

Sec 14.1-14.3

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Fourier Series or Basis function expansions using sines and cosines

Through our study of the heat equation we have seen that in order to solve the PDE we require the initial condition $f(x)$ to be able to be written as a sum of sines and cosines.

$$u(x, 0) = f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi}{L}x\right) + \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}x\right)$$

We require that this be possible as our solution $u(x, t)$ was nontrivial only if it took the form:

$$u(x, t) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi}{L}x\right) e^{-\left(\frac{n\pi}{L}\right)^2 kt} + \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}x\right) e^{-\left(\frac{n\pi}{L}\right)^2 kt}$$

A natural question to ask is, what types of functions $f(x)$ can we represent with a sum of a constant and sines and cosines. This leads us to study Fourier series.

→ It turns out that the set of functions $\{1, \sin\left(\frac{\pi x}{L}\right), \sin\left(\frac{2\pi x}{L}\right), \dots, \cos\left(\frac{\pi x}{L}\right), \cos\left(\frac{2\pi x}{L}\right), \dots\}$ form an orthogonal basis for continuous functions on the interval $[-L, L]$ with respect to the norm

$$\rightarrow \langle f, g \rangle = \int_{-L}^L f(x)g(x)dx$$

Smooth
This means that if $f(x)$ is a continuous function on $[-L, L]$ it can be represented with a Fourier series.

$$f(x) = \frac{\langle 1, f \rangle}{\langle 1, 1 \rangle} 1 + \frac{\langle \sin(\frac{\pi x}{L}), f \rangle}{\langle \sin(\frac{\pi x}{L}), \sin(\frac{\pi x}{L}) \rangle} \sin\left(\frac{\pi x}{L}\right) + \frac{\langle \sin(\frac{2\pi x}{L}), f \rangle}{\langle \sin(\frac{2\pi x}{L}), \sin(\frac{2\pi x}{L}) \rangle} \sin\left(\frac{2\pi x}{L}\right) + \dots + \frac{\langle \cos(\frac{\pi x}{L}), f \rangle}{\langle \cos(\frac{\pi x}{L}), \cos(\frac{\pi x}{L}) \rangle} \cos\left(\frac{\pi x}{L}\right) + \frac{\langle \cos(\frac{2\pi x}{L}), f \rangle}{\langle \cos(\frac{2\pi x}{L}), \cos(\frac{2\pi x}{L}) \rangle} \cos\left(\frac{2\pi x}{L}\right) + \dots$$

Or

$$f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi}{L}x\right) + \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}x\right)$$

Where

$$A_0 = \frac{\langle 1, f \rangle}{\langle 1, 1 \rangle} = \frac{1}{2L} \int_{-L}^L f(x) dx$$

$$A_n = \frac{\langle \cos(\frac{n\pi x}{L}), f \rangle}{\langle \cos(\frac{n\pi x}{L}), \cos(\frac{n\pi x}{L}) \rangle} = \frac{1}{L} \int_{-L}^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx$$

$$B_n = \frac{\langle \sin(\frac{n\pi x}{L}), f \rangle}{\langle \sin(\frac{n\pi x}{L}), \sin(\frac{n\pi x}{L}) \rangle} = \frac{1}{L} \int_{-L}^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

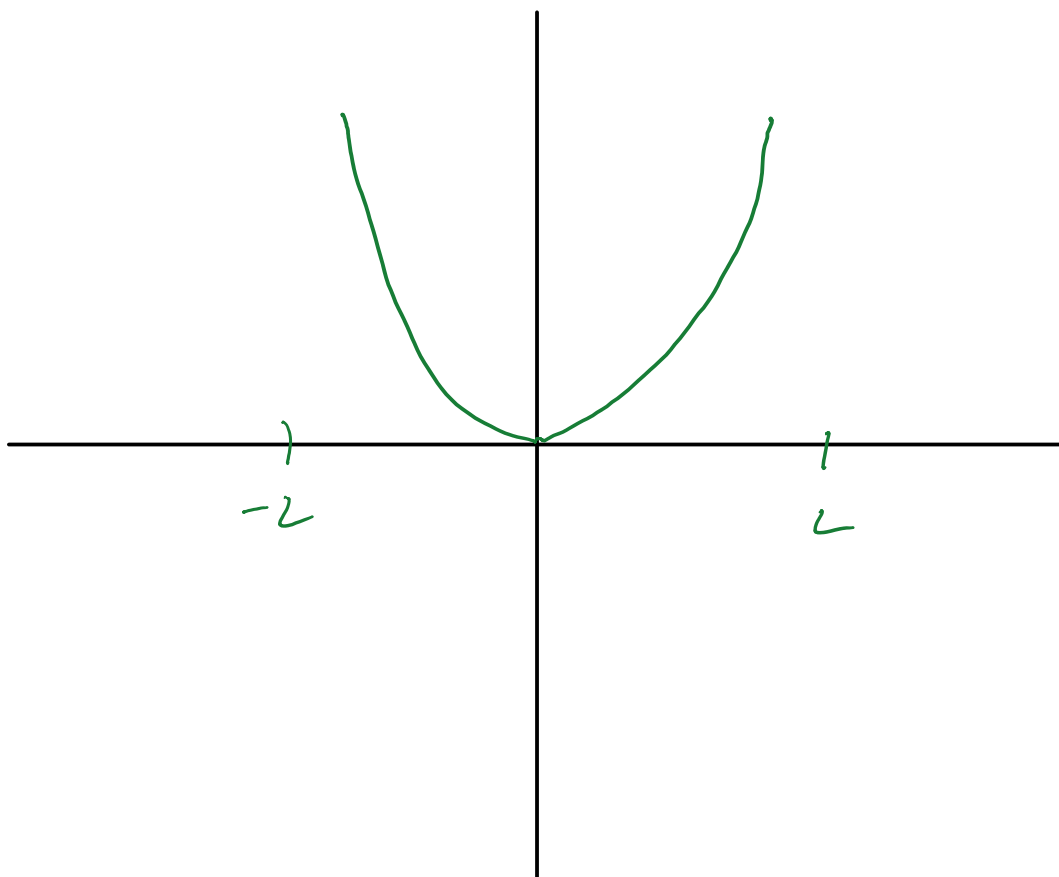
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Homework Example sec 14.2 #1(b)

Sketch the fourier series of $f(x) = x^2$ on the interval $[-L, L]$ and compare f to it's fourier series.

Solution: Since $f(x) = x^2$ is ^{smooth} continuous everywhere, it certinaly is on $[-L, L]$ thus our basis expansion will exactly match f on $[-L, L]$.



Let's look at some of the finite approximations vs the actual function in maple.

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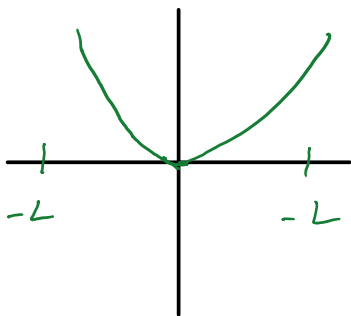
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If we recall from section 4.10 that we can use basis function expansions to represent, as closely as possible, a function which is not in the same space (at least with finite expansions). We might then think that even if $f(x)$ is not continuous we can use a Fourier series to get it's best continuous approximation. This would be the case if we had a finite sum, as the finite sum of continuous functions is continuous. However a general fourier series is an infinite sum and this presents difficulties.

It does turn out though that if $f(x)$ is a piece wise smooth function the fourier series does a very good job at approximating $f(x)$. Before we explain how, let's review the definition of a piece wise smooth function.

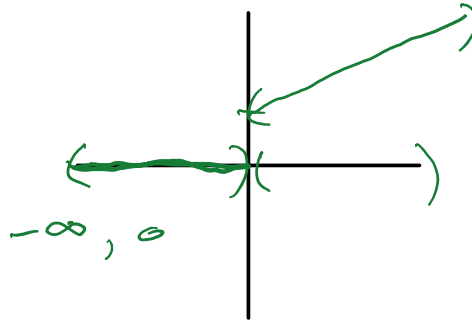
A function $f(x)$ is piecewise smooth on a given interval if that interval can be broken up in to sub intervals on which $f(x)$ and it's derivative $\frac{df}{dx}$ are continuous. Basically this means that we have no infinities in function only jump discontinuities and no infinite slopes.

Examples of piecewise smooth functions

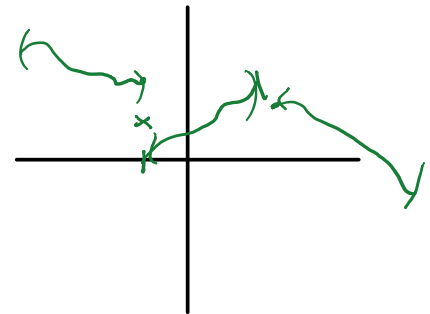


$$f(x) = x^2$$

$$f'(x) = 2x$$

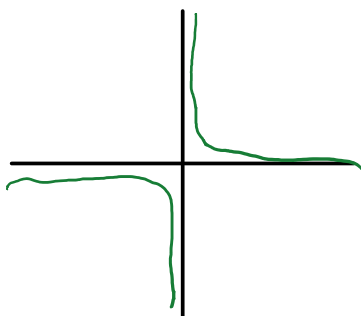


$$f(x) = \begin{cases} 0 & \text{for } x < 0 \\ 1 + x & \text{for } x > 0 \end{cases}$$

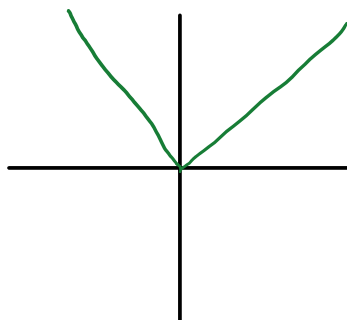


Random drawing example

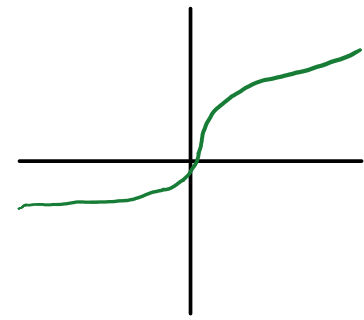
Examples of not-piecewise smooth functions



$$f(x) = \frac{1}{x}$$



$$f(x) = |x|$$



$$f(x) = x^{\frac{1}{3}}$$

$$f'(x) = \frac{1}{3} x^{-2/3} = \frac{1/3}{x^{2/3}}$$

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So what happens when with the Fourier series of $f(x)$ when it is only piecewise smooth? Let's state our Fourier Series convergence Theorem.

If $f(x)$ is piecewise smooth on the interval $[-L, L]$ then the Fourier series of $f(x)$ converges to:

- 1) The periodic extension of $f(x)$, where the periodic extension is continuous.
- 2) The average of the two limits on either side of a jump discontinuity usually

$$\frac{1}{2}(f(x+) + f(x-))$$

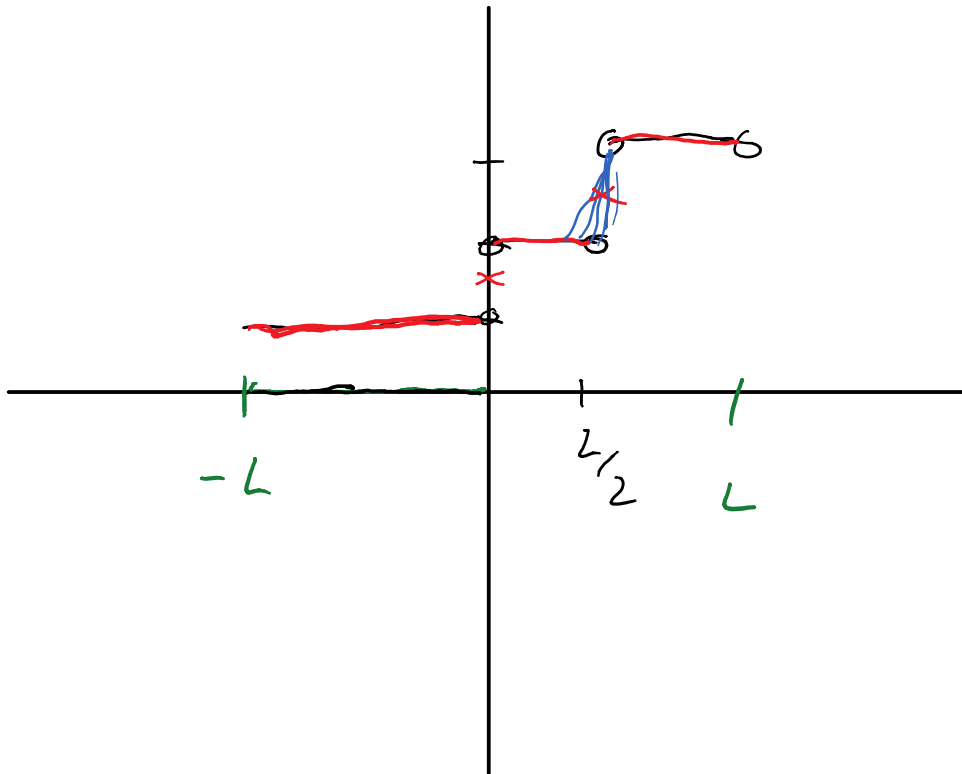
Handwritten notes and diagram illustrating the average of two limits at a jump discontinuity:

Diagram showing a jump discontinuity at $x = 1$. The function has a value of 2 for $x < 1$ and a value of 1 for $x > 1$. The average of these two values is $\frac{1+2}{2} = \frac{3}{2}$.

What we are saying is that where ever $f(x)$ is continuous the Fourier series converges exactly and takes on a value half way between the two y values at a jump discontinuity. We will talk about the periodic extension in a bit.

Example:

Draw the Fourier series of $f(x) = \begin{cases} 1 & -L < x < 0 \\ 2 & 0 < x < \frac{L}{2} \\ 3 & \frac{L}{2} < x < L \end{cases}$



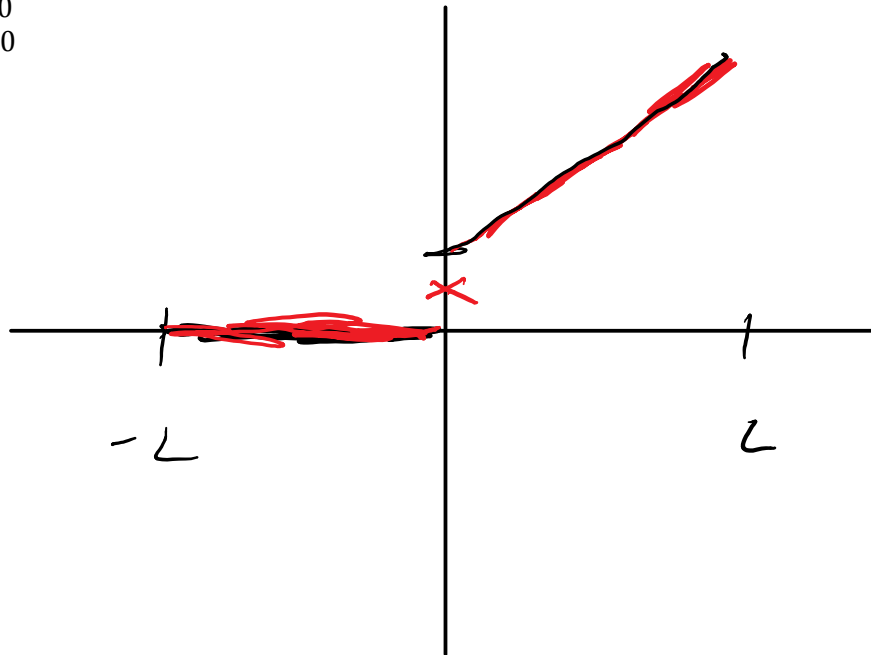
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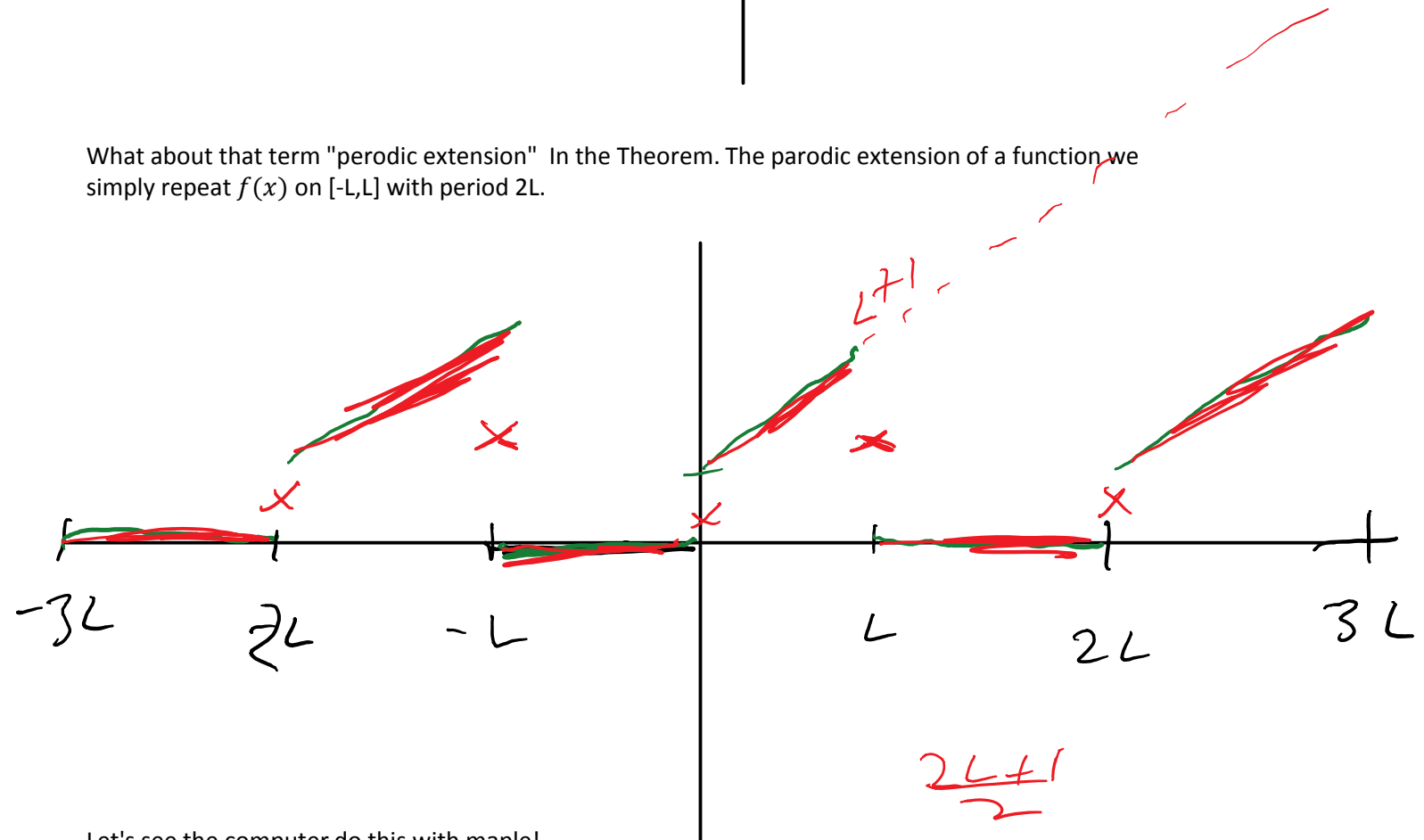
Homework Example sec 14.2#1 f

Sketch the Fourier series of $f(x)$ on $-L \leq x \leq L$ and compare it to it's Fourier series for

$$f(x) = \begin{cases} 0 & x < 0 \\ 1+x & x > 0 \end{cases}$$



What about that term "perodic extension" In the Theorem. The parodic extension of a function we simply repeat $f(x)$ on $[-L, L]$ with period $2L$.



Let's see the computer do this with maple!

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You might recall that for certain boundary conditions our Fourier series only consisted of sines or only cosines.

For homogeneous Dirichlet boundary conditions we had:

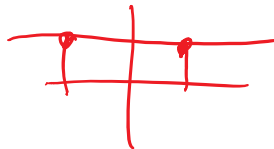
$$f(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}x\right)$$

For insulated boundary conditions we had:

$$f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi}{L}x\right)$$

So we might ask the question, when does a Fourier series only consist of sines or only cosines?

Well $\sin(x)$ is odd and $\cos(x)$ is even and you might recall that for functions:



Odd + Odd = Odd

Even + Even = Even

Odd + Even = Neither

Because of third equation it must be that :

If $f(x) = \text{even}$ it's Fourier Series can only be made up of constant and cosines

If $f(x) = \text{odd}$ it must be that it's Fourier series is made up of only sines

odd $f(x) = \sum \overset{\text{odd}}{\cancel{\cos \frac{n\pi x}{L}}} + \sum \overset{\text{odd}}{\sin \frac{n\pi x}{L}}$ neither

It is still possible however to find a Fourier sine series for an even function provided we can find an odd extension of that function.

Likewise it is possible to find a Fourier cosine series for an odd function provided we can find an even extension of that function.

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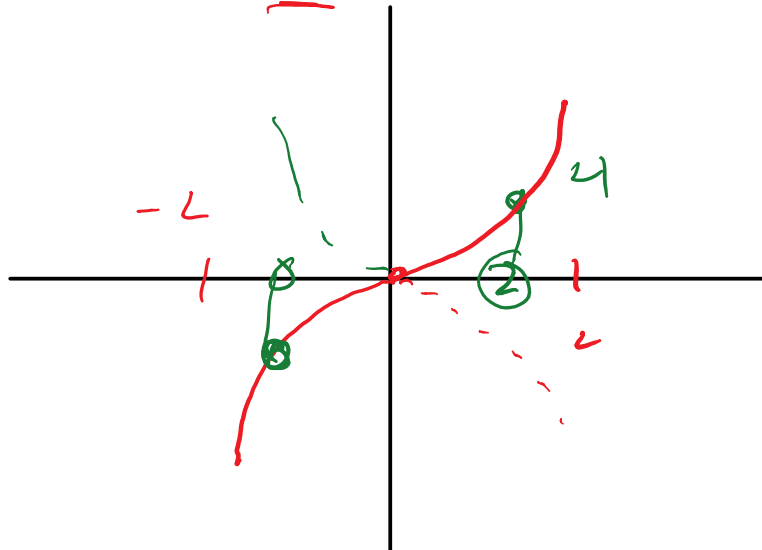
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An odd extension of an even function.

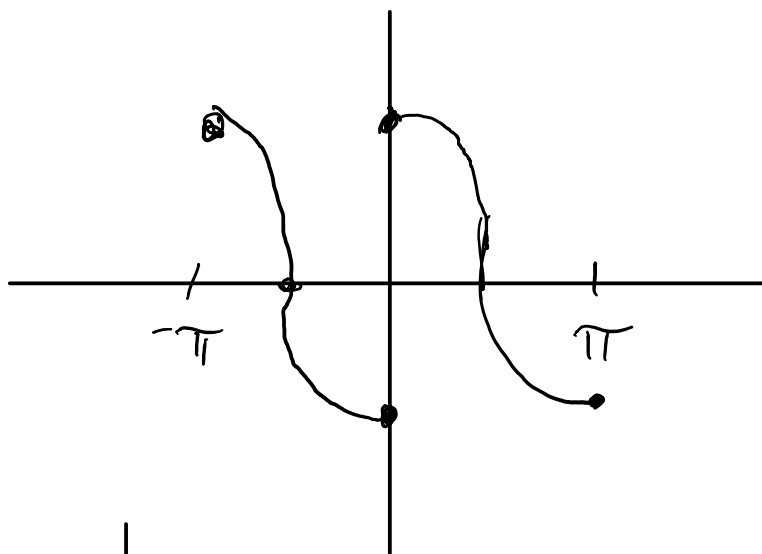
Find odd extensions for the following functions on the interval $[0, L]$ extended to the interval $[-L, L]$

$$f(x) = x^2$$

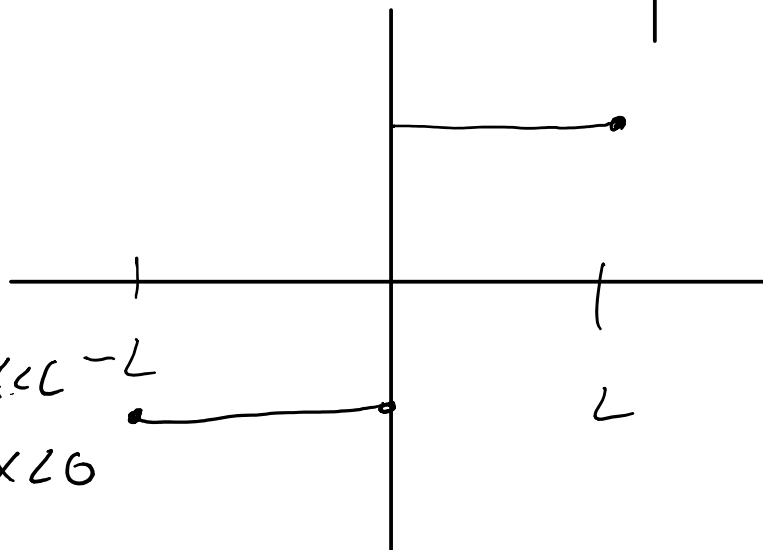
$$f(x) = \begin{cases} x^2 & 0 \leq x \leq L \\ -x^2 & -L \leq x < 0 \end{cases}$$



$$f(x) = \cos(x)$$



$$f(x) = 1$$



$$f(x) = \begin{cases} 1 & 0 \leq x \leq L \\ -1 & -L \leq x < 0 \end{cases}$$

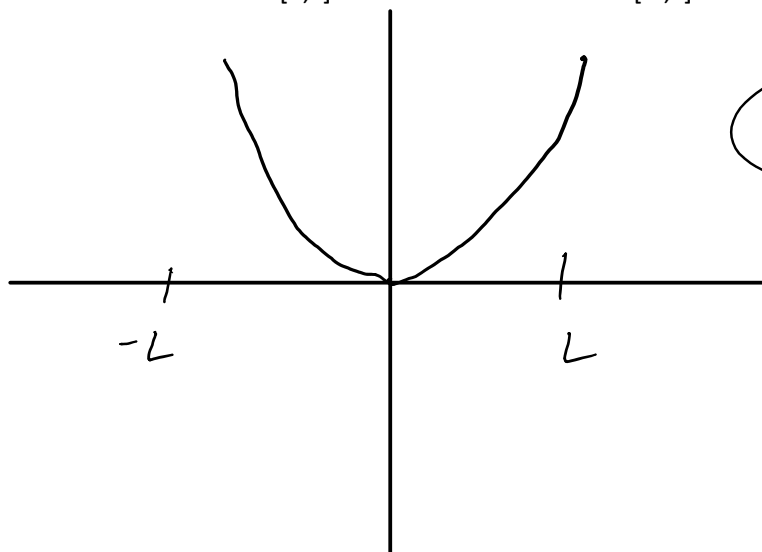
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An even extension of an odd function.

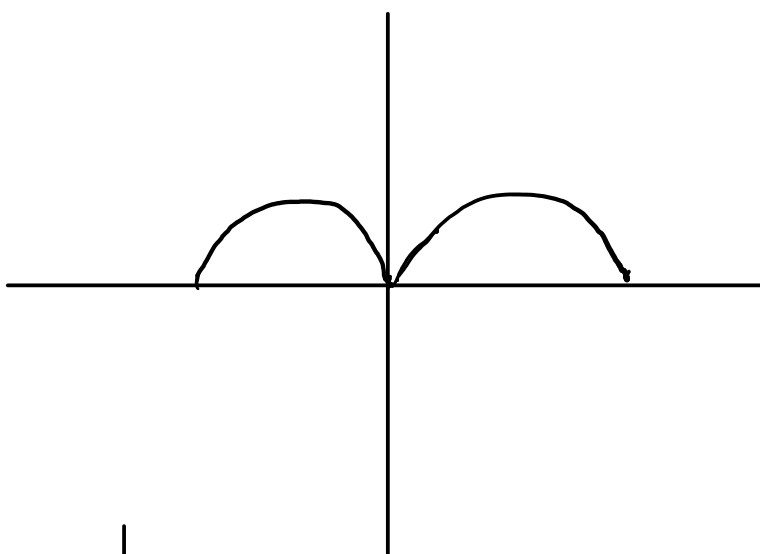
Find even extensions for the following functions on the interval $[0, L]$ extended to the interval $[-L, L]$

$$f(x) = x^3$$

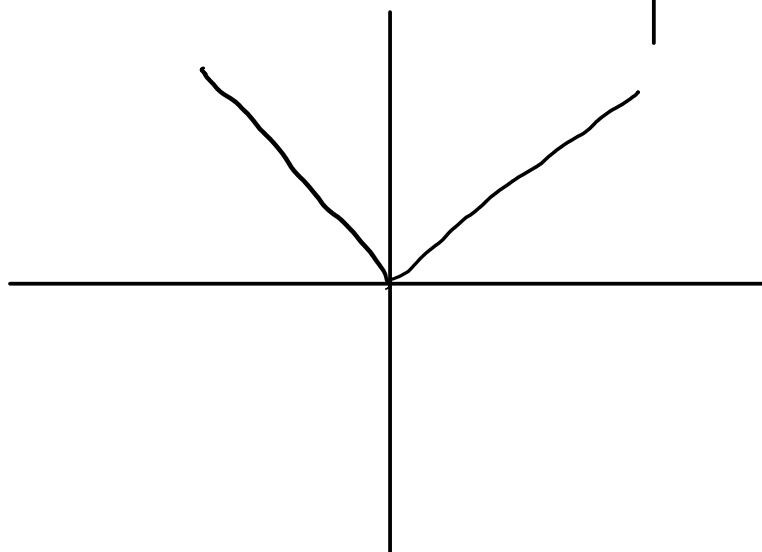


$$\underline{f(-x)} = f(x)$$

$$f(x) = \sin(x)$$



$$f(x) = x$$



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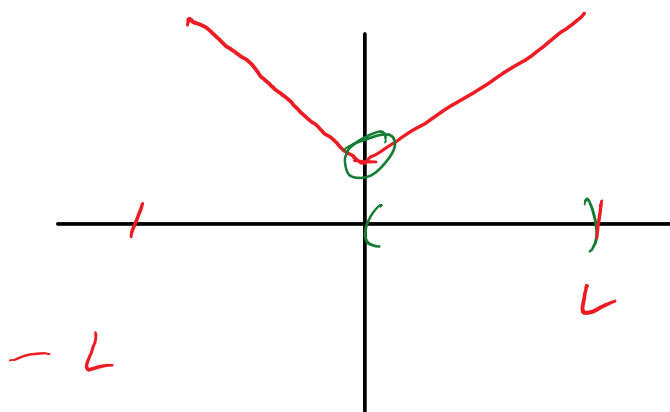
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Even and Odd extensions of functions which are neither.

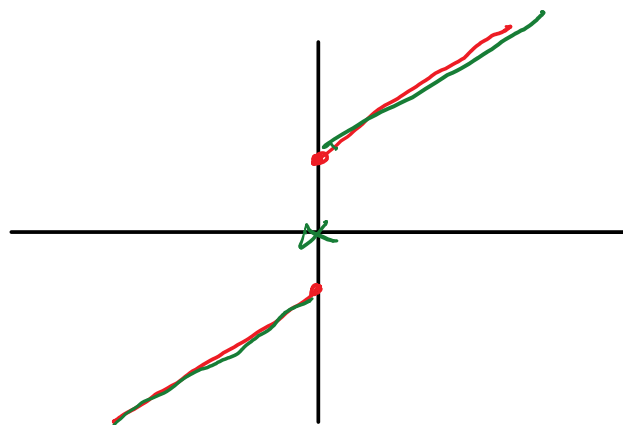
Find even and odd extensions of the following functions defined on $[0, L]$ extended to $[-L, L]$

$$f(x) = x + 1$$

even

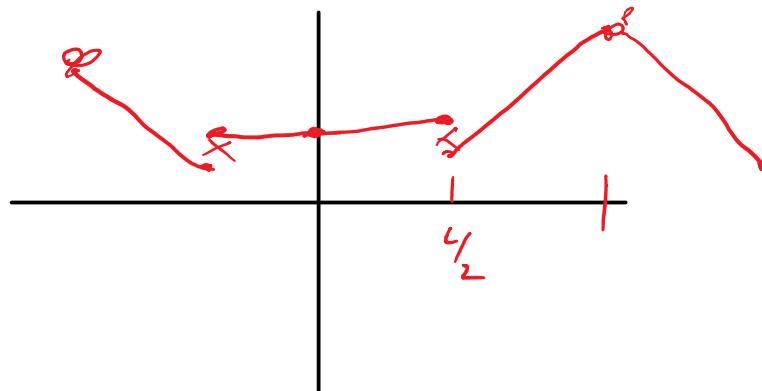
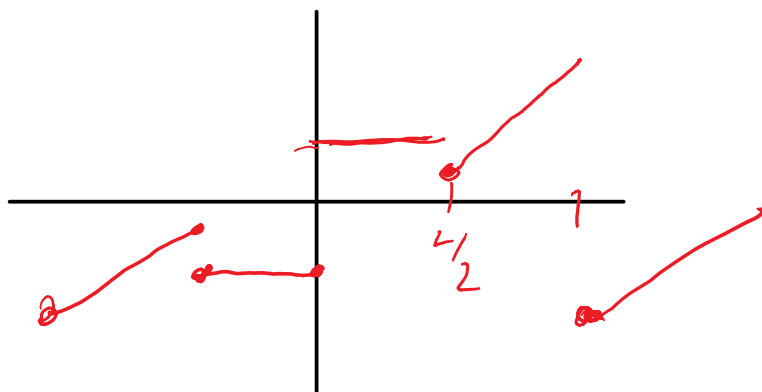


odd

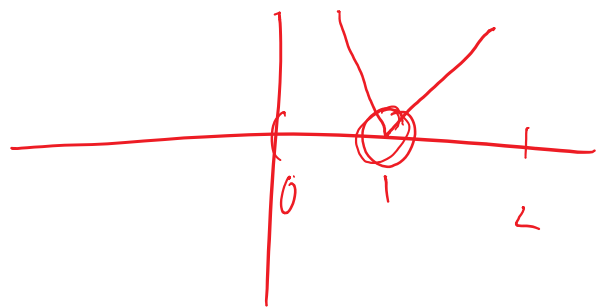


$$f(x) = \begin{cases} 1 & \text{for } 0 < x < \frac{L}{2} \\ x & \text{for } \frac{L}{2} < x < L \end{cases}$$

odd



Incidentally the Fourier series for these extensions will look very similar.



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For the 1-d rod with homogeneous Dirichlet boundary conditions we found that:

$$u(x, t) = \sum_{n=1}^{\infty} B_n e^{-k\left(\frac{n\pi}{L}\right)^2 t} \sin\left(\frac{n\pi}{L} x\right) \quad \leftarrow$$

Where

$$u(x, 0) = f(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L} x\right) \quad \text{for } 0 < x < L \quad \leftarrow$$

and

$$B_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

It turns out we actually were representing $f(x)$ with a Fourier sine series of an odd extension!

If we had used the formulas for a Fourier series with an odd extension let's see what would happen.

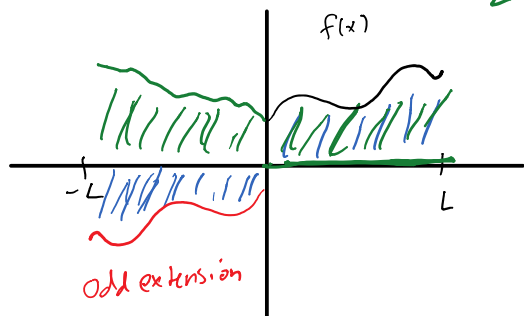
$$f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi}{L} x\right) + \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L} x\right)$$

$$A_0 = \frac{1}{2L} \int_{-L}^L f(x) dx = 0$$

$$A_n = \frac{1}{L} \int_{-L}^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx = 0 \quad \text{odd even} =$$

$$B_n = \frac{1}{L} \int_{-L}^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx \quad \text{odd odd} = \text{even}$$

$$\rightarrow 2 \frac{1}{2} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$



$$f(x) = \text{even}$$

$$g(x) = \text{odd}$$

$$g(x)f(x)$$

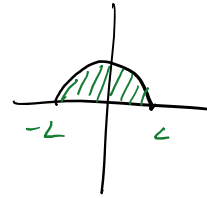
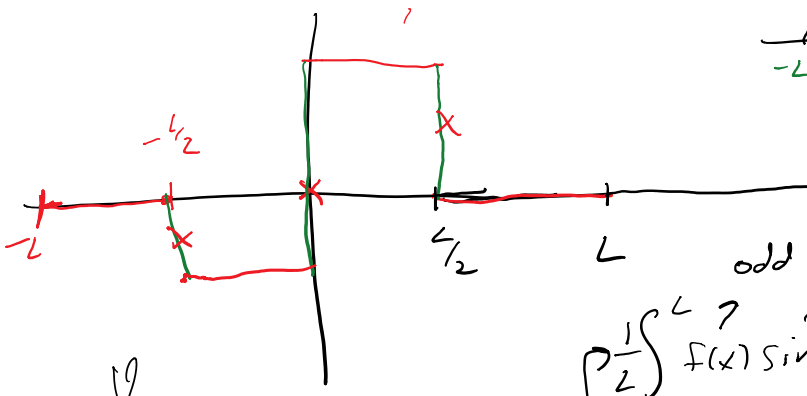
$$g(-x)f(-x) = -f(x)f(x)$$

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HW Example: sec 14.3 #2(d)

For $f(x) = \begin{cases} 1 & \text{for } x < \frac{L}{2} \\ 0 & \text{for } x > \frac{L}{2} \end{cases}$ sketch its fourier sine series and find the fourier sine series coefficients.



$$f(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}x\right)$$

$$\frac{1}{L} \int_{-L}^L f(x) \sin\left(\frac{n\pi}{L}x\right) dx$$

$$= \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi}{L}x\right) dx$$

$$B_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi}{L}x\right) dx$$

$$f(x) = \begin{cases} 1 & 0 \leq x \leq \frac{L}{2} \\ 0 & \frac{L}{2} < x \leq L \end{cases}$$

$$= \frac{2}{L} \left[\int_0^{\frac{L}{2}} (1) \sin\left(\frac{n\pi}{L}x\right) dx + \int_{\frac{L}{2}}^L 0 \sin\left(\frac{n\pi}{L}x\right) dx \right]$$

$$\frac{2}{n\pi} \cos\left(\frac{n\pi}{L}x\right) \Big|_0^{\frac{L}{2}} = -\frac{2}{n\pi} \left[\cos\left(\frac{n\pi}{L} \cdot \frac{L}{2}\right) - \cos(0) \right]$$

$$= \left[-\frac{2}{n\pi} \left[\cos\left(\frac{n\pi}{2}\right) - 1 \right] \right]$$

$$\left. \begin{array}{l} \text{even} \\ -\frac{2}{n\pi} [1 - 1] \end{array} \right| \begin{array}{l} \text{n odd} \\ -\frac{2}{n\pi} (-1) \\ = \frac{2}{n\pi} \end{array}$$

$$\cos\left(\frac{5\pi}{2}\right)$$

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For insulated boundary conditions we found that

$$u(x, t) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi}{L}x\right) e^{-\left(\frac{n\pi}{L}\right)^2 kt}$$

with

$$f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi}{L}x\right)$$

Where

$$A_0 = \frac{1}{L} \int_0^L f(x) dx$$

$$A_n = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx$$

It turns out we actually were representing $f(x)$ with a Fourier cosine series of an even extension!

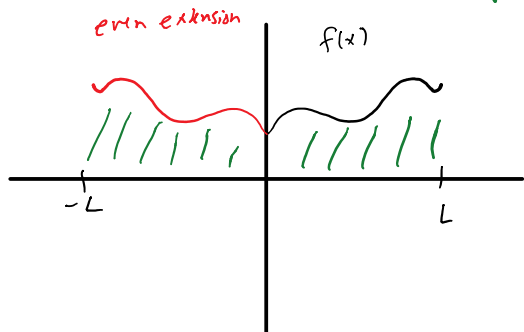
If we had used the formulas for a fourier series with an even extension let's see what would happen.

$$f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi}{L}x\right) + \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}x\right)$$

$$A_0 = \frac{1}{2L} \int_{-L}^L f(x) dx = \frac{2}{2L} \int_0^L f(x) dx = \frac{1}{L} \int_0^L f(x) dx$$

$$A_n = \frac{1}{L} \int_{-L}^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx$$

$$B_n = \frac{1}{L} \int_{-L}^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx = 0$$



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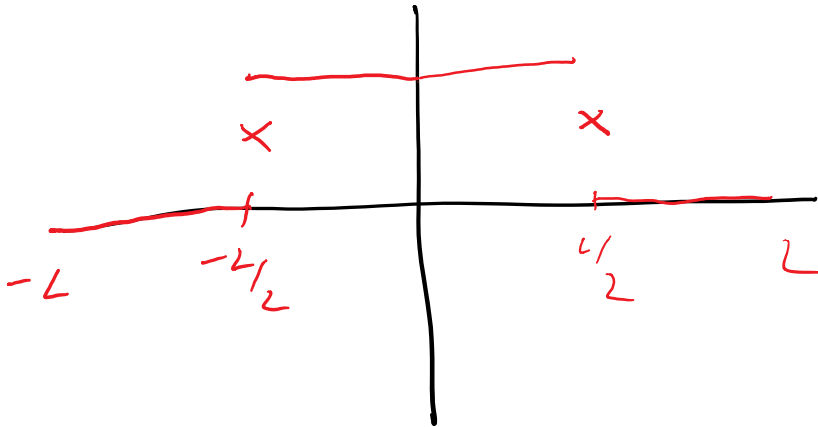
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Fourier Cosine series. If we have an even function or use an even extension we can instead use a cosine series.

Hw Example 14.3 5c

$$A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{L}$$

For $f(x) = \begin{cases} 1 & \text{for } x < \frac{L}{2} \\ 0 & \text{for } x > \frac{L}{2} \end{cases}$ sketch it's fourier cosine series and find the fourier cosine series coefficients.



$$A_0 = \frac{1}{L} \int_0^L f(x) dx = \frac{1}{L} \left[\int_0^{L/2} 1 dx + \int_{L/2}^L 0 dx \right]$$

$$A_0 = \frac{1}{L} \cdot \frac{L}{2} = \frac{1}{2}$$

$$A_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L} dx = \frac{2}{L} \left[\int_0^{L/2} (1) \cos \frac{n\pi x}{L} dx + \int_{L/2}^L 0 \cos \frac{n\pi x}{L} dx \right]$$

$$\frac{2}{L} \frac{L}{n\pi} \sin \frac{n\pi x}{L} \Big|_0^{L/2} = \frac{2}{n\pi} \sin \left(\frac{n\pi}{2} \right) = \boxed{\frac{2}{n\pi} \sin \frac{n\pi}{2}} = A_n$$

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We may be interested in a function which is neither odd nor even over the full interval $[-L, L]$ in this case we can simply use the full Fourier series.

$$f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi}{L}x\right) + \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}x\right)$$

with

$$A_0 = \frac{1}{2L} \int_{-L}^L f(x) dx = \frac{\langle f(x), 1 \rangle}{\langle 1, 1 \rangle} = \frac{\int_{-L}^L f(x) dx}{\int_{-L}^L 1 dx} = \frac{\int_{-L}^L f(x) dx}{2L}$$

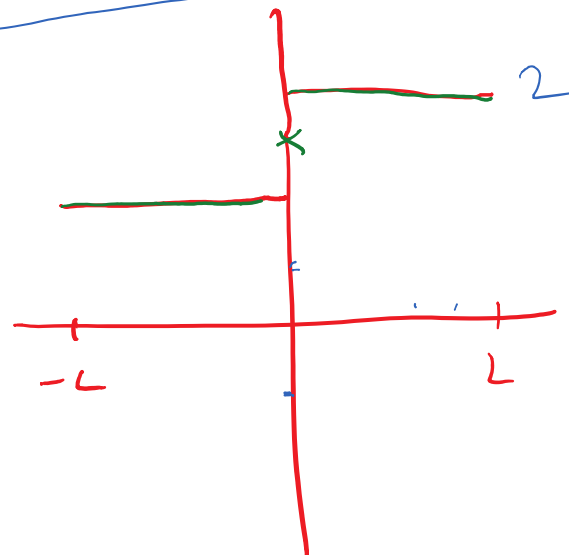
$$A_n = \frac{1}{L} \int_{-L}^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx$$

$$B_n = \frac{1}{L} \int_{-L}^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

Example: Find and sketch the Fourier series for $f(x) = \begin{cases} 1 & -L < x < 0 \\ 2 & 0 < x < L \end{cases}$

$$A_0 = \frac{1}{2L} \int_{-L}^L f(x) dx = \frac{1}{2L} \left[\int_{-L}^0 1 dx + \int_0^L 2 dx \right]$$

$$= \frac{1}{2L} L + \frac{2L}{L} = \frac{3}{2} = A_0$$



$$A_n = \frac{1}{L} \int_{-L}^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx = \frac{1}{L} \left[\int_{-L}^0 (1) \cos\left(\frac{n\pi x}{L}\right) dx + \int_0^L (2) \cos\left(\frac{n\pi x}{L}\right) dx \right]$$

$$\frac{1}{L} \frac{L}{n\pi} \sin\left(\frac{n\pi x}{L}\right) \Big|_{-L}^0 + \frac{1}{L} \frac{2L}{n\pi} \sin\left(\frac{n\pi x}{L}\right) \Big|_0^L = 0 = A_n$$

$$B_n = \frac{1}{L} \int_{-L}^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx = \frac{1}{L} \int_{-L}^0 \sin\left(\frac{n\pi x}{L}\right) dx + \int_0^L 2 \sin\left(\frac{n\pi x}{L}\right) dx$$

$$= \frac{1}{L} \left[-\frac{L}{n\pi} \cos\left(\frac{n\pi x}{L}\right) \Big|_{-L}^0 + \frac{-2L}{n\pi} \cos\left(\frac{n\pi x}{L}\right) \Big|_0^L \right] =$$

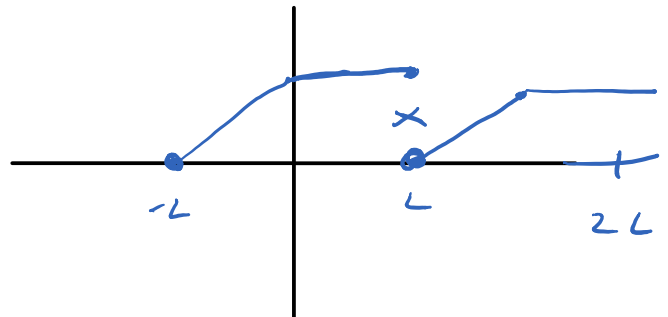
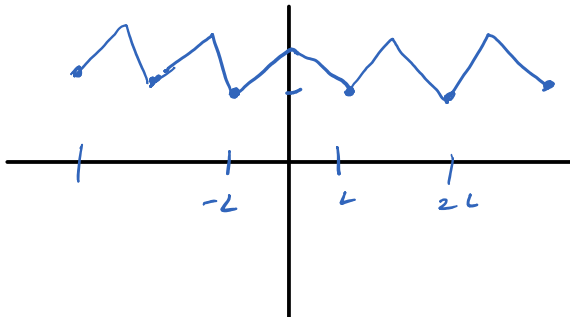
$$= \frac{1}{L} \left[\frac{-L}{n\pi} (1) + \cos(n\pi) - \frac{2L}{n\pi} \cos(n\pi) + \frac{2L}{n\pi} \right]$$

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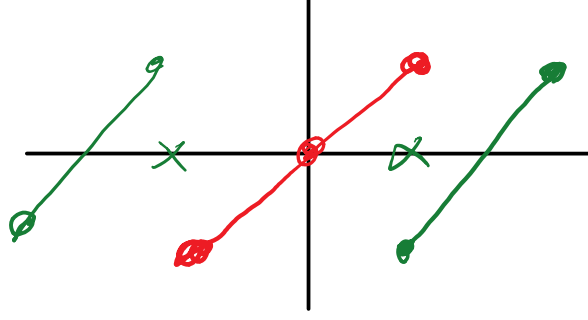
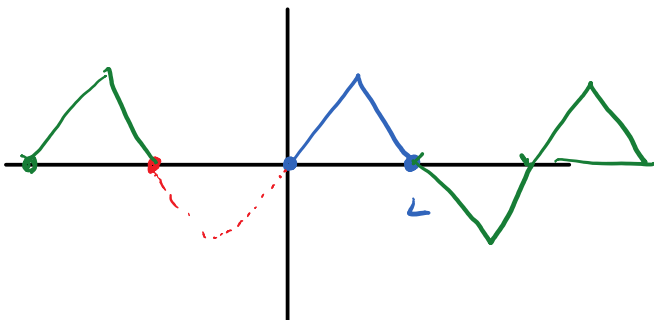
When is a Fourier Series continuous?

For a piecewise smooth $f(x)$, the Fourier series of $f(x)$ is continuous and converges to $f(x)$ on $[-L, L]$ if and only if $f(x)$ is continuous on the interval and $f(L) = f(-L)$



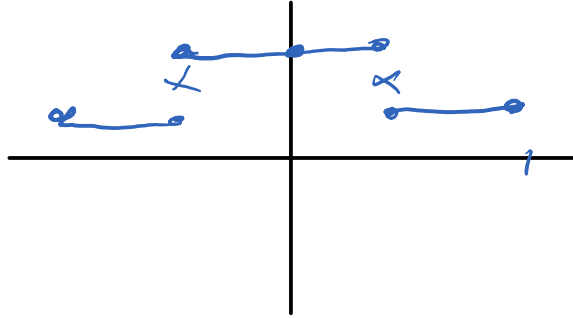
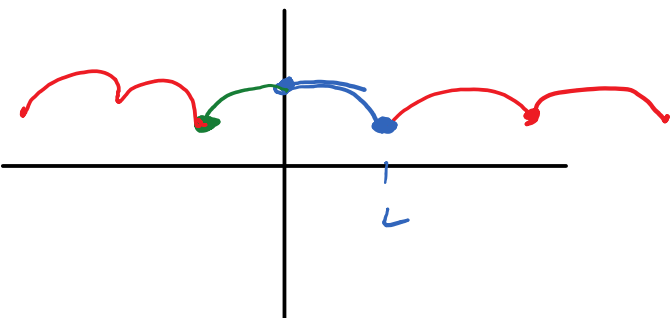
What about sine series?

For a piecewise smooth $f(x)$, the Fourier sine series of $f(x)$ is continuous and converges to $f(x)$ on $[0, L]$ if and only if $f(x)$ is continuous on the interval and $f(0) = f(L) = 0$



What about cosine series?

For a piecewise smooth $f(x)$, the Fourier cosine series of $f(x)$ is continuous and converges to $f(x)$ on $[0, L]$ if and only if $f(x)$ is continuous on the interval.



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Comparison of full, sine and cosine Fourier series

Hw example #1 (e)

Sketch the full, sine and cosine series for $f(x) = \begin{cases} 2 & \text{for } x < 0 \\ e^{-x} & \text{for } x > 0 \end{cases}$

