

## Sec 4.10

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Before we talk about inner product spaces it is useful to review the idea of a "basis", "orthogonal basis" and "orthogonal projection".

In the context of vectors a basis can be defined as a set of linearly independent vectors which span the space that they belong to.

Ex. The vectors  $\langle -1, 1 \rangle$  and  $\langle 0, 1 \rangle$  form a basis for  $\mathbb{R}^2$

Linear Independence:  $c_1 = c_2 = 0$

$$c_1 \langle -1, 1 \rangle + c_2 \langle 0, 1 \rangle = \langle 0, 0 \rangle$$

$$\langle -c_1, c_1 \rangle + \langle 0, c_2 \rangle = \langle 0, 0 \rangle$$

$$-c_1 + 0 = 0 \quad c_1 + c_2 = 0$$

They "span"  $\mathbb{R}^2$

$$c_1 = 0 \quad 0 + c_2 = 0 \quad c_2 = 0$$

$$\langle a, b \rangle$$

$$c_1 = -a$$

$$c_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$$

$$-c_1 + 0 = a$$

$$c_1 + c_2 = b$$

$$-a + c_2 = b \quad c_2 = b + a$$

An orthogonal basis is a basis of vectors for a given space which are orthogonal.

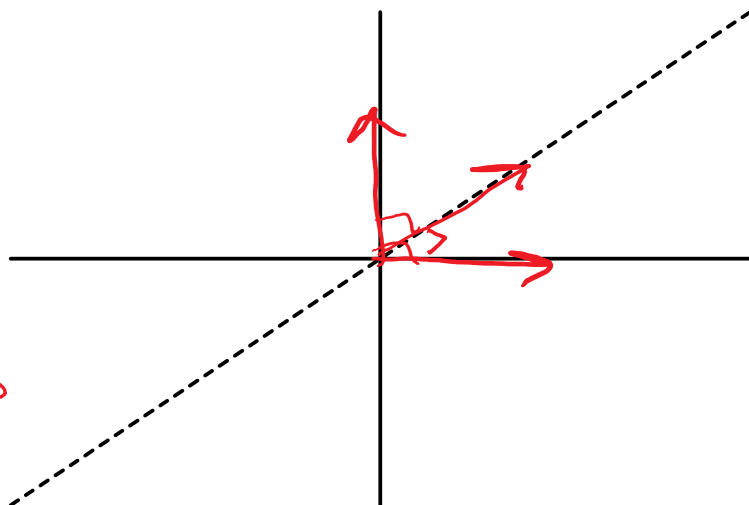
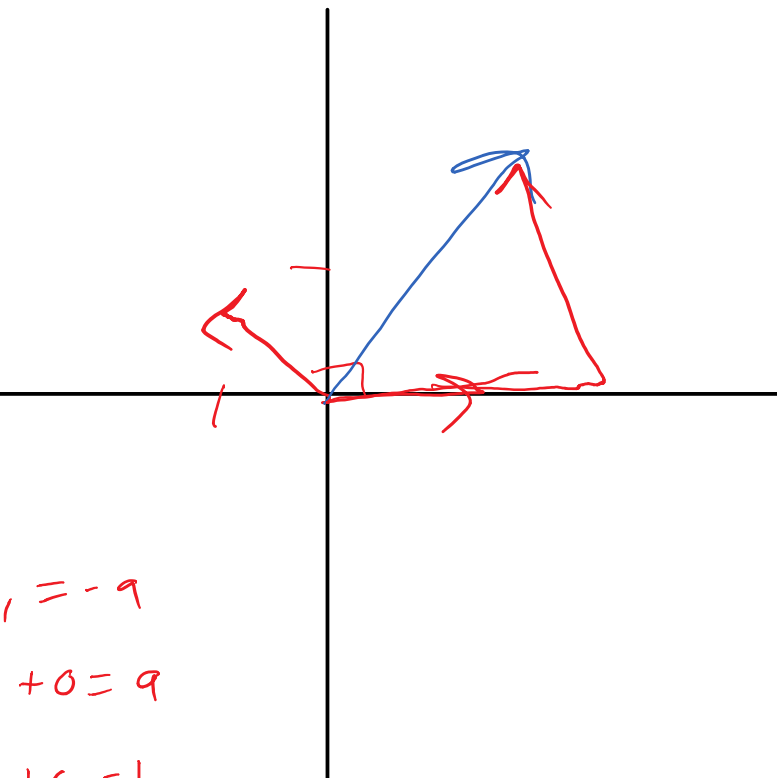
Example:  $\langle 1, 0, 0 \rangle$ ,  $\langle 0, 1, 0 \rangle$ ,  $\langle 0, 0, 1 \rangle$  form an orthogonal basis for  $\mathbb{R}^3$

Let's show that they are orthogonal.

$$\vec{a} \cdot \vec{b} = 0$$

$$\langle 1, 0, 0 \rangle \cdot \langle 0, 1, 0 \rangle = 0$$

$$\langle 0, 1, 0 \rangle \cdot \langle 0, 0, 1 \rangle = 0$$



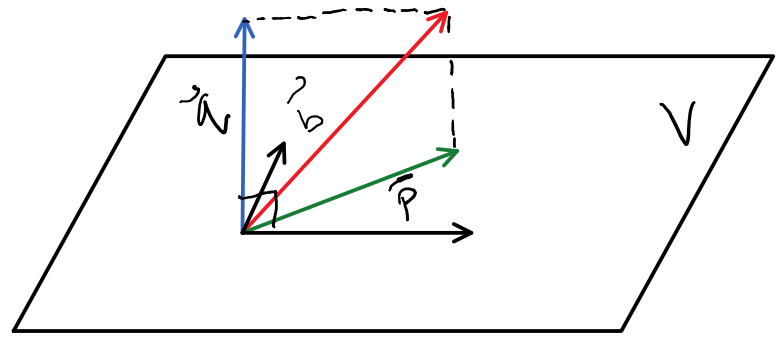
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Orthogonal Projection Problem:

Given a vector  $\vec{b}$  not in a subspace  $V$ , find vectors  $\vec{p}$  in  $V$  and  $\vec{q}$  orthogonal to  $V$  such that  $\vec{b} = \vec{p} + \vec{q}$ .

Further we wish to represent  $\vec{p}$  with the basis of  $V$   $\{\vec{a}_0, \vec{a}_1, \vec{a}_2, \dots, \vec{a}_n\}$  ideally an orthogonal basis.



Notice that  $\vec{p}$  is the closest vector to  $\vec{b}$  that lies in  $V$

Let's first suppose that we have an orthogonal basis for a subspace  $V$   $\{\vec{a}_0, \vec{a}_1, \vec{a}_2, \dots, \vec{a}_n\}$  This means that  $\vec{a}_i \cdot \vec{a}_j = 0$  if  $i \neq j$

Recall that the projection of one vector onto another is defined by  $\text{comp}_{\vec{a}}(\vec{b}) = \frac{\vec{a} \cdot \vec{b}}{||\vec{a}||}$

That is it is a measure of how much of the vector  $\vec{b}$  is in the direction of  $\vec{a}$

We could then take this amount and scale  $\vec{a}$  to make a vector in the same direction as  $\vec{a}$  call it  $\vec{p}$

How? Well take  $\vec{a}$  and divide it by its magnitude to create a unit vector in the same direction  $\frac{\vec{a}}{||\vec{a}||}$  then multiply it by  $\vec{b}$  is in the direction of  $\vec{a}$  namely the scalar we get from  $\text{comp}_{\vec{a}}(\vec{b}) = \frac{\vec{a} \cdot \vec{b}}{||\vec{a}||}$

We then have

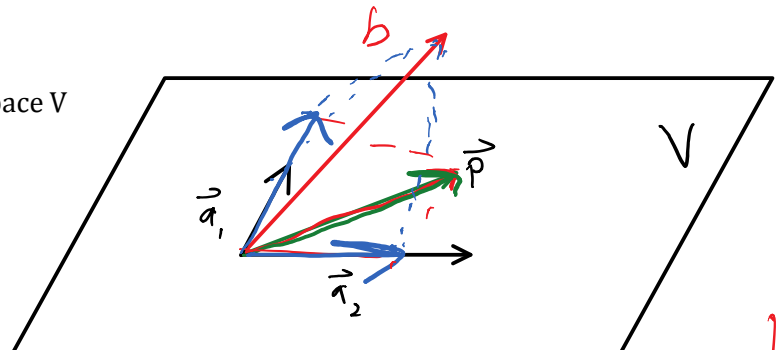
$$\vec{p} = \frac{\vec{a} \cdot \vec{b}}{||\vec{a}||} \frac{\vec{a}}{||\vec{a}||} = \frac{\vec{a} \cdot \vec{b}}{\vec{a} \cdot \vec{a}} \vec{a}$$

$$(||\vec{a}||)^2 = (\sqrt{\vec{a} \cdot \vec{a}})^2 = \vec{a} \cdot \vec{a}$$

It turns out that if we do this to  $\vec{b}$  with each vector in our orthogonal basis  $\{\vec{a}_0, \vec{a}_1, \vec{a}_2, \dots, \vec{a}_n\}$  and add them up, we end up with the orthogonal projection of  $\vec{b}$  on to the subspace  $V$  in terms of the orthogonal basis of  $V$ .

The orthogonal projection of the vector  $\vec{b}$  on to the subspace  $V$  Is given by:

$$\vec{p} = \frac{\vec{a}_1 \cdot \vec{b}}{\vec{a}_1 \cdot \vec{a}_1} \vec{a}_1 + \frac{\vec{a}_2 \cdot \vec{b}}{\vec{a}_2 \cdot \vec{a}_2} \vec{a}_2 + \dots + \frac{\vec{a}_n \cdot \vec{b}}{\vec{a}_n \cdot \vec{a}_n} \vec{a}_n$$



$\{ \dots \}$   $\sigma = \sigma_1 \oplus \sigma_2$

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It turns out that the vector  $\vec{p}$  that we found is the best approximation of  $\vec{b}$  using the basis of  $V$   $\{\vec{a}_0, \vec{a}_1, \vec{a}_2, \dots, \vec{a}_n\}$

While it may not seem particularly useful to approximate a known vector it does have its uses in inverse problems where we can't find an exact answer. For now though we might consider the case where  $\vec{b}$  lies in the subspace  $V$ . In this case  $\vec{p} = \vec{b}$  and our formula decomposes  $\vec{b}$  into a linear combo of our orthogonal basis. This idea becomes most useful for PDE's when we extend our notion of a "vector space" and a "dot product".

**Definition:** A *vector space* is a set  $V$  on which two operations  $+$  and  $\cdot$  are defined, called *vector addition* and *scalar multiplication*. **Note this  $\cdot$  is not the dot product here**

The operation  $+$  (vector addition) must satisfy the following conditions:

*Closure:* If  $\mathbf{u}$  and  $\mathbf{v}$  are any vectors in  $V$ , then the sum  $\mathbf{u} + \mathbf{v}$  belongs to  $V$ .

- (1) *Commutative law:* For all vectors  $\mathbf{u}$  and  $\mathbf{v}$  in  $V$ ,  $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
- (2) *Associative law:* For all vectors  $\mathbf{u}, \mathbf{v}, \mathbf{w}$  in  $V$ ,  $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$
- (3) *Additive identity:* The set  $V$  contains an *additive identity* element, denoted by  $\mathbf{0}$ , such that for any vector  $\mathbf{v}$  in  $V$ ,  $\mathbf{0} + \mathbf{v} = \mathbf{v}$  and  $\mathbf{v} + \mathbf{0} = \mathbf{v}$ .
- (4) *Additive inverses:* For each vector  $\mathbf{v}$  in  $V$ , the equations  $\mathbf{v} + \mathbf{x} = \mathbf{0}$  and  $\mathbf{x} + \mathbf{v} = \mathbf{0}$  have a solution  $\mathbf{x}$  in  $V$ , called an *additive inverse* of  $\mathbf{v}$ , and denoted by  $-\mathbf{v}$ .

The operation  $\cdot$  (scalar multiplication) is defined between real numbers (or scalars) and vectors, and must satisfy the following conditions:

*Closure:* If  $\mathbf{v}$  is any vector in  $V$ , and  $c$  is any real number, then the product  $c \cdot \mathbf{v}$  belongs to  $V$ .

- (5) *Distributive law:* For all real numbers  $c$  and all vectors  $\mathbf{u}, \mathbf{v}$  in  $V$ ,  $c \cdot (\mathbf{u} + \mathbf{v}) = c \cdot \mathbf{u} + c \cdot \mathbf{v}$
- (6) *Distributive law:* For all real numbers  $c, d$  and all vectors  $\mathbf{v}$  in  $V$ ,  $(c+d) \cdot \mathbf{v} = c \cdot \mathbf{v} + d \cdot \mathbf{v}$
- (7) *Associative law:* For all real numbers  $c, d$  and all vectors  $\mathbf{v}$  in  $V$ ,  $c \cdot (d \cdot \mathbf{v}) = (cd) \cdot \mathbf{v}$
- (8) *Unitary law:* For all vectors  $\mathbf{v}$  in  $V$ ,  $1 \cdot \mathbf{v} = \mathbf{v}$

Notice in the above definition  $V$  was just a set of general objects not necessarily vectors as we have used them.

In fact we can actually define a vector space of functions. For example let's take the set of all polynomials  $P$  on the real line and define a vector space. Let  $p(x), q(x) \in P$

Vector addition:  $(p + q)(x) = p(x) + q(x)$

Scalar multiplication:  $c \cdot f(x) = cf(x)$

Let's just check a few properties:

Is it closed under addition and scalar multiplication?

$$p = x^2 + 2 \quad q = 3x^2 + 4x + 1$$

$$x^2 + 2 + 3x^2 + 4x + 1 = 4x^2 + 4x + 3$$

(1)

(5)

(2)

(6)

(3)  $\emptyset$

(7)

$$(4) \quad -x^2 + 2x + 1 + x^2 - 2x = 0$$

(8)

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What about a basis for  $P$ ? Well recall that a basis is a set of objects in our set which are linearly independent and span the set. For the set of all polynomials one basis is:

$$\{1, x, x^2, x^3, x^4, \dots\}$$

$$c_1 = c_2 = c_3 = \dots = c_n = 0$$

Are they linearly independent?

$$c_1 \cdot 1 + c_2 x + c_3 x^2 + c_4 x^3 + \dots + c_n x^n = 0$$

$$c_i = 0$$

Do they span the set of all polynomials?

$$2(1)$$

$$3x^2 + 4x + 1$$

Ok so we have a basis for our vector space of functions. One natural question now is what does our "dot product" look like?

Well it turns out that the dot product is a special case of what we call an "inner product"

An **inner product** on a vector space  $V$  is a function that associates with each pair of vectors  $u$  and  $v$  in  $V$  a scalar number  $\langle u, v \rangle$  such that:

$$(1) \langle u, v \rangle = \langle v, u \rangle$$

$$(2) \langle u, v + w \rangle = \langle u, v \rangle + \langle u, w \rangle$$

$$(3) \langle cu, v \rangle = c \langle u, v \rangle$$

$$(4) \langle u, u \rangle \geq 0 \text{ and } \langle u, u \rangle = 0 \text{ if and only if } u = 0$$

$$u = \langle 1, 1 \rangle \quad \langle 2, 2 \rangle = v$$

$$\langle u, v \rangle = u \cdot v$$

$$\langle 3u, v \rangle$$

$$\langle 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \end{bmatrix} \rangle$$

$$3 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

$$= 6 + 6 = 12$$

$$3 \left( \begin{bmatrix} 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 2 \end{bmatrix} \right) = 3(2+2) = 3(4) = 12$$

If we have a vector space  $V$  with together with an inner product we call this an inner product space. For function vector spaces our inner products are usually integrals of products of functions.

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An example of an inner product space: Let  $P^n[-1,1]$  be the set of continuous polynomials of degree at most  $n$  on the interval  $[-1,1]$ .

We define the inner product between two polynomials  $p(x), q(x) \in P^n[-1,1]$  to be:

$$\langle p(x), q(x) \rangle = \int_{-1}^1 p(x)q(x)dx$$

Let's verify that this is indeed an inner product:

$$(1) \langle u, v \rangle = \langle v, u \rangle$$

$$\langle p(x), q(x) \rangle = \langle q(x), p(x) \rangle \rightarrow (2) \langle u, v+w \rangle = \langle u, v \rangle + \langle u, w \rangle$$

$$(3) \langle cu, v \rangle = c \langle u, v \rangle$$

$$(4) \langle u, u \rangle \geq 0 \text{ and } \langle u, u \rangle = 0 \text{ if and only if } u = 0$$

$$\int_{-1}^1 p(x)q(x)dx = \int_{-1}^1 q(x)p(x)dx \checkmark$$

$$\langle p(x), q(x)+w(x) \rangle = \langle p(x), q(x) \rangle + \langle p(x), w(x) \rangle$$

$$\int_{-1}^1 p(x)(q(x)+w(x))dx = \int_{-1}^1 p(x)q(x)dx + \int_{-1}^1 p(x)w(x)dx$$

$$\int_{-1}^1 p(x)q(x)dx + \int_{-1}^1 p(x)w(x)dx$$

$$\langle c p(x), q(x) \rangle = c \langle p(x), q(x) \rangle$$

$$\int_{-1}^1 c p(x)q(x)dx = c \int_{-1}^1 p(x)q(x)dx$$

$$(4) \langle p(x), p(x) \rangle = \int_{-1}^1 p(x)p(x)dx = \int_{-1}^1 (p(x))^2 dx \geq 0$$

$$\int_{-1}^1 (p(x))^2 dx = 0 \Leftrightarrow p(x) = 0$$

Q.E.D.

In checking (4) we might think we can use the inner product to measure the "length" of a function in fact we can, we define the norm of a function  $f$  on our space as:

$$\|u\| = \sqrt{u \cdot u} = \sqrt{\langle u, u \rangle}$$

$$\|f\| = \left( \int_{-1}^1 (f(x))^2 dx \right)^{\frac{1}{2}} = \sqrt{\langle f, f \rangle}$$

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As we stated before the dot product we are used to is simply a special case of an **inner product** we might also recall that for conventional vectors if their dot product is zero they are orthogonal.

We extend this notion to all inner product spaces and say that:

Two general vectors  $u$  and  $v$  are orthogonal if and only if  $\langle u, v \rangle = 0$

Let's see if our basis for the set of all polynomials on  $[-1, 1]$   $\{1, x, x^2, x^3, x^4, \dots\}$  is an orthogonal basis:

Let's try  $x^2$  and  $x$   $\langle p(x), q(x) \rangle = \int_{-1}^1 p(x)q(x) dx$

$$\langle x^2, x \rangle = \int_{-1}^1 x^2 x dx = \int_{-1}^1 x^3 dx = \frac{x^4}{4} \Big|_{-1}^1 = \frac{1^4}{4} - \frac{(-1)^4}{4} = 0$$

Let's try  $x^2$  and  $x^4$

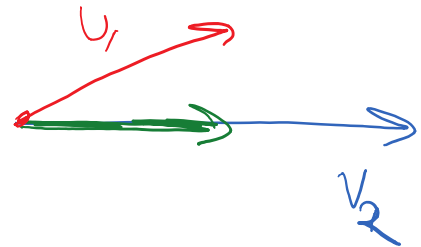
$$\int_{-1}^1 x^2 x^4 dx = \int_{-1}^1 x^6 dx = \frac{x^7}{7} \Big|_{-1}^1 = \frac{1^7}{7} - \frac{(-1)^7}{7} = \frac{2}{7} \neq 0$$

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### Gram-Schmidt Orthogonalization:

$$\begin{aligned} u_1 &= v_1 \\ u_2 &= v_2 - \frac{\langle u_1, v_2 \rangle}{\langle u_1, u_1 \rangle} u_1 \\ u_3 &= v_3 - \frac{\langle u_1, v_3 \rangle}{\langle u_1, u_1 \rangle} u_1 - \frac{\langle u_2, v_3 \rangle}{\langle u_2, u_2 \rangle} u_2 \\ &\vdots \end{aligned}$$

$$\underline{u_n} = \underline{v_n} - \frac{\langle u_1, v_2 \rangle}{\langle u_1, u_1 \rangle} u_1 - \frac{\langle u_2, v_3 \rangle}{\langle u_2, u_2 \rangle} u_2 - \dots - \frac{\langle u_{n-1}, v_n \rangle}{\langle u_{n-1}, u_{n-1} \rangle} u_{n-1}$$

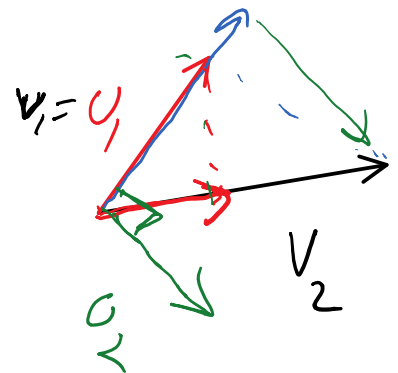

$$= v_1$$
  

$$= v_2 - \frac{u_1 \cdot v_2}{u_1 \cdot u_1} u_1 \rightarrow v_2 + \frac{u_1 \cdot v_2}{u_1 \cdot u_1} u_1 = v_2$$
  

$$= v_3 - \frac{u_1 \cdot v_2}{u_1 \cdot u_1} u_1 - \frac{u_2 \cdot v_3}{u_2 \cdot u_2} u_2$$
  

$$= v_n - \frac{u_1 \cdot v_2}{u_1 \cdot u_1} u_1 - \frac{u_2 \cdot v_3}{u_2 \cdot u_2} u_2 - \dots - \frac{u_{n-1} \cdot v_n}{u_{n-1} \cdot u_{n-1}} u_{n-1}$$

$$\vec{p} = \frac{\vec{a} \cdot \vec{b}}{||\vec{a}||} \frac{\vec{a}}{||\vec{a}||} = \frac{\vec{a} \cdot \vec{b}}{\vec{a} \cdot \vec{a}} \vec{a}$$



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$$\langle p(x), q(x) \rangle = \int_{-1}^1 p(x)q(x) dx$$

If we apply this algorithm to our set of basis vectors for  $P^n[-1,1]$   $\{1, x, x^2, x^3, x^4, \dots\}$

we obtain multiples of a famous set of polynomials called Legendre Polynomials of which the first 6 are:

Gram-Schmidt Orthogonalization:

Given any set of linearly independent vectors  $v_1, v_2, v_3, \dots, v_n$  we can obtain a set of orthogonal vectors  $u_1, u_2, u_3, \dots, u_n$  that span the same space with the following algorithm.

$$\begin{aligned} u_1 &= v_1 \\ u_2 &= v_2 - \frac{\langle u_1, v_2 \rangle}{\langle u_1, u_1 \rangle} u_1 \\ u_3 &= v_3 - \frac{\langle u_1, v_3 \rangle}{\langle u_1, u_1 \rangle} u_1 - \frac{\langle u_2, v_3 \rangle}{\langle u_2, u_2 \rangle} u_2 \\ &\vdots \\ u_n &= v_n - \frac{\langle u_1, v_n \rangle}{\langle u_1, u_1 \rangle} u_1 - \frac{\langle u_2, v_n \rangle}{\langle u_2, u_2 \rangle} u_2 - \dots - \frac{\langle u_{n-1}, v_n \rangle}{\langle u_{n-1}, u_{n-1} \rangle} u_{n-1} \end{aligned}$$

$$u_3 = x^2 - 0 - \frac{\int_{-1}^1 x x^2 dx}{\int_{-1}^1 x^2 dx} x$$

Let's calculate the first two, you get to do the rest in your homework. :)

$$u_1 = 1$$

$$u_2 = v_2 - \frac{\langle u_1, v_2 \rangle}{\langle u_1, u_1 \rangle} u_1$$

$$u_2 = x - \frac{\langle 1, x \rangle}{\langle 1, 1 \rangle} (1)$$

$$u_2 = x - \frac{\int_{-1}^1 x dx}{\int_{-1}^1 1 dx} (1) = x - \frac{x^2 \Big|_{-1}^1}{x \Big|_{-1}^1}$$

$$= x - \frac{\frac{1}{2} - \frac{1}{2}}{2} = x$$

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Ok so what is the point of all of this math... Recall earlier that if we had a vector  $\vec{p}$  in a space  $V$  we could decompose it into a linear combo the orthogonal basis vectors for  $V$  with the formula

$$\vec{p} = \frac{\vec{a}_1 \cdot \vec{b}}{\vec{a}_1 \cdot \vec{a}_1} \vec{a}_1 + \frac{\vec{a}_2 \cdot \vec{b}}{\vec{a}_2 \cdot \vec{a}_2} \vec{a}_2 + \dots + \frac{\vec{a}_n \cdot \vec{b}}{\vec{a}_n \cdot \vec{a}_n} \vec{a}_n$$

$a_1, \dots, a_n$   $V$   $\underbrace{\quad}_{\vec{b}}$

It turns out we can do the same thing with functions! All we have to do is replace the dot product with the appropriate inner product that is for a given function  $f(x) \in C([a, b])$  (the set of continuous functions on the interval  $[a, b]$ ) if we have an orthogonal basis for  $C([a, b])$   $\{g_1(x), g_2(x), \dots\}$

We can write:

$$f(x) = \frac{\langle g_1, f \rangle}{\langle g_1, g_1 \rangle} g_1 + \frac{\langle g_2, f \rangle}{\langle g_2, g_2 \rangle} g_2 + \dots + \frac{\langle g_n, f \rangle}{\langle g_n, g_n \rangle} g_n + \dots$$

*Handwritten note:  $\sin(\frac{n\pi x}{L})$  is written above the first term.*

This means that we can represent a given function as a linear combination of basis functions!

We have actually seen this already, if you recall we were able to represent our initial conditions to the heat equation as:

For homogeneous Dirichlet boundary conditions we had

$$u(x, 0) = f(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}x\right)$$

Where

$$B_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi}{L}x\right) dx = \frac{\int_0^L f(x) \sin\left(\frac{n\pi}{L}x\right) dx}{\int_0^L \sin^2\left(\frac{n\pi}{L}x\right) dx} = \frac{\langle \sin\left(\frac{n\pi x}{L}\right), f(x) \rangle}{\langle \sin\left(\frac{n\pi x}{L}\right), \sin\left(\frac{n\pi x}{L}\right) \rangle}$$

*Handwritten note:  $\frac{L}{2}$  is written below the denominator integral.*

If we define the inner product

$$\langle f, g \rangle = \int_0^L f(x)g(x)dx \quad \text{we are simply saying:} \quad f(x) = \frac{\langle g_1, f \rangle}{\langle g_1, g_1 \rangle} g_1 + \frac{\langle g_2, f \rangle}{\langle g_2, g_2 \rangle} g_2 + \dots + \frac{\langle g_n, f \rangle}{\langle g_n, g_n \rangle} g_n + \dots$$

With  $g_n = \sin\left(\frac{n\pi x}{L}\right)$  which are indeed orthogonal since we we showed that:

$$\langle \sin\left(\frac{n\pi x}{L}\right), \sin\left(\frac{m\pi x}{L}\right) \rangle = \int_0^L \sin\left(\frac{n\pi}{L}x\right) \sin\left(\frac{m\pi}{L}x\right) dx = \begin{cases} 0 & \text{when } m \neq n \\ \frac{L}{2} & \text{when } m = n \end{cases}$$

$n \neq m$   $\int_0^L \sin\frac{n\pi x}{L} \sin\frac{m\pi x}{L} = 0$

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It turns out that an orthogonal basis for the set of ~~piecewise~~ smooth functions on the interval  $[-L, L]$  is:

$$\{1, \sin\left(\frac{\pi x}{L}\right), \sin\left(\frac{2\pi x}{L}\right), \dots, \cos\left(\frac{\pi x}{L}\right), \cos\left(\frac{2\pi x}{L}\right), \dots\}$$

This means if we have any ~~piecewise~~ smooth function on the interval  $[-L, L]$  we could represent it as a sum of a constant and sines and cosines with the appropriate coefficients determined by an inner product.

For the interval  $[-L, L]$  we would use the innerproduct  $\langle f, g \rangle = \int_{-L}^L f(x)g(x)dx$

That is we can write

$$\begin{aligned} f(x) &= \frac{\langle 1, f \rangle}{\langle 1, 1 \rangle} 1 + \frac{\langle \sin\left(\frac{\pi x}{L}\right), f \rangle}{\langle \sin\left(\frac{\pi x}{L}\right), \sin\left(\frac{\pi x}{L}\right) \rangle} \sin\left(\frac{\pi x}{L}\right) + \frac{\langle \sin\left(\frac{2\pi x}{L}\right), f \rangle}{\langle \sin\left(\frac{2\pi x}{L}\right), \sin\left(\frac{2\pi x}{L}\right) \rangle} \sin\left(\frac{2\pi x}{L}\right) + \dots \\ &+ \frac{\langle \cos\left(\frac{\pi x}{L}\right), f \rangle}{\langle \cos\left(\frac{\pi x}{L}\right), \cos\left(\frac{\pi x}{L}\right) \rangle} \cos\left(\frac{\pi x}{L}\right) + \frac{\langle \cos\left(\frac{2\pi x}{L}\right), f \rangle}{\langle \cos\left(\frac{2\pi x}{L}\right), \cos\left(\frac{2\pi x}{L}\right) \rangle} \cos\left(\frac{2\pi x}{L}\right) + \dots \end{aligned}$$

And

$$\frac{\langle 1, f \rangle}{\langle 1, 1 \rangle} = \frac{\int_{-L}^L f(x)dx}{\int_{-L}^L dx} = \frac{\int_{-L}^L f(x)dx}{2L} = \frac{1}{2L} \int_{-L}^L f(x)dx = A_0$$

$$\frac{\langle \sin\left(\frac{n\pi x}{L}\right), f \rangle}{\langle \sin\left(\frac{n\pi x}{L}\right), \sin\left(\frac{n\pi x}{L}\right) \rangle} = \frac{\int_{-L}^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx}{\int_{-L}^L \sin^2\left(\frac{n\pi x}{L}\right) dx} = \frac{\int_{-L}^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx}{L} = \frac{1}{L} \int_{-L}^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx = B_n$$

$$\frac{\langle \cos\left(\frac{n\pi x}{L}\right), f \rangle}{\langle \cos\left(\frac{n\pi x}{L}\right), \cos\left(\frac{n\pi x}{L}\right) \rangle} = \frac{\int_{-L}^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx}{\int_{-L}^L \cos^2\left(\frac{n\pi x}{L}\right) dx} = \frac{\int_{-L}^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx}{L} = \frac{1}{L} \int_{-L}^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx = A_n$$

So we can then represent our sum in the more familiar form we saw with periodic boundary conditions of

$$f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi}{L}x\right) + \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}x\right)$$

$$\begin{aligned} \phi(-L) &= \phi(L) \\ \frac{d\phi}{dx}(-L) &= \frac{d\phi}{dx}(L) \end{aligned}$$

With this choice of basis functions we call this a Fourier series!

# Sec 4.10

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