

Wave Equation on the Real Line

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We have solved the wave equation for a string of length L , but waves move through more than string they also move through space! We might ask what the solution to 1-d wave equation looks like on an infinite domain of $(-\infty, \infty)$

We are going to solve:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \text{For } x \in (-\infty, \infty)$$

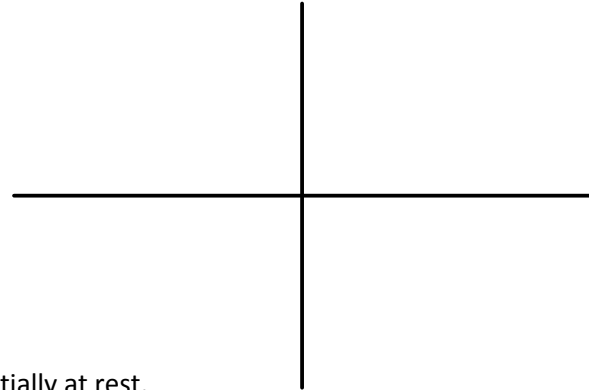
We will have an initial shape of the wave given by:

$$u(x, 0) = f(x)$$

And Initial speed of the wave given by:

$$\frac{\partial u}{\partial t}(x, 0) = 0 \quad \text{That is it is initially at rest.}$$

To do this we will take the Fourier Transform of our PDE to obtain a nice ODE!



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So we have:

$$\frac{\partial^2 \bar{U}}{\partial t^2} = -c^2 \omega^2 \bar{U}$$

With initial conditions:

$$\bar{U}(x, 0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{i\omega x} dx$$

$$\frac{\partial}{\partial t} \bar{U}(\omega, 0) = 0$$

Let's solve it:

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So we have:

$$u(x, t) = \int_{-\infty}^{\infty} \bar{U}(\omega, 0) \cos(c\omega t) e^{-i\omega x} d\omega$$

However a clever application of Euler's formula can simplify this greatly. Recall that $e^{ix} = \cos(x) + i \sin(x)$.

This means that $\cos(x) = \frac{1}{2} [e^{ix} + e^{-ix}]$ Let's just verify that...

So then we could write $\cos(c\omega t) = \frac{1}{2} (e^{i\omega ct} + e^{-i\omega ct})$ What does this say about our integral solution?

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$$\text{So } u(x, t) = \frac{1}{2} [f(x - ct) + f(x + ct)]$$

That it is just the average of the initial wave moving to the left and to the right with speed c !
The way to interpret this physically, is that half of the initial wave form moves to the right and half to the left. After the wave can travel either direction.

Let's see what this would actually look like!