

Sec 14.1-14.2

Fourier Series or Basis function expansions using sines and cosines

Through our study of the heat equation we have seen that in order to solve the PDE we require the initial condition $f(x)$ to be able to be written as a sum of sines and cosines.

$$u(x, 0) = f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi}{L}x\right) + \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}x\right)$$

We require that this be possible as our solution $u(x, t)$ was nontrivial only if it took the form:

$$u(x, t) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi}{L}x\right) e^{-\left(\frac{n\pi}{L}\right)^2 kt} + \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}x\right) e^{-\left(\frac{n\pi}{L}\right)^2 kt}$$

A natural question to ask is, what types of functions $f(x)$ can we represent with a sum of a constant and sines and cosines. This leads us to study Fourier series.

It turns out that the set of functions $\{1, \sin\left(\frac{\pi x}{L}\right), \sin\left(\frac{2\pi x}{L}\right), \dots, \cos\left(\frac{\pi x}{L}\right), \cos\left(\frac{2\pi x}{L}\right), \dots\}$ form an orthogonal basis for smooth functions on the interval $[-L, L]$ with respect to the norm

f continuous
 $\frac{df}{dx}$ is continuous

$$\langle f, g \rangle = \int_{-L}^L f(x)g(x)dx$$

This means that if $f(x)$ is a smooth function on $[-L, L]$ it can be represented with a fourier series.

$$f(x) = \frac{\langle 1, f \rangle}{\langle 1, 1 \rangle} 1 + \frac{\langle \sin\left(\frac{\pi x}{L}\right), f \rangle}{\langle \sin\left(\frac{\pi x}{L}\right), \sin\left(\frac{\pi x}{L}\right) \rangle} \sin\left(\frac{\pi x}{L}\right) + \frac{\langle \sin\left(\frac{2\pi x}{L}\right), f \rangle}{\langle \sin\left(\frac{2\pi x}{L}\right), \sin\left(\frac{2\pi x}{L}\right) \rangle} \sin\left(\frac{2\pi x}{L}\right) + \dots + \frac{\langle \cos\left(\frac{\pi x}{L}\right), f \rangle}{\langle \cos\left(\frac{\pi x}{L}\right), \cos\left(\frac{\pi x}{L}\right) \rangle} \cos\left(\frac{\pi x}{L}\right) + \frac{\langle \cos\left(\frac{2\pi x}{L}\right), f \rangle}{\langle \cos\left(\frac{2\pi x}{L}\right), \cos\left(\frac{2\pi x}{L}\right) \rangle} \cos\left(\frac{2\pi x}{L}\right) + \dots$$

Or

$$f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi}{L}x\right) + \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}x\right)$$

Where

$$A_0 = \frac{\langle 1, f \rangle}{\langle 1, 1 \rangle} = \frac{1}{2L} \int_{-L}^L f(x) dx$$

$$A_n = \frac{\langle \cos\left(\frac{n\pi x}{L}\right), f \rangle}{\langle \cos\left(\frac{n\pi x}{L}\right), \cos\left(\frac{n\pi x}{L}\right) \rangle} = \frac{1}{L} \int_{-L}^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx$$

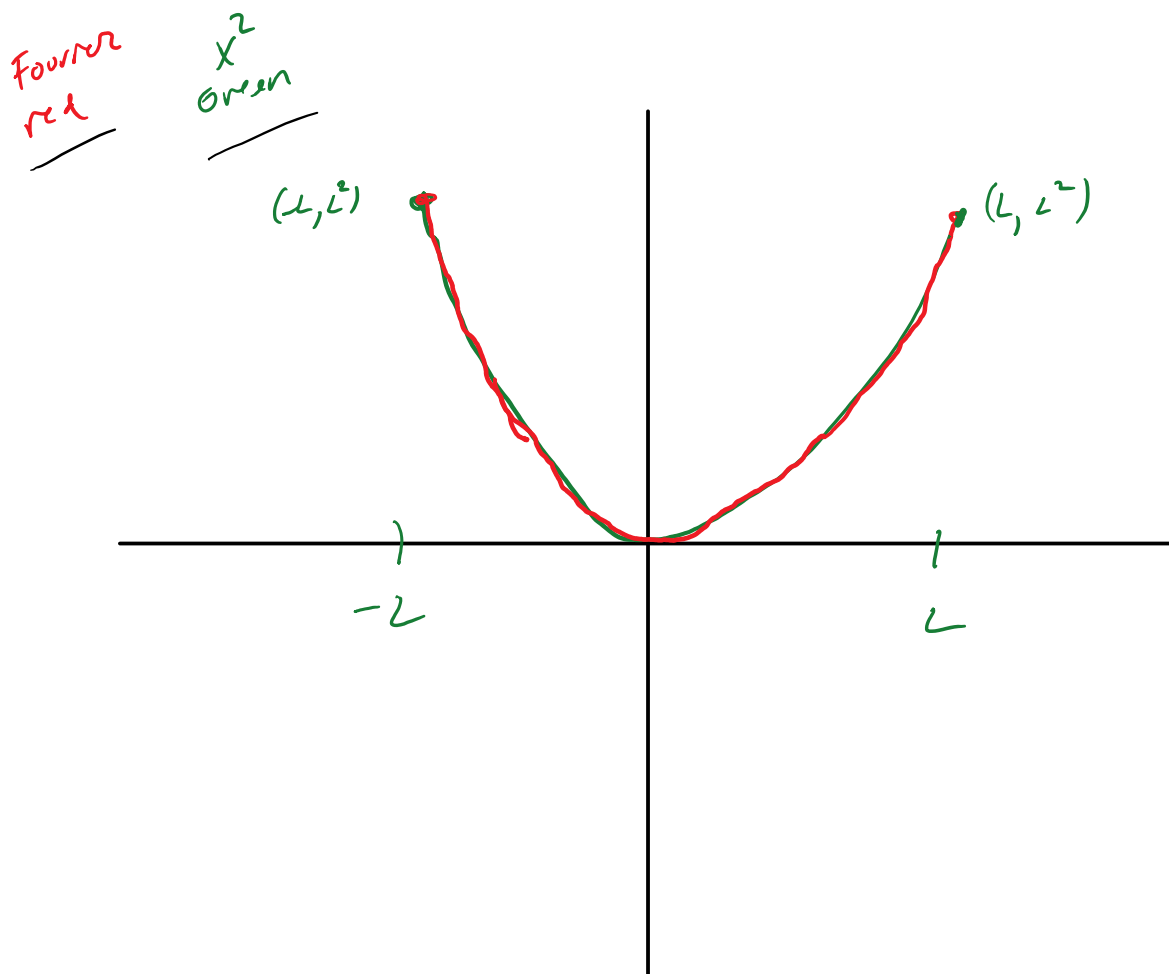
$$B_n = \frac{\langle \sin\left(\frac{n\pi x}{L}\right), f \rangle}{\langle \sin\left(\frac{n\pi x}{L}\right), \sin\left(\frac{n\pi x}{L}\right) \rangle} = \frac{1}{L} \int_{-L}^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

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Homework Example sec 14.2 #1(b)

Sketch the fourier series of $f(x) = x^2$ on the interval $[-L, L]$ and compare f to it's fourier series.

Solution: Since $f(x) = x^2$ is smooth everywhere, it certinaly is on $[-L, L]$ thus our basis expansion will exactly match f on $[-L, L]$.



Let's look at some of the finite approximations vs the actual function in maple.

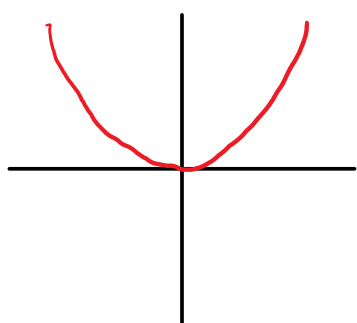
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If we recall from section 4.10 that we can use basis function expansions to represent, as closely as possible, a function which is not in the same space (at least with finite expansions). We might then think that even if $f(x)$ is not continuous we can use a Fourier series to get its best continuous approximation. This would be the case if we had a finite sum, as the finite sum of continuous functions is continuous. However a general Fourier series is an infinite sum and this presents difficulties.

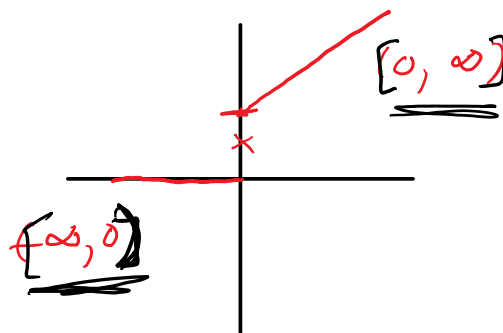
It does turn out though that if $f(x)$ is a piecewise smooth function the Fourier series does a very good job at approximating $f(x)$. Before we explain how, let's review the definition of a piecewise smooth function.

A function $f(x)$ is piecewise smooth on a given interval if that interval can be broken up into sub intervals on which $f(x)$ and its derivative $\frac{df}{dx}$ are continuous. Basically this means that we have no infinities in the function only jump discontinuities and no infinite slopes.

Examples of piecewise smooth functions



$$f(x) = x^2$$

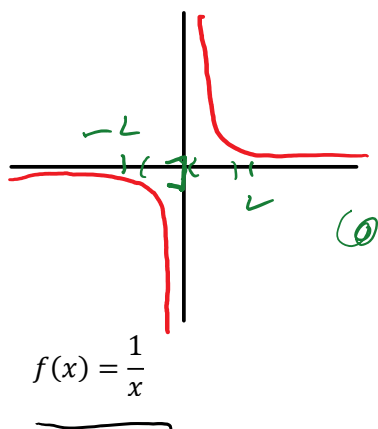


$$f(x) = \begin{cases} 0 & \text{for } x < 0 \\ 1 + x & \text{for } x > 0 \end{cases}$$

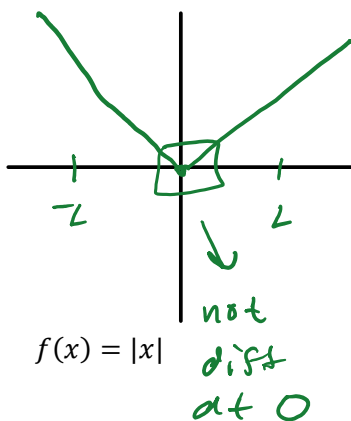


Random drawing example

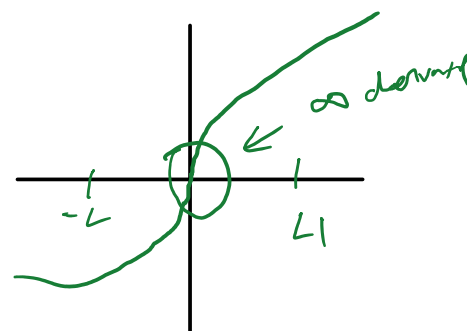
Examples of **not**-piecewise smooth functions



$$f(x) = \frac{1}{x}$$



$$f(x) = |x|$$



$$f(x) = x^{1/3}$$

$$f'(x) = \frac{1}{3} x^{-2/3} \rightarrow \frac{1}{2} \frac{1}{x^{2/3}}$$

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So what happens when with the Fourier series of $f(x)$ when it is only piecewise smooth? Let's state our Fourier Series convergence Theorem.

If $f(x)$ is piecewise smooth on the interval $[-L, L]$ then the Fourier series of $f(x)$ converges to:

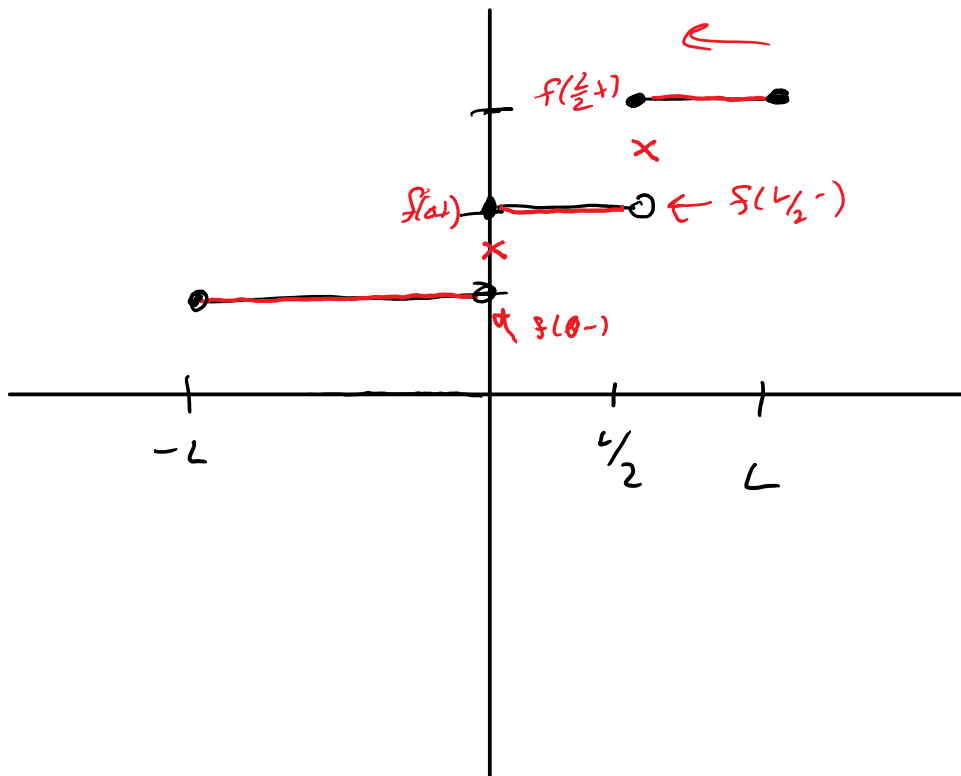
- 1) The periodic extension of $f(x)$, where the periodic extension is continuous.
- 2) The average of the two limits on either side of a jump discontinuity usually

$$\frac{1}{2}(f(x+) + f(x-))$$

What we are saying is that where ever $f(x)$ is fully smooth the Fourier series converges exactly and takes on a value half way between the two y values at a jump discontinuity. We will talk about the periodic extension in a bit.

Example:

Draw the Fourier series of $f(x) = \begin{cases} 1 & -L < x < 0 \\ 2 & 0 \leq x < \frac{L}{2} \\ 3 & \frac{L}{2} \leq x < L \end{cases}$



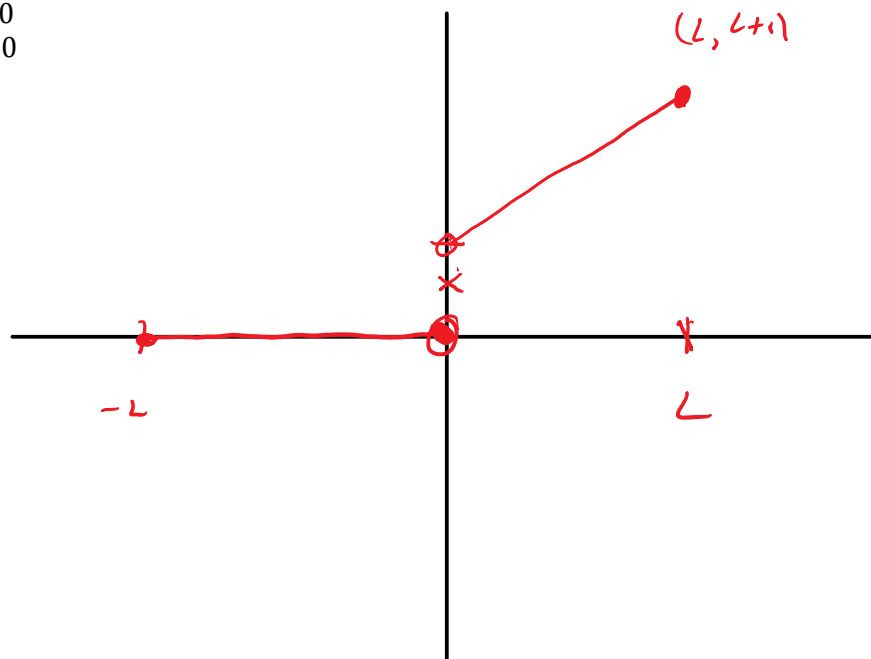
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Homework Example sec 14.2#1 f

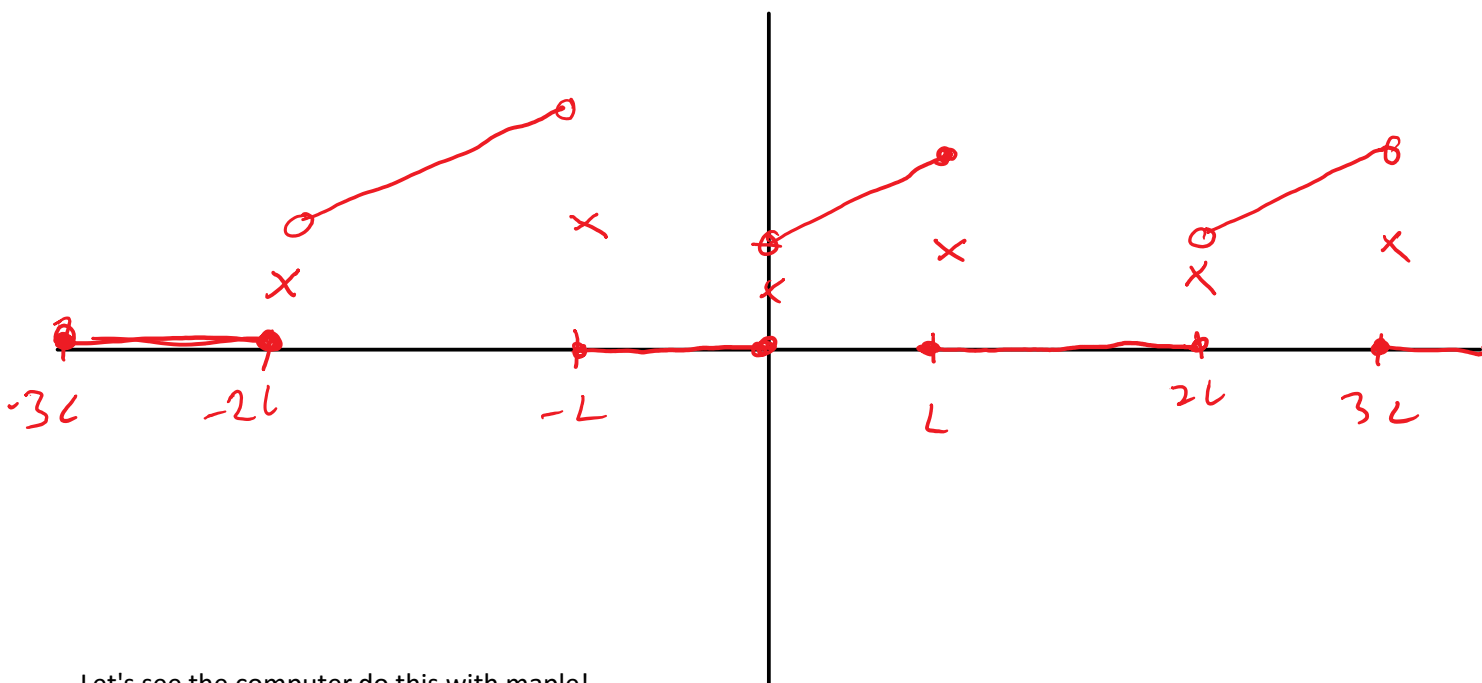
Sketch the Fourier series of $f(x)$ on $-L \leq x \leq L$ and compare it to its Fourier series for

$$f(x) = \begin{cases} 0 & x \leq 0 \\ 1+x & x > 0 \end{cases}$$

$$y = x + 1$$



What about that term "periodic extension" In the Theorem. The periodic extension of a function we simply repeat $f(x)$ on $[-L, L]$ with period $2L$.



Let's see the computer do this with maple!