary conditions.

Let's solve the PDE for u_1 first.

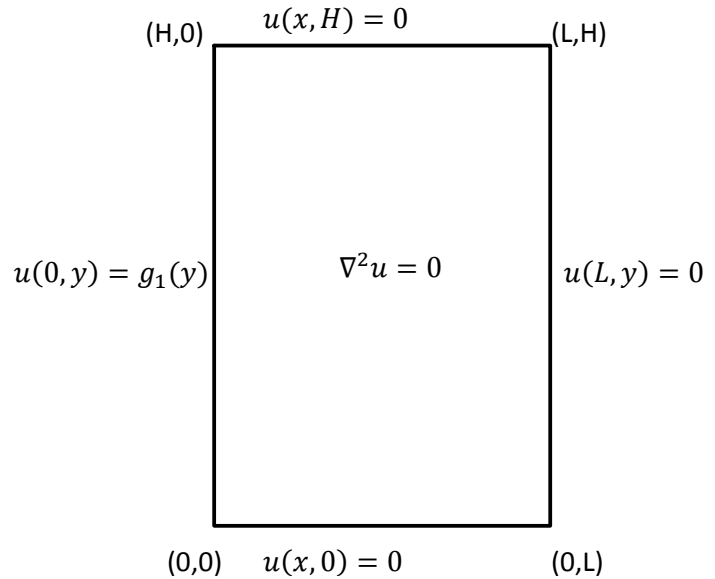
$$\nabla^2 u_1 = 0$$

$$u_1(0, y) = g_1(y)$$

$$u_1(L, y) = 0$$

$$u_1(x, 0) = 0$$

$$u_1(x, H) = 0$$



From separation of variables we have:

$$\frac{1}{h} \frac{d^2 h}{dx^2} = -\frac{1}{\phi} \frac{d^2 \phi}{dy^2} = \lambda \quad \text{Or} \quad -\frac{1}{h} \frac{d^2 h}{dx^2} = \frac{1}{\phi} \frac{d^2 \phi}{dy^2} = \lambda$$

If we take $\lambda > 0$ then which ever of these two ODE's has a minus sign is going to give us our sines and cosines. We want to choose this based on which variable has the homogeneous boundary conditions. For this first one we have homogeneous boundary conditions in y so we will stick the minus sign on ϕ so we can have sines and cosines there which will allow us to find λ .

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Solve for u_1 Cont.

Now we need to solve $\frac{1}{h} \frac{d^2 h}{dx^2} = \lambda$ and we have that $\lambda = \left(\frac{n\pi}{H}\right)^2$

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So now we have $u_{1n}(x) = A_n \sin\left(\frac{n\pi}{H}y\right) \sinh\left(\frac{n\pi}{H}(x-L)\right)$ we again apply the superposition principle and obtain:

$$u_1(x, y) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{H}y\right) \sinh\left(\frac{n\pi}{H}(x-L)\right)$$

We also still have one boundary condition to use $u_1(0, y) = g_1(y)$. Applying it we get the equation:

$$u_1(0, y) = g_1(y) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{H}y\right) \sinh\left(\frac{n\pi}{H}(-L)\right)$$

But we still need to find those A_n 's .. We play the same game as before, multiply bothsides by $\sin\left(\frac{m\pi}{L}x\right)$ and integrate using the fact that the sines are orthogonl that is:

$$\int_0^H \sin\left(\frac{n\pi}{H}x\right) \sin\left(\frac{m\pi}{H}x\right) dx = \begin{cases} 0 & \text{when } m \neq n \\ \frac{H}{2} & \text{when } m = n \end{cases}$$

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Let's take a look at the situation for u_3

Now the homogeneous BCS are in x.

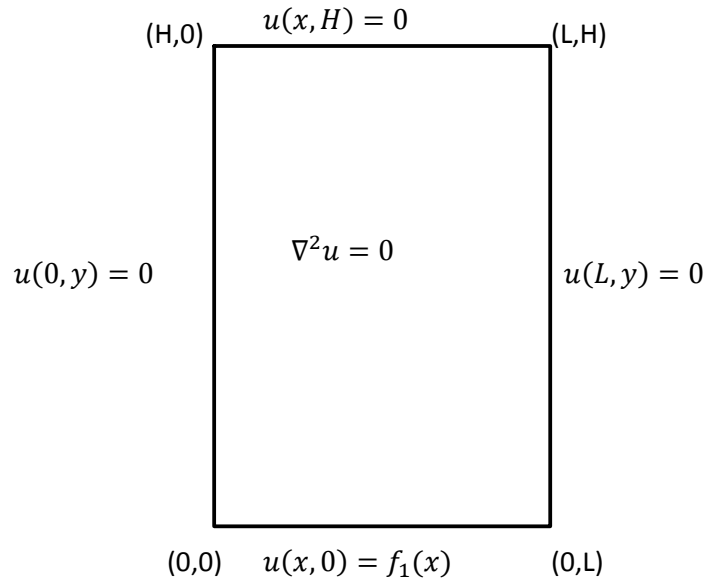
$$\nabla^2 u_3 = 0$$

$$u_3(0, y) = 0$$

$$u_3(L, y) = 0$$

$$u_3(x, 0) = f_1(x)$$

$$u_3(x, H) = 0$$



From separation of variables we have:

$$\frac{1}{h} \frac{d^2 h}{dx^2} = -\frac{1}{\phi} \frac{d^2 \phi}{dy^2} = \lambda \quad \text{Or} \quad -\frac{1}{h} \frac{d^2 h}{dx^2} = \frac{1}{\phi} \frac{d^2 \phi}{dy^2} = \lambda$$

Here we have the homogeneous boundary conditions in x. As a result we want our function of x to be made up of the sines and cosines so that we can find λ . For this case this means that we want

$$h(x) = c_1 \cos(\sqrt{\lambda} x) + c_2 \sin(\sqrt{\lambda} x)$$

So we can just move the minus sign over to there. Another option which your book suggests is that you keep the minus sign on ϕ :

$$\frac{1}{h} \frac{d^2 h}{dx^2} = -\frac{1}{\phi} \frac{d^2 \phi}{dy^2} = \lambda$$

But that if you have your homogeneous boundary conditions are in x you instead solve:

$$\frac{1}{h} \frac{d^2 h}{dy^2} = -\frac{1}{\phi} \frac{d^2 \phi}{dx^2} = \lambda$$

Either way we are just paring that minus sign with the variable that has the homogeneous boundary condition so that we can find λ .

That said we would now have to solve the other 3 PDE's get sour solutions and add them up to get $u(x, y)$

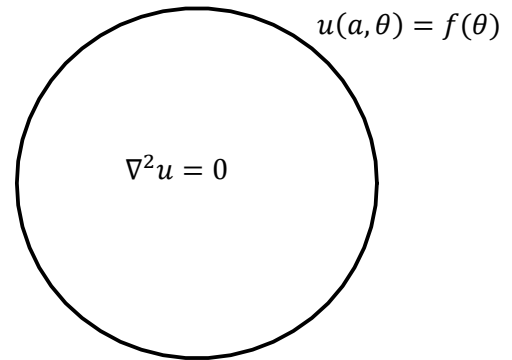
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Laplace's Equation on a circular disk

$$\nabla^2 u(r, \theta) = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = 0$$

BC1 $u(a, \theta) = f(\theta)$



Before we try to solve this bad boy we will need to consider some additional conditions on our solution.

Polar coordinates are singular at the origin $r = 0$ so we will enforce the condition on our solution that:

$$\lim_{r \rightarrow 0} |u(r, \theta)| < \infty$$

$$u(r, -\pi) = u(r, \pi)$$

For mathematical reasons, we will also require:

$$\frac{\partial u}{\partial \theta}(r, -\pi) = \frac{\partial u}{\partial \theta}(r, \pi)$$

These are called the Periodicity conditions and imply that at $\theta = \pi$ we have continuous temperature and heat flow.

If we have these conditions we can apply separation of variables that is we will look for a solution:

$$u(r, \theta) = \phi(\theta)G(r)$$

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Separation of variables for Laplace's equation in a disk.

We have: $\nabla^2 u(r, \theta) = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = 0$

$$u(r, \theta) = \phi(\theta)G(r)$$

So we now have:

$$\frac{1}{r} \frac{d}{dr} \left(r \frac{dG}{dr} \right) = -\frac{1}{\phi} \frac{d^2 \phi}{d\theta^2} = \lambda$$

We have two cases here that don't give us trivial solutions namely $\lambda = 0$ and $\lambda > 0$

Let's solve the ODE which is new to us first.

$$\frac{1}{r} \frac{d}{dr} \left(r \frac{dG}{dr} \right) = \lambda$$

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$$\frac{1}{r} \frac{d}{dr} \left(r \frac{dG}{dr} \right) = \lambda$$

Case 1 $\lambda = 0$

Keeping in mind that we need $\lim_{n \rightarrow \infty} |u(r, \theta)| < \infty$

Case 2 $\lambda > 0$ Expanding our ODE out we can write:

$$r^2 \frac{d^2 G}{dr^2} + r \frac{dG}{dr} - \lambda G = 0$$

Let's guess a solution of $G(r) = r^p$

Keeping in mind that we need: $\lim_{n \rightarrow \infty} |u(r, \theta)| < \infty$

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Now let's solve our eigen value problem

$$-\frac{1}{\phi} \frac{d^2 \phi}{d\theta^2} = \lambda \quad \phi(-\pi) = \phi(\pi)$$
$$\frac{\partial \phi}{\partial \theta}(-\pi) = \frac{\partial \phi}{\partial \theta}(\pi)$$

Case 1 $\lambda = 0$

Case 2 $\lambda > 0$

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So we have solutions of:

$$G(r) = c_1 r^{\sqrt{\lambda}} \text{ with } \lambda = n^2 \text{ we have } G_n(r) = c_1 r^n$$

and

$$\phi_n = A_n \cos(n\theta) + B_n \sin(n\theta)$$

$$\text{Thus } u_n = G_n(r)\phi_n(\theta) = A_n r^n \cos(n\theta) + B_n r^n \sin(n\theta)$$

Applying Superposition once again we obtain:

$$u(r, \theta) = \sum_{n=0}^{\infty} A_n r^n \cos(n\theta) + \sum_{n=1}^{\infty} B_n r^n \sin(n\theta)$$

Again we have to find our A_n and B_n . Applying our last BC gives us:

$$f(\theta) = \sum_{n=0}^{\infty} A_n a^n \cos(n\theta) + \sum_{n=1}^{\infty} B_n a^n \sin(n\theta)$$

Applying our usual tricks we find that

$$\begin{aligned} A_0 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx \\ A_n a^n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos\left(\frac{n\pi}{L} x\right) dx \\ B_n a^n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin\left(\frac{n\pi}{L} x\right) dx \end{aligned}$$

You may notice we did now hw problems in this lecture, that is due to time constraints. I will be posting a video with some examples from your homework to help you out.