

Sec 14.3

You might recall that for certain boundary conditions our Fourier series only consisted of sines or only cosines.

For homogeneous Dirichlet boundary conditions we had:

$$f(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}x\right)$$

For insulated boundary conditions we had:

$$f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi}{L}x\right)$$

So we might ask the question, when does a Fourier series only consist of sines or only cosines?

Well $\sin(x)$ is odd and $\cos(x)$ is even and you might recall that for functions:

$$\underline{\text{Odd}} + \underline{\text{Odd}} = \underline{\text{Odd}}$$

$$\underline{\text{Even}} + \underline{\text{Even}} = \underline{\text{Even}}$$

$$\underline{\text{Odd}} + \underline{\text{Even}} = \underline{\text{Neither}}$$

Because of third equation it must be that :

If $f(x) = \text{even}$ it's Fourier Series can only be made up of constant and cosines

If $f(x) = \text{odd}$ it must be that it's Fourier series is made up of only sines

It is still possible however to find a Fourier sine series for an even function provided we can find an odd extension of that function.

Likewise it is possible to find a Fourier cosine series for an odd function provided we can find an even extension of that function.

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odd: $f(-x) = -f(x)$

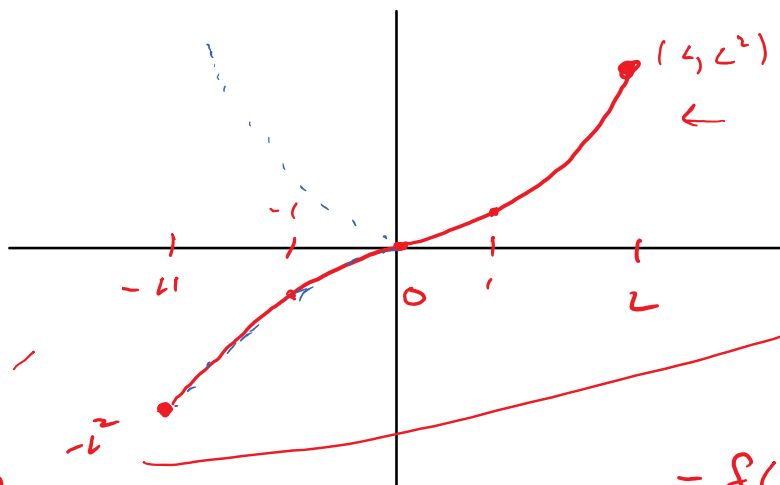
An odd extension of an even function.

reflections: $f(x) \rightarrow f(-x)$ reflection y-axis
 $f(x) \rightarrow -f(x)$ reflection x-axis

Find odd extensions for the following functions on the interval $[0, L]$ extended to the interval $[-L, L]$

$f(x) = x^2$

$f(L) = L^2$
 $f(-L) = -L^2$



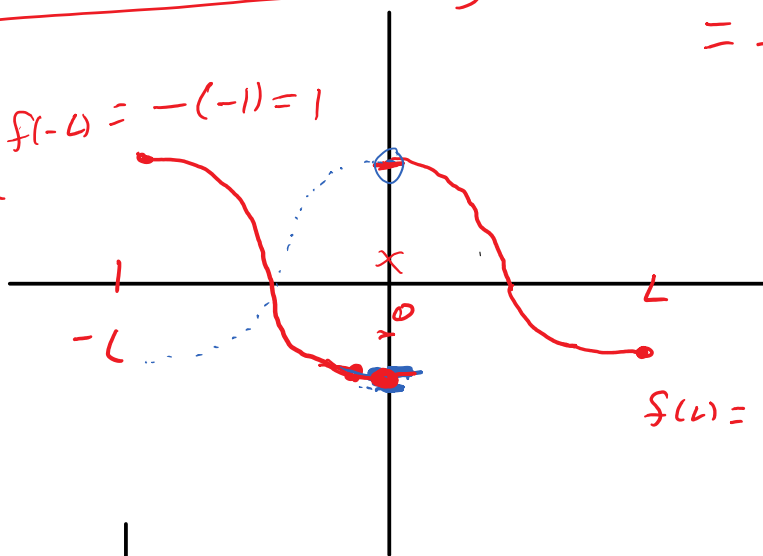
$-f(-x)$
 $= -(-x)^2$
 $= -x^2$

$f(x) = \begin{cases} x^2 & x > 0 \\ -x^2 & x < 0 \end{cases}$
 odd extension

$f(x) = \cos(x)$

$f(x) = \begin{cases} \cos(x) & 0 < x < L \\ -\cos(x) & -L < x < 0 \end{cases}$

$f(-L) = -(-1) = 1$

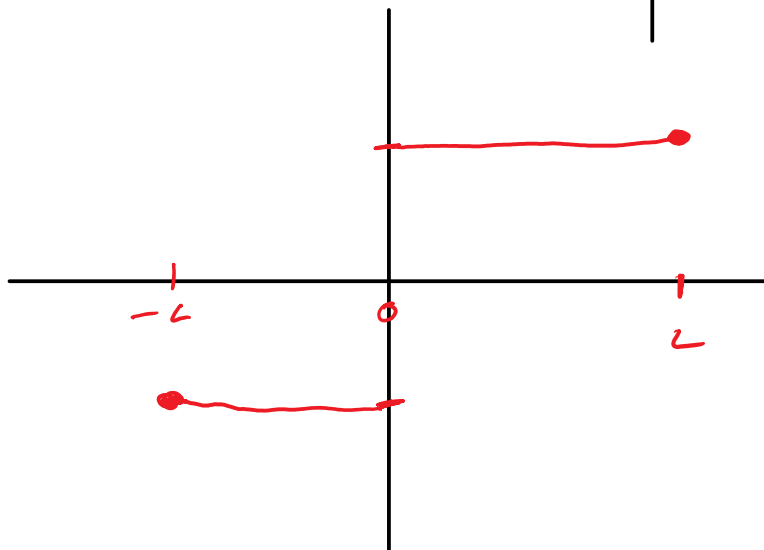


$-f(-x) = -\cos(-x)$
 $= -\cos(x)$

$f(L) = -1$

$f(x) = 1$

$f(x) = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \end{cases}$



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even: $f(-x) = f(x)$

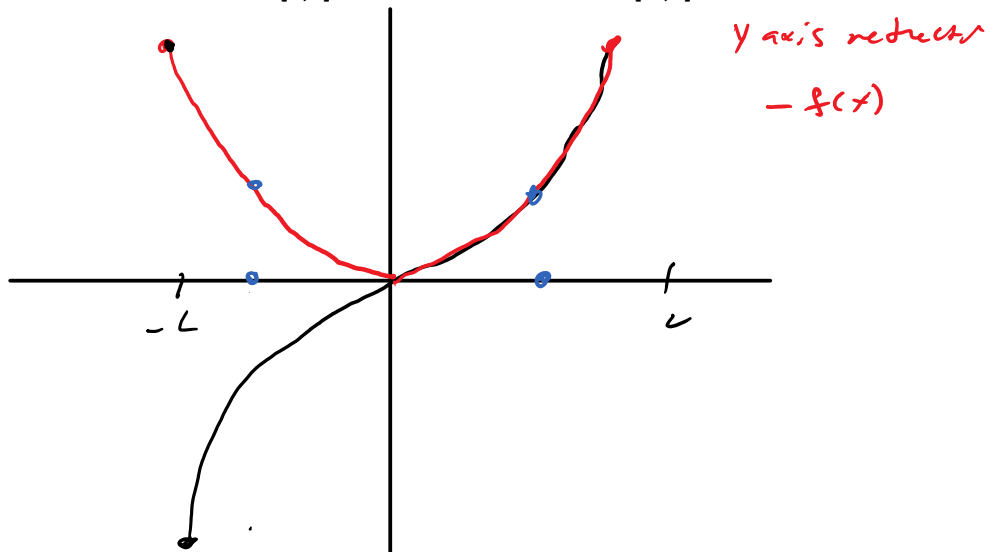
An even extension of an odd function.

$$f(L) = L^3 \quad f(-L) = L^3$$

Find even extensions for the following functions on the interval $[0, L]$ extended to the interval $[-L, L]$

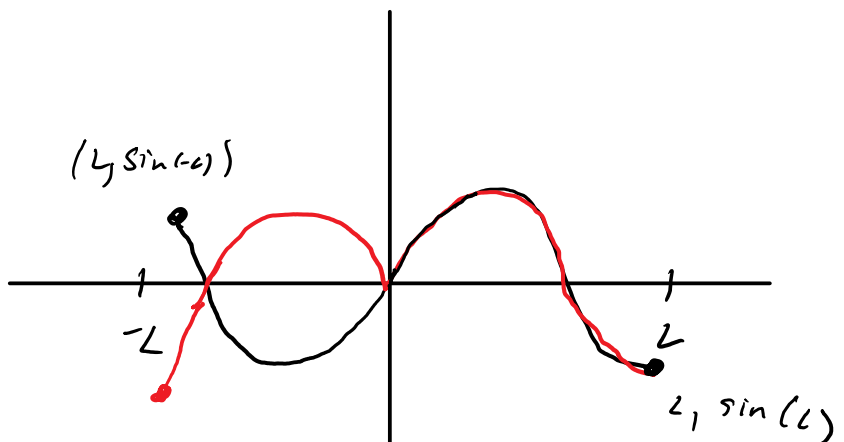
$f(x) = x^3$

$$f(x) = \begin{cases} x^3 & 0 \leq x \leq L \\ -x^3 & -L \leq x < 0 \end{cases}$$



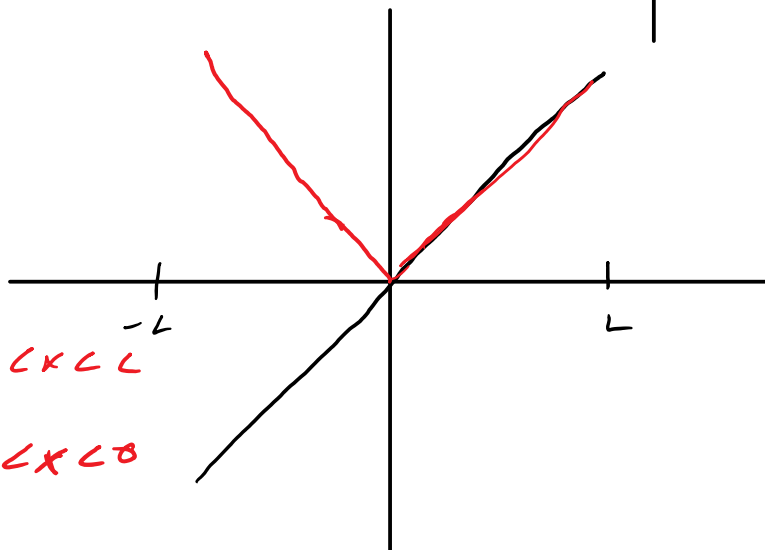
$f(x) = \sin(x)$

$$f(x) = \begin{cases} \sin(x) & 0 \leq x \leq L \\ -\sin(x) & -L \leq x < 0 \end{cases}$$



$f(x) = x$

$$f(x) = \begin{cases} x & 0 \leq x \leq L \\ -x & -L \leq x < 0 \end{cases}$$

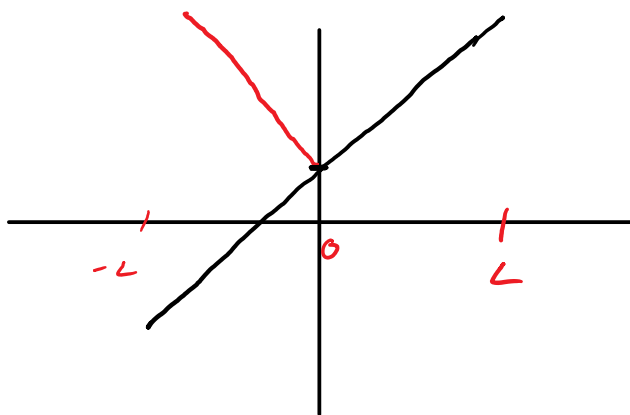


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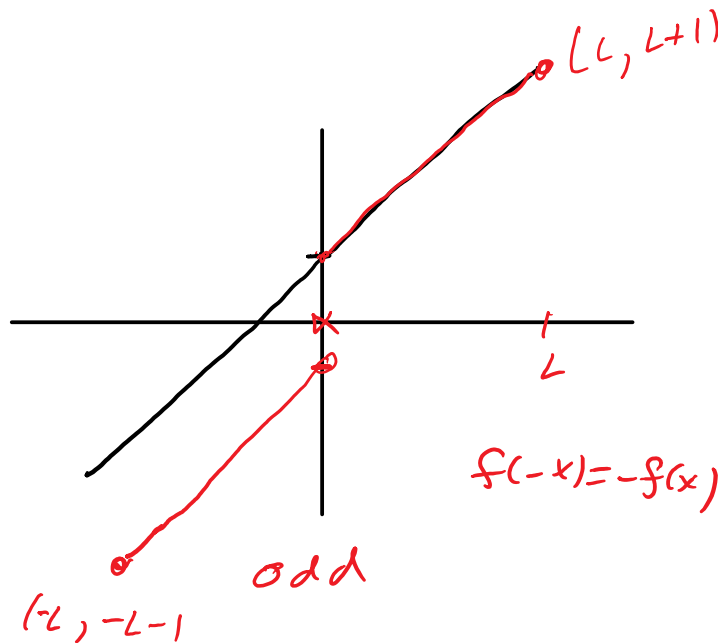
Even and Odd extensions of functions which are neither.

Find even and odd extensions of the following functions defined on $[0, L]$ extended to $[-L, L]$

$$f(x) = x + 1$$

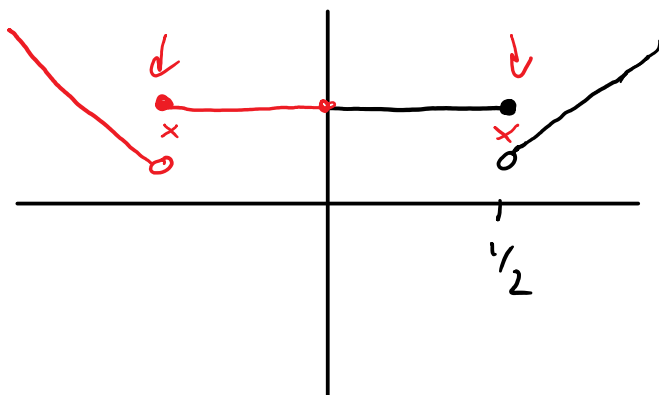


even

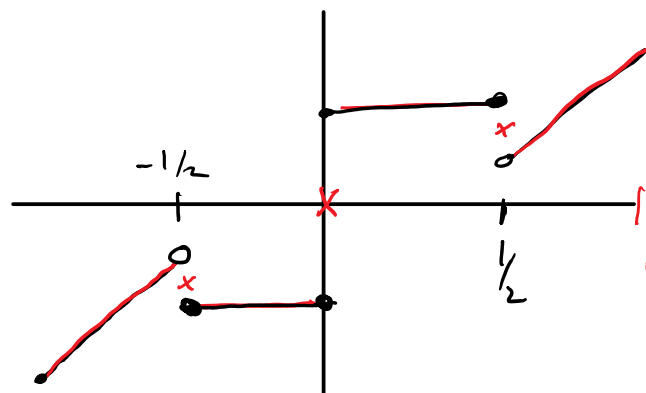


odd

$$f(x) = \begin{cases} 1 & \text{for } 0 < x < \frac{1}{2} \\ x & \text{for } \frac{1}{2} < x < 1 \end{cases}$$



even



odd

Incidentally the Fourier series for these extensions will look very similar.

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For the 1-d rod with homogeneous Dirichlet boundary conditions we found that:

$$u(x, t) = \sum_{n=1}^{\infty} B_n e^{-k\left(\frac{n\pi}{L}\right)^2 t} \sin\left(\frac{n\pi}{L} x\right)$$

Where

$$u(x, 0) = f(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L} x\right) \text{ for } 0 < x < L$$

and

$$B_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

It turns out we actually were representing $f(x)$ with a Fourier sine series of an odd extension!

If we had used the formulas for a Fourier series with an odd extension let's see what would happen.

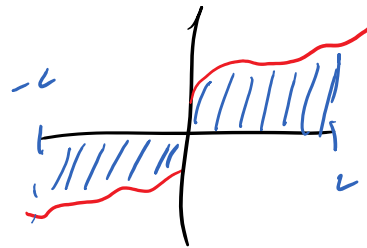
$$f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi}{L} x\right) + \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L} x\right)$$

$$A_0 = \frac{1}{2L} \int_{-L}^L f(x) dx \rightarrow \frac{1}{2L} \int_{-L}^L \text{odd} dx = 0$$

$$A_n = \frac{1}{L} \int_{-L}^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx \rightarrow \frac{1}{L} \int_{-L}^L \text{odd even} = 0$$

$$B_n = \frac{1}{L} \int_{-L}^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx \rightarrow \frac{1}{L} \int_{-L}^L \text{odd odd} \neq 0$$

even



odd even = odd

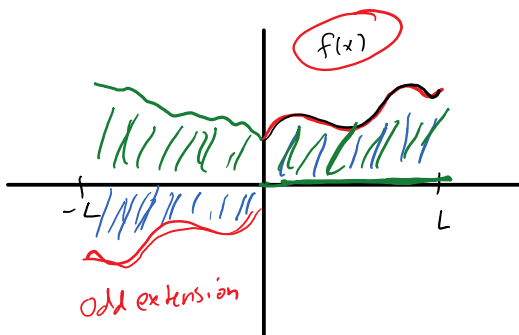
f odd g even

$$f(-x) = -f(x)$$

$$g(-x) = g(x)$$

$$f(x)g(x)$$

$$f(-x)g(-x) = -f(x)g(x)$$

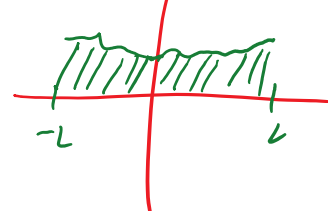


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$$B_n = \frac{1}{L} \int_{-L}^L f(x) \sin \frac{n\pi}{L} x$$

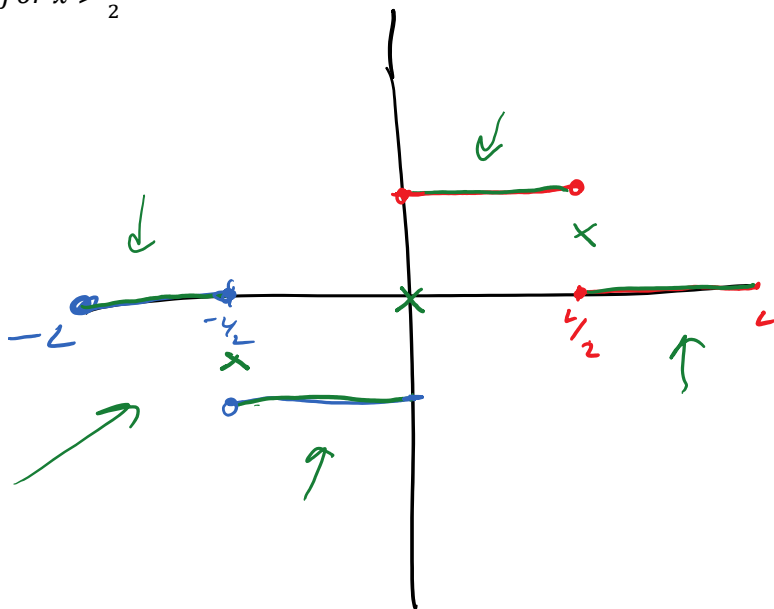
$\downarrow$ odd $\cdot$ $\downarrow$ odd = even

$\int_{-L}^L = 2 \int_0^L$



Hw Example: sec 14.3 #2(d)

For $f(x) \begin{cases} 1 & \text{for } x < \frac{L}{2} \\ 0 & \text{for } x > \frac{L}{2} \end{cases}$ sketch its fourier sine series and find the fourier sine series coefficients.



$$B_n = \frac{2}{L} \int_0^L f(x) \sin \left(\frac{n\pi}{L} x \right)$$

$$f(x) = \begin{cases} 1 & x < L/2 \\ 0 & x > L/2 \end{cases}$$

$$= \frac{2}{L} \int_0^{L/2} \sin \left(\frac{n\pi}{L} x \right) + \frac{2}{L} \int_{L/2}^L 0 \sin \left(\frac{n\pi}{L} x \right) dx$$

$$= \frac{2}{L} \int_0^{L/2} \sin \left(\frac{n\pi}{L} x \right) = \frac{2}{L} \left(-\frac{L}{n\pi} \right) \cos \left(\frac{n\pi}{L} x \right) \Big|_0^{L/2}$$

$$= \frac{2}{L} \left(-\frac{L}{n\pi} \right) \left[\cos \left(\frac{n\pi}{L} \frac{L}{2} \right) - \cos(0) \right]$$

$$B_n = \frac{-2}{n\pi} \left[\cos \left(\frac{n\pi}{2} \right) - 1 \right] \Rightarrow f(x) = \sum_{n=1}^{\infty} \frac{-2}{n\pi} \left[\cos \left(\frac{n\pi}{2} \right) - 1 \right] \sin \left(\frac{n\pi}{L} x \right)$$

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For insulated boundary conditions we found that

$$u(x, t) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi}{L}x\right) e^{-\left(\frac{n\pi}{L}\right)^2 kt}$$

with

$$f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi}{L}x\right)$$

even extension

Where

$$A_0 = \frac{1}{L} \int_0^L f(x) dx$$

$$A_n = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx$$

It turns out we actually were representing $f(x)$ with a Fourier cosine series of an even extension!

If we had used the formulas for a fourier series with an even extension let's see what would happen.

$$f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi}{L}x\right) + \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}x\right)$$

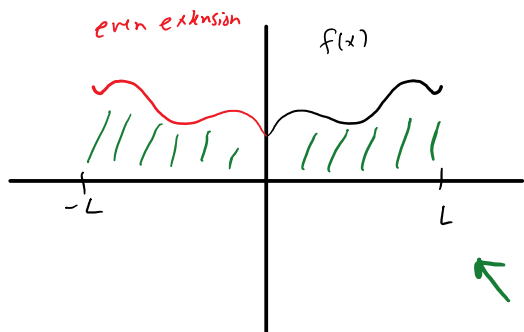
$$A_0 = \frac{1}{2L} \int_{-L}^L f(x) dx \rightarrow \frac{1}{2L} \int_{-L}^L f(x) dx = 2 \cdot \frac{1}{2L} \int_0^L f(x) dx = \frac{1}{L} \int_0^L f(x) dx$$

$$A_n = \frac{1}{L} \int_{-L}^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx \rightarrow 2 \cdot \frac{1}{L} \int_0^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx$$

even · even = even

$$B_n = \frac{1}{L} \int_{-L}^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx = \frac{1}{L} \int_{-L}^L \text{odd} dx = 0$$

even · odd = odd

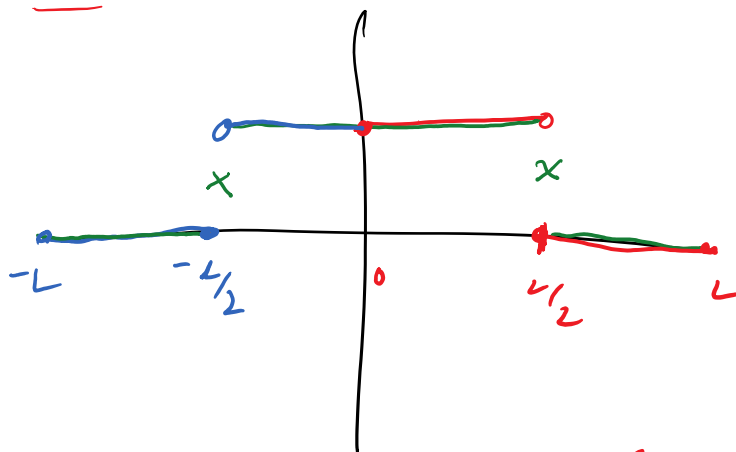


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Fourier Cosine series. If we have an even function or use an even extension we can instead use a cosine series.

Hw Example 14.3 5c

For $f(x) = \begin{cases} 1 & \text{for } x < \frac{L}{2} \\ 0 & \text{for } x > \frac{L}{2} \end{cases}$ sketch its fourier cosine series and find the fourier cosine series coefficients.



$$A_0 = \frac{1}{L} \int_0^L f(x) dx = \frac{1}{L} \int_0^{L/2} 1 dx + \frac{1}{L} \int_{L/2}^L 0 dx = \frac{1}{L} x \Big|_0^{L/2} = \frac{1}{L} \left(\frac{L}{2} - 0 \right) = \frac{1}{2}$$

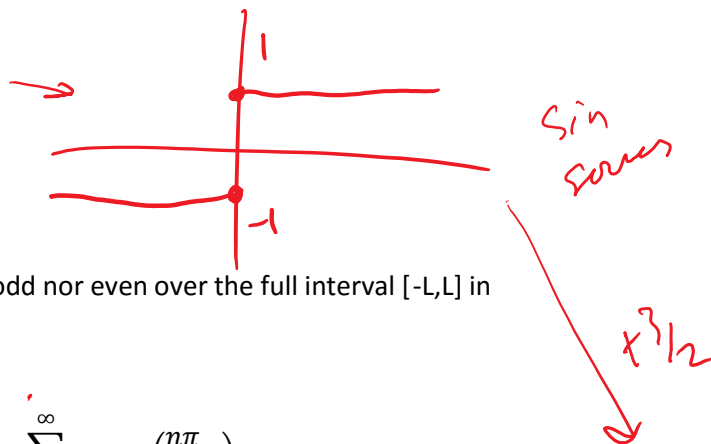
$$A_n = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx = \frac{2}{L} \int_0^{L/2} 1 \cos\left(\frac{n\pi x}{L}\right) dx + \frac{2}{L} \int_{L/2}^L 0 \cos\left(\frac{n\pi x}{L}\right) dx$$

$$= \frac{2}{L} \left[\frac{L}{n\pi} \sin\left(\frac{n\pi x}{L}\right) \right]_0^{L/2} = \frac{2}{n\pi} \left[\sin\left(\frac{n\pi}{2}\right) - \sin(0) \right]$$

$$= \frac{2}{n\pi} \left[\sin\left(\frac{n\pi}{2}\right) \right] = A_n$$

$$f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{n\pi} \sin\left(\frac{n\pi}{2}\right) \cos\left(\frac{n\pi x}{L}\right)$$

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We may be interested in a function which is neither odd nor even over the full interval $[-L, L]$ in this case we can simply use the full Fourier series.

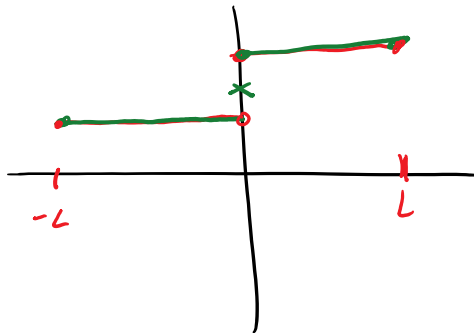
$$f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi}{L}x\right) + \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}x\right)$$

with

$$A_0 = \frac{1}{2L} \int_{-L}^L f(x) dx$$

$$A_n = \frac{1}{L} \int_{-L}^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx$$

$$B_n = \frac{1}{L} \int_{-L}^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$



Example: Find and sketch the Fourier series for $f(x) = \begin{cases} 1 & -L < x < 0 \\ 2 & 0 < x < L \end{cases}$

$$\frac{1}{2L} [0 - (-L)] + \frac{1}{2L} [2L - 0]$$

$$A_0 = \frac{1}{2L} \int_{-L}^L f(x) dx = \frac{1}{2L} \int_{-L}^0 1 dx + \frac{1}{2L} \int_0^L 2 dx = \frac{1}{2L} (x|_{-L}^0) + \frac{1}{2L} (2x|_0^L)$$

$$= \frac{L}{2L} + 1 = \frac{1}{2} + 1 = \frac{3}{2}$$

$$A_n = \frac{1}{L} \int_{-L}^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx = \frac{1}{L} \int_{-L}^0 1 \cos\left(\frac{n\pi x}{L}\right) dx + \frac{1}{L} \int_0^L 2 \cos\left(\frac{n\pi x}{L}\right) dx$$

$$= \frac{1}{L} \left[\frac{L}{n\pi} \sin\left(\frac{n\pi x}{L}\right) \right]_{-L}^0 + \frac{1}{L} 2 \left[\frac{L}{n\pi} \sin\left(\frac{n\pi x}{L}\right) \right]_0^L$$

$$= \frac{1}{n\pi} [\sin(0) - \sin(-n\pi)] + \frac{2}{n\pi} [\sin(n\pi) - \sin(0)]$$

$$A_n = 0$$

$$B_n = \frac{1}{L} \int_{-L}^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx = \frac{1}{L} \int_{-L}^0 1 \sin\left(\frac{n\pi x}{L}\right) dx + \frac{1}{L} \int_0^L 2 \sin\left(\frac{n\pi x}{L}\right) dx$$

$$= \frac{1}{L} \left[-\frac{L}{n\pi} \cos\left(\frac{n\pi x}{L}\right) \right]_{-L}^0 + \frac{2}{L} \left[-\frac{L}{n\pi} \cos\left(\frac{n\pi x}{L}\right) \right]_0^L = \frac{-1}{n\pi} [\cos(0) - \cos(-n\pi)] - \frac{2}{n\pi} [\cos(n\pi) - \cos(0)]$$

$$= \frac{-1}{n\pi} [1 - \cos(n\pi)] - \frac{2}{n\pi} [\cos(n\pi) - 1] = \frac{-1}{n\pi} [1 - \cos(n\pi)] + \frac{2}{n\pi} [1 - \cos(n\pi)]$$

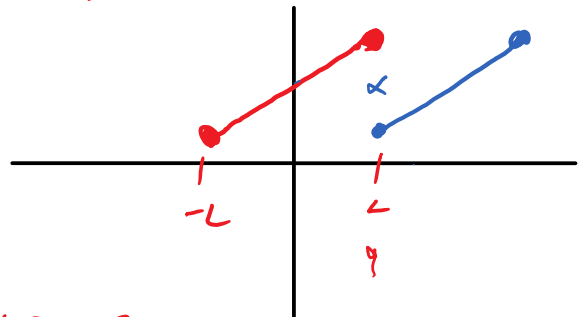
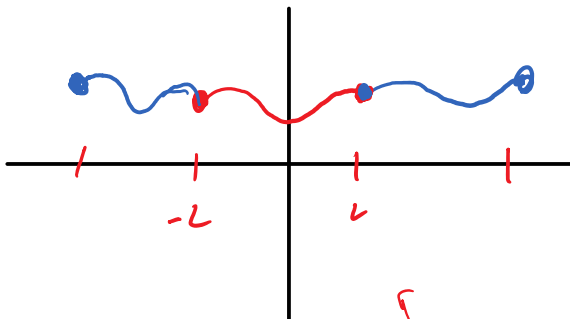
$$= \frac{1}{n\pi} [1 - \cos(n\pi)] = \frac{1}{n\pi} (1 - (-1)^n) = B_n$$

$$S(x) = \frac{3}{2} + \sum_{n=1}^{\infty} \frac{1}{n\pi} (1 - (-1)^n) \sin\left(\frac{n\pi}{L}x\right)$$

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When is a Fourier Series continuous?

For a piecewise smooth $f(x)$, the Fourier series of $f(x)$ is continuous and converges to $f(x)$ on $[-L, L]$ if and only if $f(x)$ is continuous on the interval and $f(L) = f(-L)$



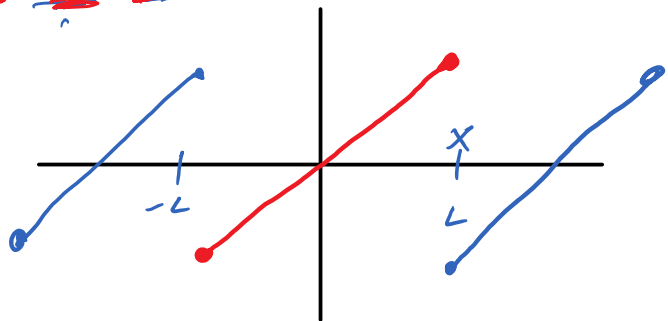
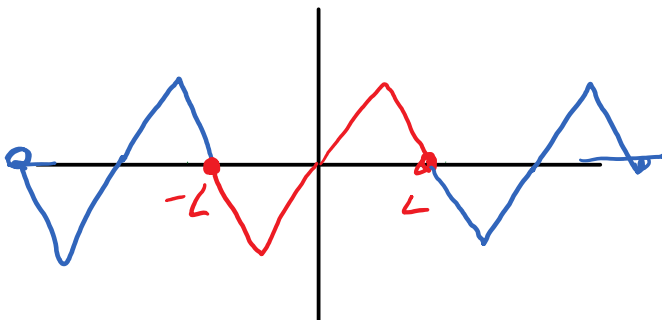
$$f(L) = f(-L)$$

$$-f(L) = f(L)$$

$$f(-L) = f(L)$$

What about sine series?

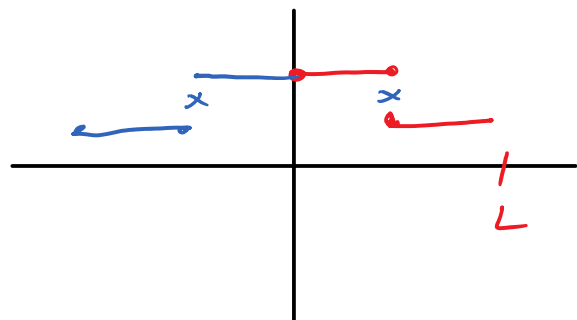
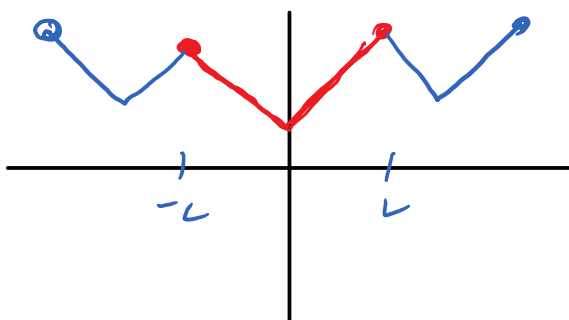
For a piecewise smooth $f(x)$, the Fourier sine series of $f(x)$ is continuous and converges to $f(x)$ on $[0, L]$ if and only if $f(x)$ is continuous on the interval and $f(0) = f(L) = 0$



What about cosine series?

For a piecewise smooth $f(x)$, the Fourier cosine series of $f(x)$ is continuous and converges to $f(x)$ on $[0, L]$ if and only if $f(x)$ is continuous on the interval.

$$f(L) = f(-L)$$



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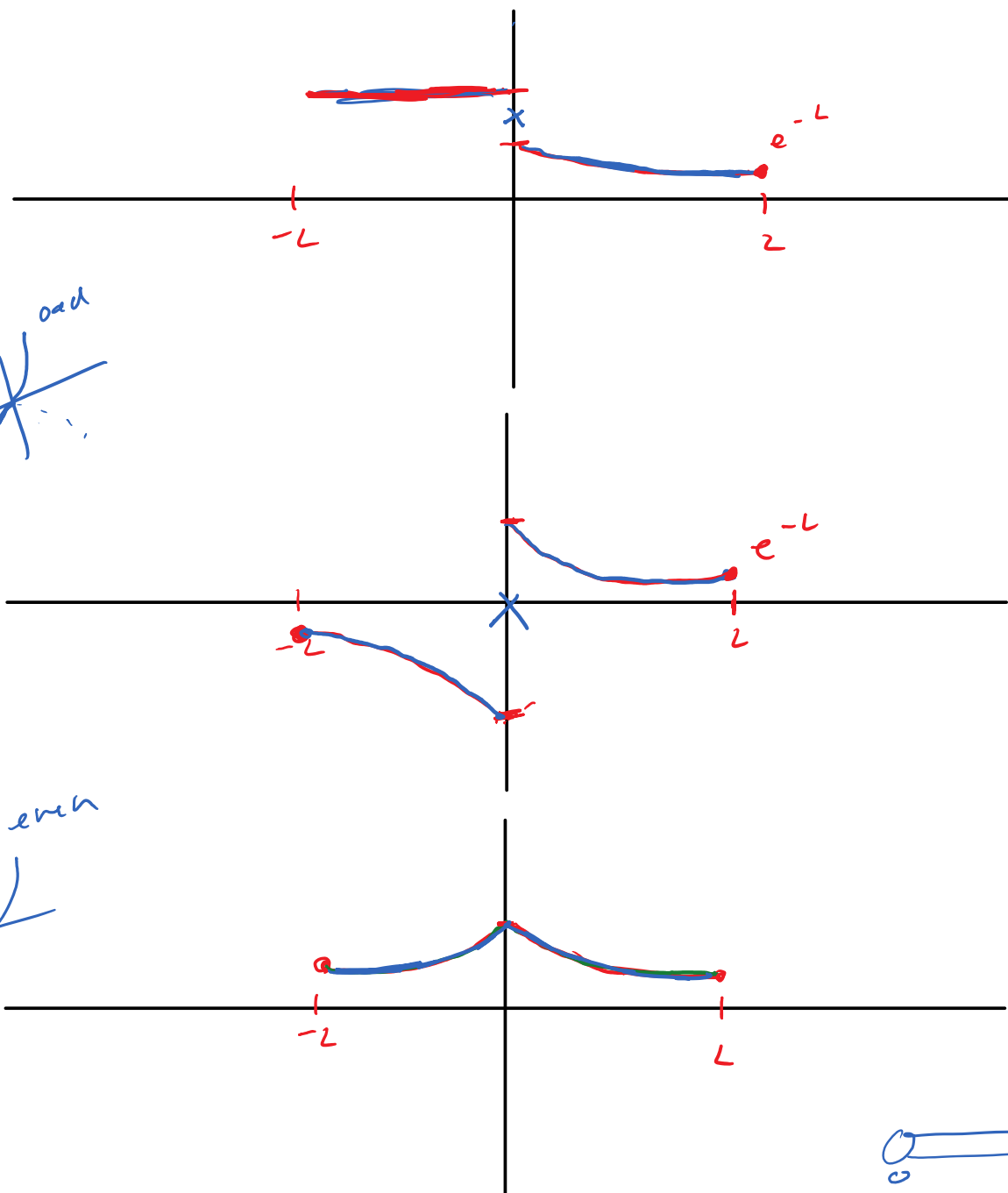
Comparison of full, sine and cosine Fourier series

Hw example #1 (e)

Sketch the full, sine and cosine series for $f(x) = \begin{cases} 2 & \text{for } x < 0 \\ e^{-x} & \text{for } x > 0 \end{cases}$



already odd



If already even



Something From 14.4(Method Of Eigen Function Expansion)

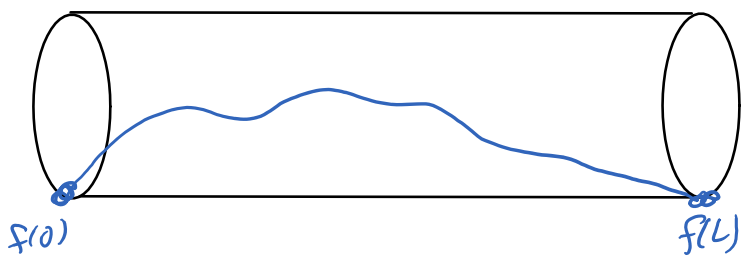
Another way to solve a PDE...

Suppose we wanted to solve our old friend the 1-d heat equation with homogeneous boundary conditions.

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}$$

$$u(0, t) = 0 \quad u(L, t) = 0$$

$$u(x, 0) = f(x)$$



When we look at what the boundary conditions say about the initial condition at the ends, $f(0) = 0, f(L) = 0$ we might think, hey that is a lot like the condition for the convergence of a sine series

For a piecewise smooth $f(x)$, the Fourier sine series of $f(x)$ is continuous and converges to $f(x)$ on $[0, L]$ if and only if $f(x)$ is continuous on the interval and $f(0) = f(L) = 0$

In this case we would realize that at time $t = 0$ the our solution to the PDE could be represented with a sine series, that is

$$u(x, 0) = f(x) = \sum_{n=1}^{\infty} b_n(0) \sin\left(\frac{n\pi}{L} x\right)$$

$$\text{Where } b_n(0) = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi}{L} x\right)$$

At a given time other than zero we might think that our solution would have the same form, just with different coefficients. That is to say, perhaps our solution has the form:

$$u(x, t) = \sum_{n=1}^{\infty} b_n(t) \sin\left(\frac{n\pi}{L} x\right)$$

Let's assume that is the case and see if we can't figure out what $b_n(t)$ needs to be in order for the $u(x, t)$ above to be the solution to the PDE satisfying the boundary conditions. What we will do is plug in our assumed form of the solution into the PDE and find out what the $b_n(t)$'s have to be for it to be a solution.

Something From 14.4 (Method Of Eigen Function Expansion)

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}$$

$$u(x, t) = \sum_{n=1}^{\infty} b_n(t) \sin\left(\frac{n\pi}{L} x\right)$$

$$\frac{\partial}{\partial t} \sum_{n=1}^{\infty} b_n(t) \sin\left(\frac{n\pi}{L} x\right) = k \frac{\partial^2}{\partial x^2} \sum_{n=1}^{\infty} b_n(t) \sin\left(\frac{n\pi}{L} x\right)$$

$$\sum_{n=1}^{\infty} \frac{db_n}{dt} \sin\left(\frac{n\pi}{L} x\right) = k \sum_{n=1}^{\infty} b_n(t) \frac{\partial^2}{\partial x^2} \sin\left(\frac{n\pi}{L} x\right)$$

$$\frac{\partial}{\partial x} \left(\frac{n\pi}{L} \cos\left(\frac{n\pi}{L} x\right) \right) = -\left(\frac{n\pi}{L}\right)^2 \sin\left(\frac{n\pi}{L} x\right)$$

$$\sum_{n=1}^{\infty} \frac{db_n}{dt} \sin\left(\frac{n\pi}{L} x\right) = k \sum_{n=1}^{\infty} b_n(t) \left(-\left(\frac{n\pi}{L}\right)^2\right) \sin\left(\frac{n\pi}{L} x\right)$$

$$\sum_{n=1}^{\infty} \frac{db_n}{dt} \sin\left(\frac{n\pi}{L} x\right) = \sum_{n=1}^{\infty} \left(-\left(\frac{n\pi}{L}\right)^2 k b_n(t)\right) \sin\left(\frac{n\pi}{L} x\right)$$

need to be same

$$\frac{db_n}{dt} = -\left(\frac{n\pi}{L}\right)^2 k b_n(t)$$

So we have a statement that

$$\sum_{n=1}^{\infty} \frac{db_n}{dt} \sin\left(\frac{n\pi}{L} x\right) = \sum_{n=1}^{\infty} -k \left(\frac{n\pi}{L}\right)^2 b_n(t) \sin\left(\frac{n\pi}{L} x\right)$$

Remembering that these $\sin\left(\frac{n\pi}{L} x\right)$ are all linearly independent we can match the coefficients for like values of n , this means that:

$$\frac{db_n}{dt} = -k \left(\frac{n\pi}{L}\right)^2 b_n(t)$$

Something From 14.4 (Method Of Eigen Function Expansion)

If we solve this ODE for b_n

$$\frac{db_n}{dt} = -k \left(\frac{n\pi}{L} \right)^2 b_n(t)$$

With the boundary condition that $b_n(0) = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi}{L} x\right)$

$$\frac{d}{dt} e^{kx} = k e^{kx}$$

$$\frac{db_n}{dt} = -k \left(\frac{n\pi}{L} \right)^2 b_n(t)$$

$$\int \frac{db_n}{b_n} = \int -k \left(\frac{n\pi}{L} \right)^2 dt$$

$$\ln(b_n) = -k \left(\frac{n\pi}{L} \right)^2 t + C$$

$$b_n = e^{-k \left(\frac{n\pi}{L} \right)^2 t + C} = e^C e^{-k \left(\frac{n\pi}{L} \right)^2 t} = C e^{-k \left(\frac{n\pi}{L} \right)^2 t}$$

$$b_n(0) = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi}{L} x$$

$$b_n(0) = C e^0 = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi}{L} x = C = b_n(0)$$

So $b_n(t) = b_n(0) e^{-k \left(\frac{n\pi}{L} \right)^2 t}$

With $b_n(0) = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi}{L} x\right)$

Putting this into our solution $u(x, t) = \sum_{n=1}^{\infty} b_n(t) \sin\left(\frac{n\pi}{L} x\right)$ we get finally that

$$u(x, t) = \sum_{n=1}^{\infty} b_n(0) e^{-k \left(\frac{n\pi}{L} \right)^2 t} \sin\left(\frac{n\pi}{L} x\right)$$

With $b_n(0) = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi}{L} x\right)$

The exact same solution we got from separation of variables!

Something From 14.4 (Method Of Eigen Function Expansion)

The method of Eigen function expansion is quite useful in the world of differential equations. It can be used to solve a large class of problems where separation of variables might fail. It is useful in situations where we have a source or sink in our 1-d rod, or when we don't have homogeneous boundary conditions.

However it is important to note that we can't just willy-nilly take time or spatial derivatives of Fourier series... Here is a summary of the rules for term by term differentiation of a Fourier series.

Full Fourier Series

If $f(x)$ is piecewise smooth, then the Fourier series of a continuous function $f(x)$ can be differentiated term by term if $f(-L) = f(L)$

For Cosine Series

If $f'(x)$ is piecewise smooth, then the Fourier cosine series of a continuous function $f(x)$ can be differentiated term by term.

For Sine Series

If $f'(x)$ is piecewise smooth then the Fourier sine series of a continuous function $f(x)$ can be differentiated term by term only if $f(0) = 0$ and $f(L) = 0$

$f(x)$
 $f'(x)$

Time Dependent Series

The Fourier series for a continuous function $u(x, t)$ which depends on t

$$u(x, t) = a_0(t) + \sum_{n=1}^{\infty} a_n(t) \cos\left(\frac{n\pi}{L}x\right) + \sum_{n=1}^{\infty} b_n(t) \sin\left(\frac{n\pi}{L}x\right)$$

Can be differentiated term by term with respect to t if $\frac{\partial u}{\partial t}$ is piecewise smooth. Where

$$u(x, t) \sim a'_0(t) + \sum_{n=1}^{\infty} a'_n(t) \cos\left(\frac{n\pi}{L}x\right) + \sum_{n=1}^{\infty} b'_n(t) \sin\left(\frac{n\pi}{L}x\right)$$

Something From 14.4 (Method Of Eigen Function Expansion)

Homework Example 14.4 #9

Consider the 1-d heat equation with a known source $q(x, t)$

$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} + q(x, t)$
with $u(0, t) = 0, u(L, t) = 0$



Assume that $q(x, t)$ is piecewise smooth function of x for each $t > 0$. Assume that term by term differentiation of the sine series is justified.

What ODE do you get for $b_n(t)$? Note that you won't be able to solve it since we did not specify q , the trick is to expand q in a Fourier series as well.

$$u(x, t) = \sum_{n=1}^{\infty} b_n(t) \sin\left(\frac{n\pi}{L} x\right)$$

$$\frac{\partial}{\partial t} \sum_{n=1}^{\infty} b_n(t) \sin\left(\frac{n\pi}{L} x\right) = k \frac{\partial^2}{\partial x^2} \sum_{n=1}^{\infty} b_n(t) \sin\left(\frac{n\pi}{L} x\right) + q(x, t)$$

$$\frac{\partial}{\partial t} \sum_{n=1}^{\infty} b_n(t) \sin\left(\frac{n\pi}{L} x\right) = k \frac{\partial^2}{\partial x^2} \sum_{n=1}^{\infty} b_n(t) \sin\left(\frac{n\pi}{L} x\right) + \sum_{h=1}^{\infty} q_h \sin\left(\frac{h\pi}{L} x\right)$$

where

$$q_n = \frac{2}{L} \int_0^L q(x, t) \sin\left(\frac{n\pi}{L} x\right) dx$$

$$\sum_{n=1}^{\infty} \frac{db_n}{dt} \sin\left(\frac{n\pi}{L} x\right) = \sum_{n=1}^{\infty} -k \left(\frac{n\pi}{L}\right)^2 b_n(t) \sin\left(\frac{n\pi}{L} x\right) + \sum_{n=1}^{\infty} q_n \sin\left(\frac{n\pi}{L} x\right)$$

$$\sum_{n=1}^{\infty} \frac{db_n}{dt} \sin\left(\frac{n\pi}{L} x\right) = \sum_{n=1}^{\infty} -k \left(\frac{n\pi}{L}\right)^2 b_n(t) \sin\left(\frac{n\pi}{L} x\right) + q_n \sin\left(\frac{n\pi}{L} x\right)$$

$$\sum_{n=1}^{\infty} \frac{db_n}{dt} \sin\left(\frac{n\pi}{L} x\right) = \sum_{n=1}^{\infty} \left(-k \left(\frac{n\pi}{L}\right)^2 b_n(t) + q_n \right) \sin\left(\frac{n\pi}{L} x\right)$$

need to be the same

$$\underline{db_n} = -k \left(\frac{n\pi}{L}\right)^2 b_n(t) + q_n$$

$$\frac{d}{dt} - \frac{2}{L} \sin(\frac{n\pi x}{L})$$

$$\frac{db_n}{dt} + K \left(\frac{n\pi}{L} \right)^2 b_n(t) = \frac{2}{L} \int_0^L q(x,t) \sin\left(\frac{n\pi x}{L}\right)$$