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Section 12.1

What is a PDE?

Strictly speaking, a PDE is an equation that relates the partial derivatives of one or more functions of several variables.

Example:

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} \quad \text{or in another notation } \frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}$$

This PDE relates the time rate of change of a quantity u to its second spatial derivative. This is the 1-d heat equation which we will derive today.

Just as with ODE's we need some initial conditions to be able to find a unique solution. For a PDE rather than just an initial value we have an initial function and usually some boundary conditions.

ODE:

$$\frac{du}{dt} = au \quad \int \frac{du}{u} = \int a dt$$

Where $u = u(t)$

Initial Condition

$$u_0 = u(0) = 2$$

$$u(t) = ce^{at} \quad u(0) = ce^0 = 2 \quad c = 2$$

PDE

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}$$

Where $u = u(x, t)$

Initial Condition

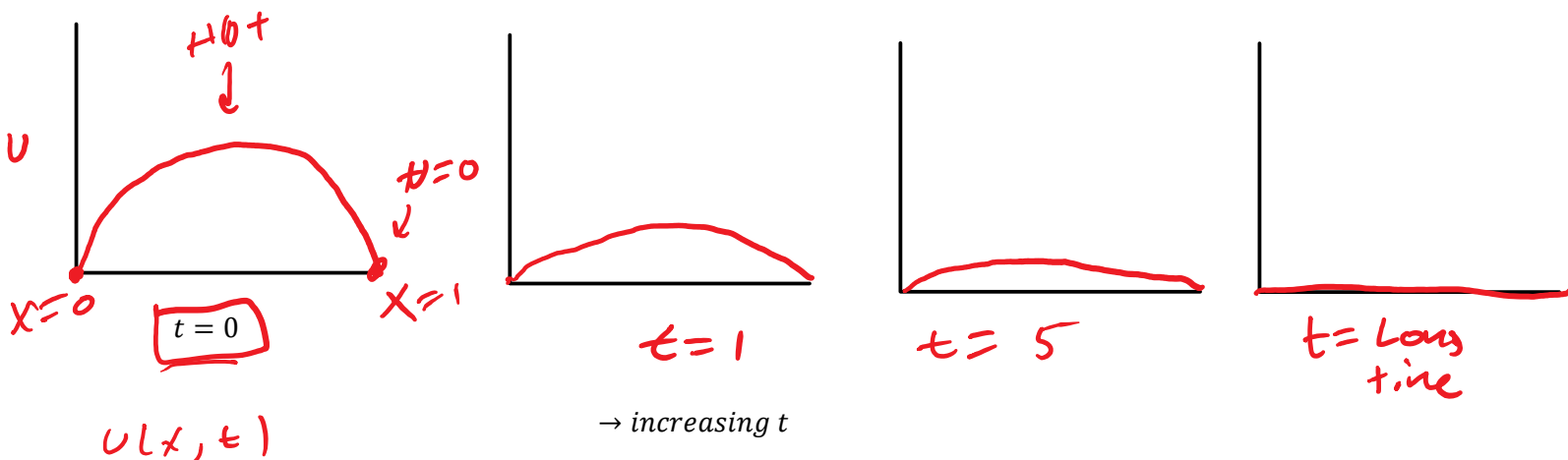
$$u(x, 0) = f(x) = -x^2 + x$$

Boundary Conditions

$$u(0, t) = 0 \quad u(1, t) = 0$$

$$x=0 \quad x=1$$

What does the solution to this 1-d heat equation look like?



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Section 12.2

Derivation of the 1-d Heat equation:

For this derivation we employ physical principles and arguments to give you a sense of what the heat equation actually says:

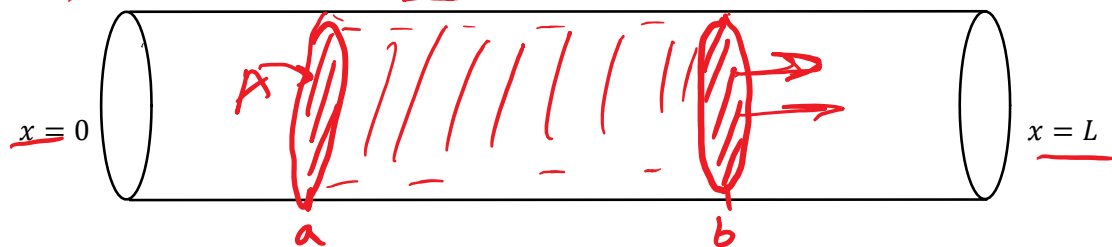
Thermal Energy Density

Let's define the thermal energy density as the amount of thermal energy e in joules per unit volume m^3

Let $e(x, t)$ = thermal energy density.

$$\frac{J}{m^3}$$

Let's first think of a metal rod and derive an equation for heat flow with in the rod. Let's consider the total energy in a section of the rod, say from $x = a$ to $x = b$



We will assume that the rod is perfectly insulated except at the ends so that heat can only flow in 1 direction in this case we will call it the x direction.

Since we have the energy density we could calculate the total energy in our section from a to b..

$$\text{Total Energy in our section: } E = A \int_a^b e(x, t) dx \leftarrow m$$

$$m^2 \cdot \frac{J}{m^3} = J$$

We also know from physics that energy is a conserved quantity, this means that the only way the amount of energy can change in our section is if it comes in our out of the surfaces at the ends of our section. That is $\frac{dE}{dt} = \text{net flux}$

Let's define the heat flux field

$\phi(x, t)$ = amount thermal energy flowing to the right per unit time per unit surface area

Let's relate the two quantities. Recall that if we integrate the flux field over the surface of the section of our rod this tells us how much of a quantity is leaving through the surface. In our case heat can only leave through the ends. We then obtain

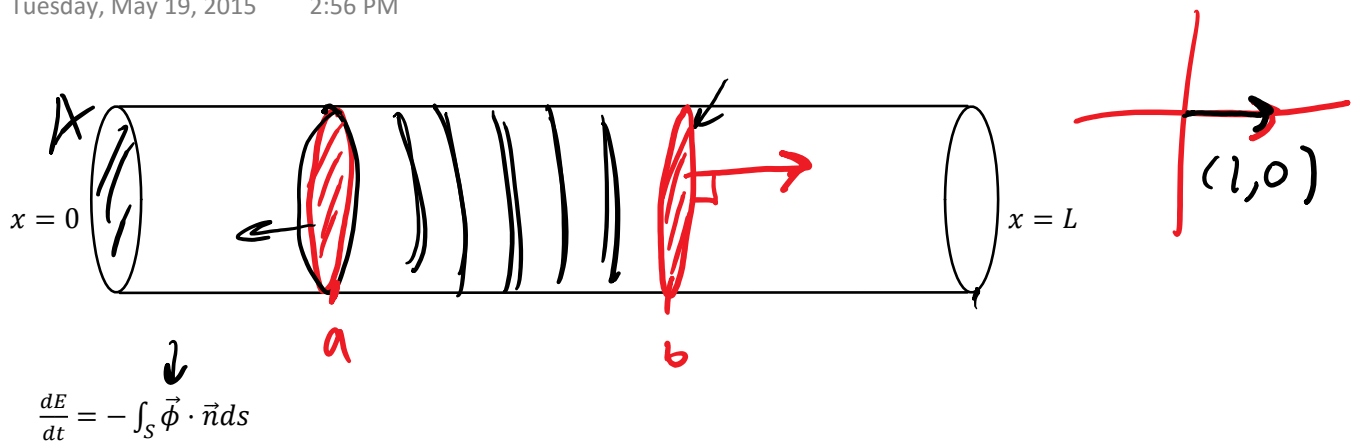
$$\frac{dE}{dt} = \int_S \vec{\phi} \cdot \vec{n} ds = \vec{\phi}(b, t) A - \vec{\phi}(a, t) A$$

where A is the surface area of the ends of our section.

Let's derive the last part of that equality,

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$$\frac{dE}{dt} = - \int_S \vec{\phi} \cdot \vec{n} ds$$

$$= - \int_S \phi(x,t) \cdot (1,0) ds = - \int_S \phi ds$$

$$= - \int_a \phi(a,t) ds + - \int_b \phi(b,t) ds$$

$$= - \phi(a,t) \underbrace{\int_a ds}_A + (-\phi(b,t)) \underbrace{\int_b ds}_A = -A\phi(a,t) - A\phi(b,t)$$

$$= A(\phi(a,t)) - A(\phi(b,t))$$

Now that we have this we can employ the fundamental theorem of calculus which yields

$$-A(\phi(b,t) - \phi(a,t)) = A \int_a^b \frac{\partial \phi}{\partial x} dx$$

We now have the statement that

$$\boxed{\frac{dE}{dt} = -A \int_a^b \frac{\partial \phi}{\partial x} dx}$$

$$f = \frac{dF}{dx}$$

$$\int_a^b f(x) dx = F(b) - F(a)$$

This is nice and all but we have no idea what the flux actually is, we need an equation we can actually solve, we are headed for a PDE.

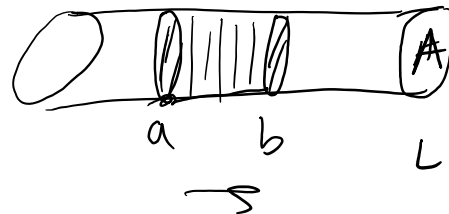
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Recall that we defined $e(x, t) = \text{thermal energy density}$. The thermal energy per unit volume

With this $E = A \int_a^b e(x, t) dx$

$m^2 \quad J/m^3 \quad m = J$



Let's Examine what this says about $\frac{dE}{dt}$

$$\frac{dE}{dt} = \frac{d}{dt} A \int_a^b e(x, t) dx$$

$$\frac{dE}{dt} = A \int_a^b \frac{de}{dt} dx$$

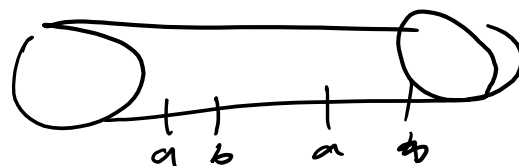
So we now have that:

$$A \int_a^b \frac{\partial e}{\partial t} dx = -A \int_a^b \frac{\partial \phi}{\partial x} dx$$

$$\int_a^b \frac{de}{dt} dx = - \int_a^b \frac{d\phi}{dx} dx$$

$$\int_a^b \frac{de}{dt} + \frac{d\phi}{dx} dx = 0$$

$$= \int_a^b \frac{de}{dt} + \frac{d\phi}{dx} dx = 0$$



$$\frac{de}{dt} + \frac{d\phi}{dx} = 0$$

$$\frac{de}{dt} = - \frac{d\phi}{dx}$$

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So now we have a PDE namely that: $\frac{\partial e}{\partial t} = \frac{\partial \phi}{\partial x}$

However, we are still not quite at something we can solve.. Here we have an equation relating the partials of two different functions what we want is an equation relating the partials of the *same* function

Temperature and specific heat:

Define: $u(x, t) = \text{temperature}$, $c = \text{specific heat}$, and $\rho(x) = \text{mass density}$

Specific heat is the amount of energy required to raise a unit of mass of a material one degree and has units of $\frac{\text{energy}}{\text{mass temperature}}$

Thus we can say: $\frac{\partial e}{\partial t} = c(x)\rho(x)\frac{\partial u}{\partial t} = \frac{\text{energy}}{\text{mass temp}} \times \frac{\text{mass}}{\text{vol}} \times \text{temp} = \frac{\text{energy}}{\text{vol}}$

$= c(x)\rho(x)\frac{\partial u}{\partial t}(x, t)$

So far we have:

$$\frac{\partial e}{\partial t} = c(x)\rho(x)\left(\frac{\partial u}{\partial t}\right) = -\frac{\partial \phi}{\partial x}$$

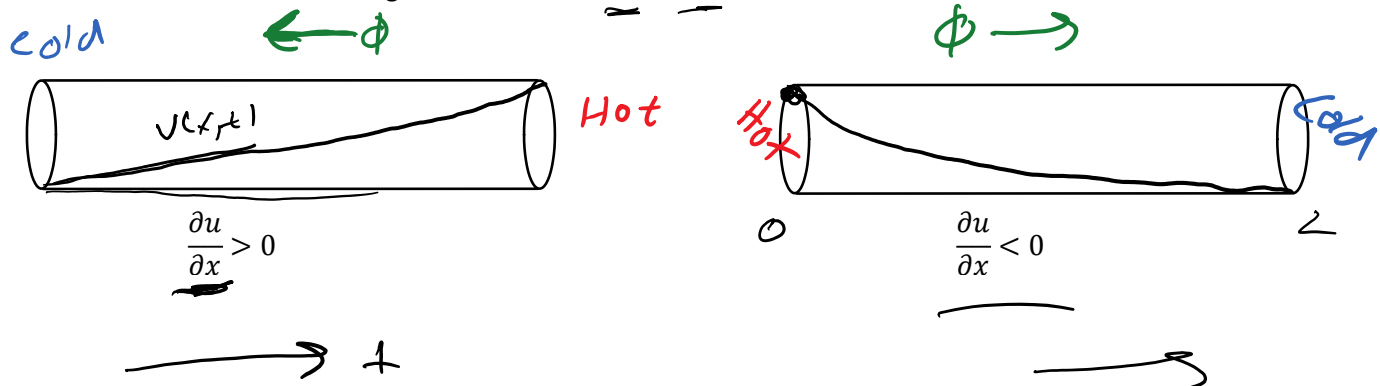
If we can write ϕ in terms of the temperature then we will finally have a PDE in terms of one function's partials.

Fourier's Law: Heat flows from hot to cold, and the greater the difference the faster the flow

$$\phi = -K_0 \frac{\partial u}{\partial x}$$

Recall that a gradient points in the direction of greatest change
This is the 1-d gradient of temperature.

What's with the minus sign? This ensures hot to cold flow.



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Now we have:

$$\frac{\partial e}{\partial t} = -\frac{\partial \phi}{\partial x} \quad \frac{\partial \phi}{\partial x} = \frac{\partial}{\partial x} - K_0 \frac{\partial u}{\partial x}$$

$$c(x)\rho(x) \frac{\partial u}{\partial t} = -\frac{\partial \phi}{\partial x} \quad \text{and} \quad \frac{\partial \phi}{\partial x} = -K_0 \frac{\partial u}{\partial x}$$

Combining these we obtain the heat equation in 1-d!

$$c(x)\rho(x) \frac{\partial u}{\partial t} = \frac{\partial}{\partial x} \left(-K_0 \frac{\partial u}{\partial x} \right) \quad \text{and if } K_0 \text{ happens to be constant in } x$$

$$K_0 \frac{\partial}{\partial x} \frac{\partial u}{\partial x} = K_0 \frac{\partial^2 u}{\partial x^2}$$

$$c(x)\rho(x) \frac{\partial u}{\partial t} = K_0 \frac{\partial^2 u}{\partial x^2}$$

Often we will deal with the case where both $c(x)$ and $\rho(x)$ also constant allowing us to divide both sides by them and obtain a new constant k

$$\frac{\partial u}{\partial t} = \left(\frac{K_0}{c\rho} \right) \frac{\partial^2 u}{\partial x^2}$$

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}$$

We have derived the heat equation for the case where there were no internal sources or sinks. If there is an internal source of heat or heat loss in the rod we can simply add it to the left hand side of the equation.

$$c(x)\rho(x) \frac{\partial u}{\partial t} = K_0 \frac{\partial^2 u}{\partial x^2} + Q$$

