

Fourier Transform (16.1-16.2)

Before We begin With Fourier Transforms Let's take a look at a homework problem from chapter 15.

Sec 15.4 #9

Derive conservation of energy for a vibrating string. $\frac{dE}{dt} = c^2 \frac{\partial u}{\partial x} \frac{\partial u}{\partial t} \Big|_0^L = c^2 \frac{\partial u}{\partial x}(L,t) \frac{\partial u}{\partial t}(L,t) - c^2 \frac{\partial u}{\partial x}(0,t) \frac{\partial u}{\partial t}(0,t)$

Using the fact that the total energy E is the sum of the kinetic energy $KE = \int_0^L \frac{1}{2} \left(\frac{\partial u}{\partial t} \right)^2 dx$ and the potential energy $PE = \int_0^L \frac{c^2}{2} \left(\frac{\partial u}{\partial x} \right)^2 dx$

To do this we will use both the wave equation $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ and integration by parts.

$$E = KE + PE = \int_0^L \frac{1}{2} \left(\frac{\partial u}{\partial t} \right)^2 dx + \int_0^L \frac{c^2}{2} \left(\frac{\partial u}{\partial x} \right)^2 dx$$

$$\frac{dE}{dt} = \frac{d}{dt} \int_0^L \frac{1}{2} \left(\frac{\partial u}{\partial t} \right)^2 dx + \frac{d}{dt} \int_0^L \frac{c^2}{2} \left(\frac{\partial u}{\partial x} \right)^2 dx$$

$$= \frac{1}{2} \int_0^L \frac{\partial}{\partial t} \left(\frac{\partial u}{\partial t} \right)^2 dx + \int_0^L \frac{c^2}{2} \frac{\partial}{\partial t} \left(\frac{\partial u}{\partial x} \right)^2 dx$$

$$= \frac{1}{2} \int_0^L 2 \frac{\partial u}{\partial t} \cdot \frac{\partial^2 u}{\partial t^2} dx + \int_0^L \frac{c^2}{2} 2 \frac{\partial u}{\partial x} \cdot \frac{\partial^2 u}{\partial x \partial t} dx$$

$$= \frac{1}{2} \int_0^L 2 \frac{\partial u}{\partial t} c^2 \frac{\partial^2 u}{\partial x^2} dx$$

$$\int_0^L u dv = \frac{u \cdot v}{1} \Big|_0^L - \int_0^L v du$$

$$u = \frac{\partial u}{\partial x} \quad dv = \frac{\partial^2 u}{\partial x^2} dx$$

$$du = \frac{\partial^2 u}{\partial x^2} dx \quad v = \frac{\partial u}{\partial t}$$

$$= \int_0^L c^2 \frac{\partial u}{\partial t} \frac{\partial^2 u}{\partial x^2} dx + c^2 \frac{\partial u}{\partial t} \frac{\partial u}{\partial x} \Big|_0^L - \int_0^L c^2 \frac{\partial u}{\partial t} \frac{\partial^2 u}{\partial x^2} dx = c^2 \frac{\partial u}{\partial t} \frac{\partial u}{\partial x} \Big|_0^L$$

$$\boxed{\frac{dE}{dt} = c^2 \frac{\partial u}{\partial t} \frac{\partial u}{\partial x} \Big|_0^L}$$

$$\frac{dE}{dt} = c^2 \frac{\partial v}{\partial t} \frac{\partial v}{\partial x} \Big|_0^L$$

Fourier Transform (16.1-16.2)

13.4 #10

What happens to the total energy E of the vibrating string if:

$$\frac{dE}{dt} = c^2 \frac{\partial u}{\partial t} \frac{\partial u}{\partial x} \Big|_0^L$$

$$u(0, t) = u(L, t) = 0$$



$$\frac{dE}{dt} = \left[c^2 \frac{\partial u}{\partial t}(L, t) \frac{\partial u}{\partial x}(L, t) - \frac{\partial u}{\partial t}(0, t) \frac{\partial u}{\partial x}(0, t) \right] = 0$$

E is constant

$$\frac{\partial u}{\partial x}(0, t) = 0 \text{ and } u(L, t) = 0 \quad \frac{\partial u}{\partial t}(L, t) = 0 \quad \frac{\partial u}{\partial x}(L, 0) = 0$$

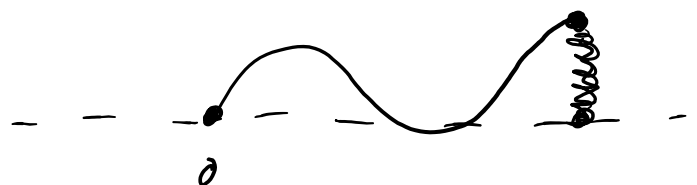
$$\frac{dE}{dt} = 0 - 0 = 0$$

constant energy

$$u(0, t) = 0 \text{ and } \frac{\partial u}{\partial x}(L, t) = -\gamma u(L, t) \quad \gamma > 0$$

$$\begin{aligned} \frac{dE}{dt} &= \left[c^2 \frac{\partial u}{\partial t}(L, t) \frac{\partial u}{\partial x}(L, t) - \frac{\partial u}{\partial t}(0, t) \frac{\partial u}{\partial x}(0, t) \right] = 0 \\ &= c^2 \frac{\partial u}{\partial t}(L, t) (-\gamma u(L, t)) = -\gamma c^2 \frac{\partial u}{\partial t}(L, t) u(L, t) \end{aligned}$$

What if $\gamma < 0$?



Fourier Transform (16.1-16.2)



We have spent a great deal of time studying the 1-d heat equation with various boundary conditions. However, in all cases we used a rod which had finite length. Being good math students we might ask, what about an infinite rod?



Let's Examine the heat equation defined on the interval $(-\infty, \infty)$

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}$$

Let's proceed in the usual way by separating variables. With our usual assumption $u(x, t) = h(t)\phi(x)$ we obtain our old friends:

$$\frac{dh}{dt} = -\lambda k h$$

$$\frac{d^2 \phi}{dx^2} = -\lambda \phi$$

$\lambda < 0$ $\lambda = 0$ $\lambda > 0$
 $\downarrow$
 $\phi = c_1 e^{\sqrt{\lambda} x} + c_2 e^{-\sqrt{\lambda} x}$
 $\lambda > 0$

With our usual solutions:

$$h(t) = c_1 e^{-\lambda k t}$$

$$\phi(x) = c_1 \cos(\sqrt{\lambda} x) + c_2 \sin(\sqrt{\lambda} x)$$

Normally at this point we would apply our boundary conditions to determine our separation constant λ alas we have no boundaries as this is an infinite interval. So instead we will make the requirement that:

$$|x+y| \leq |x| + |y|$$

$$|u(\pm\infty, t)| < \infty \leftarrow |\phi(\pm\infty)| < \infty$$

$$u(\infty) = 0 \quad u(-\infty) = 0$$

The question then becomes, for what values of λ do we satisfy $|\phi(\pm\infty)| < \infty$

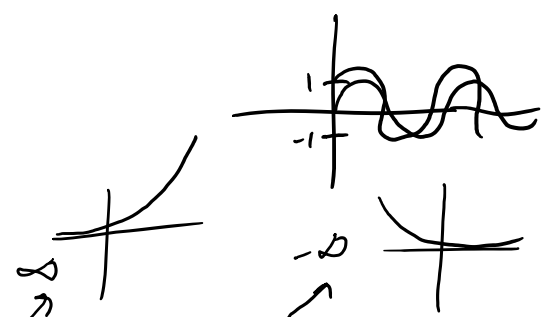
$$|\phi(x)| = |c_1 \cos \sqrt{\lambda} x + c_2 \sin \sqrt{\lambda} x| \leq |c_1 \cos \sqrt{\lambda} x| + |c_2 \sin \sqrt{\lambda} x|$$

$$\leq |c_1| |\cos(\sqrt{\lambda} x)| + |c_2| |\sin \sqrt{\lambda} x|$$

$$\leq |c_1| + |c_2| < \infty$$

or

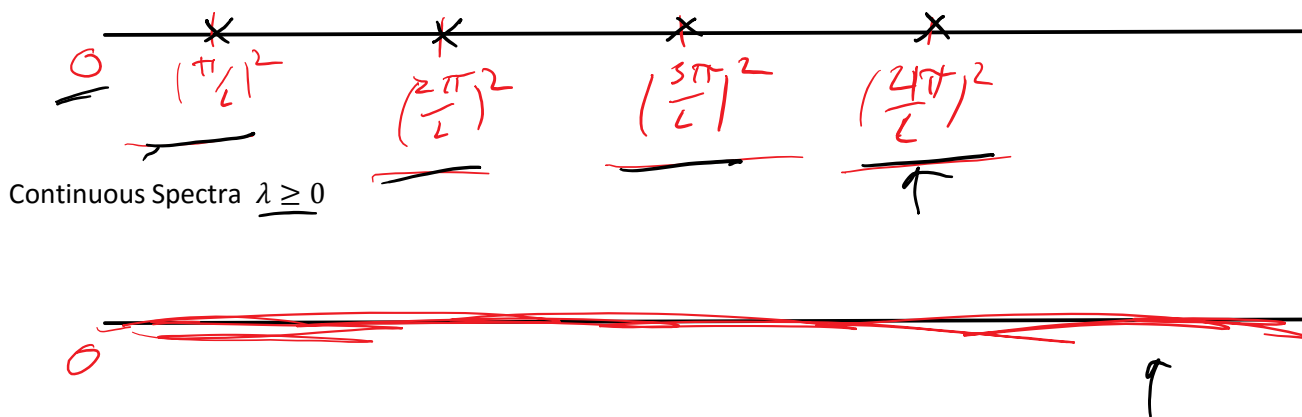
$$|\phi(x)| = |c_1 e^{\sqrt{\lambda} x} + c_2 e^{-\sqrt{\lambda} x}| \leq |c_1| e^{\sqrt{\lambda} x} + |c_2| e^{-\sqrt{\lambda} x}$$



Fourier Transform (16.1-16.2)

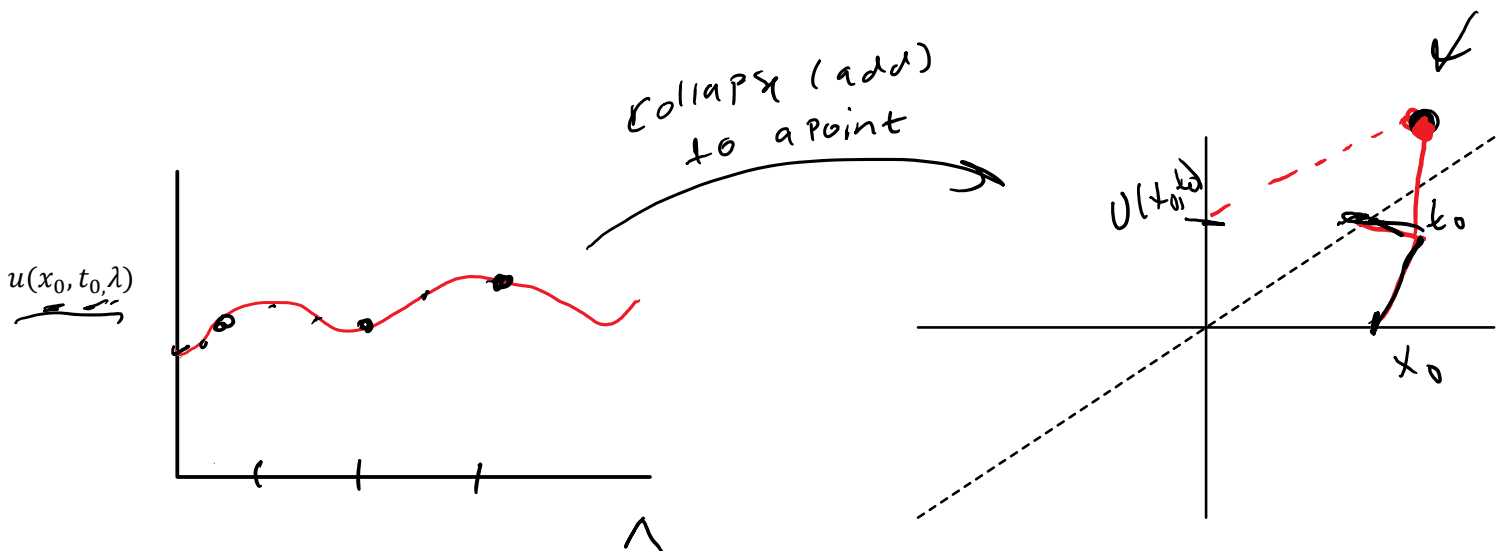
So as long as $\lambda \geq 0$ our condition of $|\phi(\pm\infty)| < \infty$ is satisfied, this is very different from the case we had before where $\lambda = \left(\frac{n\pi}{L}\right)^2$. We often refer to the set of eigen values as the spectra of a problem. When we had a finite rod we had a discrete spectra for this problem we have a continuous spectra.

Discrete Spectra $\lambda = \left(\frac{n\pi}{L}\right)^2$



So now we have a solution $u_\lambda(x, t) = h(t)\phi(x) = e^{-k\lambda t}(c_1(\lambda) \cos(\sqrt{\lambda}x) + c_2(\lambda) \sin(\sqrt{\lambda}x))$ for each value of $\lambda \geq 0$, so again we have infinitely many solutions but with a continuous spectra, the question now is how do we apply the principle of super position.

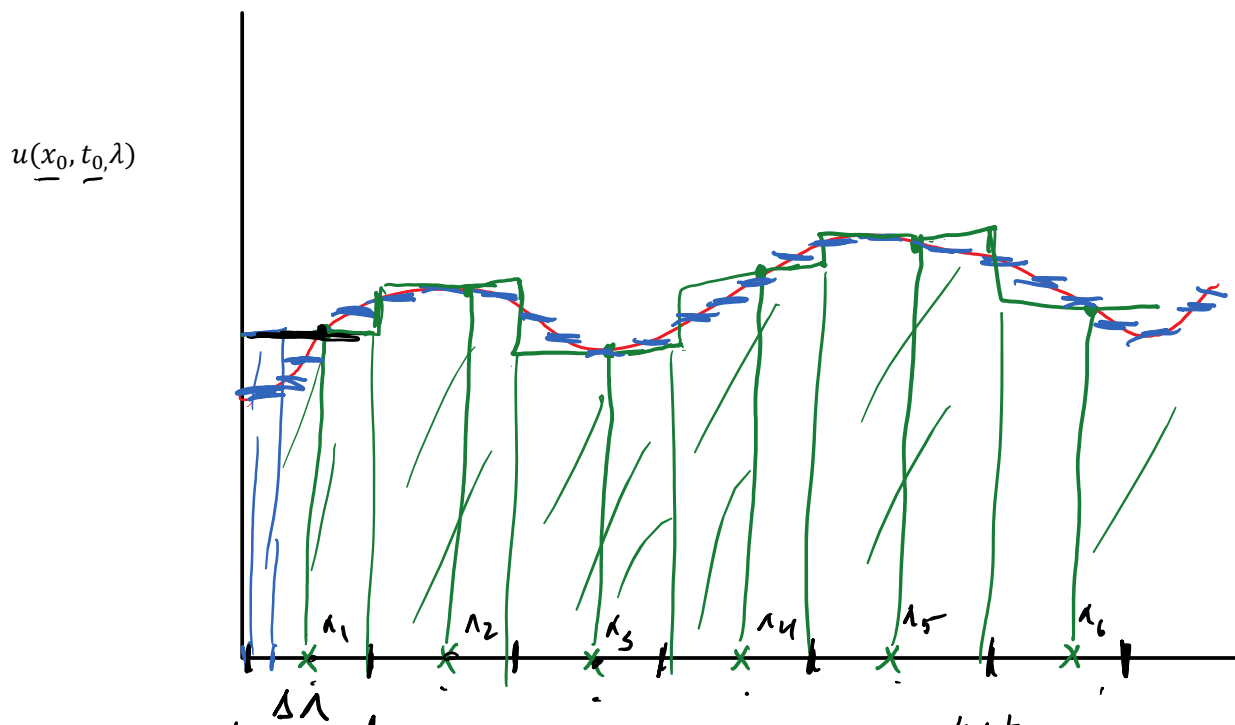
Well, let's consider our solution at a specific time t_0 and position x_0 , then we can think of u as a function of lambda, what we want to do is add up each contribution for each lambda to give us the real temperature $u(x_0, t_0)$



Fourier Transform (16.1-16.2)

One way to proceed would be to approximate our sum for each x and t with line segments for intervals

$\Delta\lambda$



What would such a sum look like?

$$u(x,t) = e^{-k\lambda t} (c_1(\lambda) \cos \sqrt{\lambda} x + c_2(\lambda) \sin \sqrt{\lambda} x)$$

$$u(x_0, t_0) \approx \sum_{i=1}^{\infty} e^{-k\lambda_i t_0} (c_1(\lambda_i) \cos(\sqrt{\lambda_i} x) + c_2(\lambda_i) \sin(\sqrt{\lambda_i} x)) \Delta\lambda$$

$$u(x_0, t_0) \approx \sum_{i=1}^{\infty} e^{-k\lambda_i t_0} (c_1(\lambda_i) \cos(\sqrt{\lambda_i} x_0) + c_2(\lambda_i) \sin(\sqrt{\lambda_i} x_0)) \Delta\lambda$$

It becomes clear, that the smaller and smaller we take $\Delta\lambda$ the more accurate this solution becomes...

So what we might want to say is that

$$u(x_0, t_0) = \lim_{\Delta\lambda \rightarrow 0} \sum_{i=1}^{\infty} e^{-k\lambda_i t_0} (c_1(\lambda_i) \cos(\sqrt{\lambda_i} x_0) + c_2(\lambda_i) \sin(\sqrt{\lambda_i} x_0)) \Delta\lambda$$

Does this remind you of anything from calculus?

$$\int_0^{\infty} e^{-k\lambda t} (c_1(\lambda) \cos(\sqrt{\lambda} x) + c_2(\lambda) \sin(\sqrt{\lambda} x)) d\lambda$$

Does this remind you of anything from calculus?



$$\int_0^{\infty} e^{-k_1 t} [c_1(\lambda) \cos(\sqrt{\lambda} x) + c_2(\lambda) \sin(\sqrt{\lambda} x)] d\lambda$$

Fourier Transform (16.1-16.2)

With these considerations instead of summing over λ we integrate over λ

$$\underline{u(x,t)} = \int_0^\infty e^{-k\lambda t} (c_1(\lambda) \cos(\sqrt{\lambda}x) + c_2(\lambda) \sin(\sqrt{\lambda}x)) \underline{d\lambda}$$

Homework Example

Verify that the solution $u(x,t) = \int_0^\infty e^{-k\lambda t} (c_1(\lambda) \cos(\sqrt{\lambda}x) + c_2(\lambda) \sin(\sqrt{\lambda}x)) d\lambda$ satisfies the 1-d heat equation:

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}$$

$$\begin{aligned} \frac{\partial}{\partial t} \int_0^\infty e^{-k\lambda t} (c_1 \cos(\sqrt{\lambda}x) + c_2 \sin(\sqrt{\lambda}x)) d\lambda &= k \frac{\partial^2}{\partial x^2} \int_0^\infty e^{-k\lambda t} (c_1 \cos(\sqrt{\lambda}x) + c_2 \sin(\sqrt{\lambda}x)) d\lambda \\ &= \int_0^\infty \frac{\partial}{\partial t} e^{-k\lambda t} (c_1 \cos(\sqrt{\lambda}x) + c_2 \sin(\sqrt{\lambda}x)) d\lambda = k \int_0^\infty e^{-k\lambda t} \frac{\partial^2}{\partial x^2} (c_1 \cos(\sqrt{\lambda}x) + c_2 \sin(\sqrt{\lambda}x)) d\lambda \\ &= \int_0^\infty -k\lambda e^{-k\lambda t} (c_1 \cos(\sqrt{\lambda}x) + c_2 \sin(\sqrt{\lambda}x)) d\lambda = k \int_0^\infty e^{-k\lambda t} (-c_1 \lambda \cos(\sqrt{\lambda}x) - c_2 \lambda \sin(\sqrt{\lambda}x)) d\lambda \\ &= \int_0^\infty -k\lambda e^{-k\lambda t} (c_1 \cos(\sqrt{\lambda}x) + c_2 \sin(\sqrt{\lambda}x)) d\lambda \stackrel{?}{=} \int_0^\infty -k\lambda e^{-k\lambda t} (c_1 \cos(\sqrt{\lambda}x) + c_2 \sin(\sqrt{\lambda}x)) d\lambda \end{aligned}$$

$$\frac{\partial}{\partial x} \left(\frac{\partial}{\partial x} (c_1 \cos(\sqrt{\lambda}x) + c_2 \sin(\sqrt{\lambda}x)) \right)$$

$$\frac{\partial}{\partial x} (-c_1 \sqrt{\lambda} \sin(\sqrt{\lambda}x) + c_2 \sqrt{\lambda} \cos(\sqrt{\lambda}x))$$

$$-c_1 \sqrt{\lambda} \sqrt{\lambda} \cos(\sqrt{\lambda}x) - c_2 \sqrt{\lambda} \sqrt{\lambda} \sin(\sqrt{\lambda}x)$$

$$-c_1 \lambda \cos(\sqrt{\lambda}x) - c_2 \lambda \sin(\sqrt{\lambda}x)$$

✓

Fourier Transform (16.1-16.2)

So now that we have our solution we need to find the coefficients. We assume we have some initial condition $u(x, 0) = f(x)$ and proceed in the usual way.

$$u(x, t) = \int_0^{\infty} e^{-k\lambda t} (c_1(\lambda) \cos(\sqrt{\lambda}x) + c_2(\lambda) \sin(\sqrt{\lambda}x)) d\lambda$$

Real quickly, since lambda is nonnegative anyway we will define $\lambda = \omega^2$ and write our solution in the form.

$$u(x, t) = \int_0^{\infty} e^{-k\omega^2 t} (A(\omega) \cos(\omega x) + B(\omega) \sin(\omega x)) d\omega$$

$$\sqrt{\lambda} = \sqrt{\omega^2} = \omega$$

Let's now apply our initial condition:

$$u(x, 0) = f(x) = \int_0^{\infty} (A(\omega) \cos(\omega x) + B(\omega) \sin(\omega x)) d\omega$$

The question then becomes how do we find $A(\omega)$ and $B(\omega)$, this is where the Fourier transform will come in.

Before we do that though we need to re-write our initial condition equation and solution using complex exponentials

It can be shown that:

$$e^{i\theta} = \cos(\theta) + i \sin(\theta)$$

$$u(x, t) = \int_0^{\infty} e^{-k\omega^2 t} (A(\omega) \cos(\omega x) + B(\omega) \sin(\omega x)) d\omega = \int_{-\infty}^{\infty} c(\omega) e^{-i\omega x} e^{-k\omega^2 t} d\omega$$

$$f(x) = \int_0^{\infty} (A(\omega) \cos(\omega x) + B(\omega) \sin(\omega x)) d\omega = \int_{-\infty}^{\infty} c(\omega) e^{-i\omega x} d\omega$$

with $A(\omega), B(\omega) \in \mathbb{R}$ and $c(\omega) \in \mathbb{C}$

Showing this is a part of your homework and we will do it at the end of class or next time but for now let's push on.

For now let's just use the fact that we could write:

$$u(x, t) = \int_{-\infty}^{\infty} c(\omega) e^{-i\omega x} e^{-k\omega^2 t} d\omega$$

$$f(x) = \int_{-\infty}^{\infty} c(\omega) e^{-i\omega x} d\omega$$

Fourier Transform (16.1-16.2)

So we can say that $f(x) = \int_{-\infty}^{\infty} c(\omega) e^{-i\omega x} d\omega$

But how do we figure out $c(\omega)$?

There are just a few steps first though. We need to rewrite our Fourier Series definition. Recall from chapter 14 that for discrete spectra,

$$\rightarrow f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi}{L}x\right) + \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}x\right)$$

$$A_0 = \frac{1}{2L} \int_{-L}^L f(x) dx$$

$$A_n = \frac{1}{L} \int_{-L}^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx$$

$$B_n = \frac{1}{L} \int_{-L}^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

Next consider Euler's formulas for sine and cosine

$$\sin(\theta) = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

$$\cos(\theta) = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

$$\begin{aligned} \cos \theta + i \sin \theta &= \frac{e^{i\theta} + e^{-i\theta}}{2} + i \frac{e^{i\theta} - e^{-i\theta}}{2i} \\ &= \frac{e^{i\theta} + e^{-i\theta} + e^{i\theta} - e^{-i\theta}}{2} \\ &= \frac{2e^{i\theta}}{2} = e^{i\theta} \end{aligned}$$

If we plug in $\frac{n\pi x}{L}$ and collect like exponentials...

$$\begin{aligned} f(x) &= A_0 + \sum_{n=1}^{\infty} A_n \frac{e^{\frac{i n \pi x}{L}} + e^{-\frac{i n \pi x}{L}}}{2} + B_n \frac{e^{\frac{i n \pi x}{L}} - e^{-\frac{i n \pi x}{L}}}{2i} \\ &= A_0 + \sum_{n=1}^{\infty} \left[\frac{A_n}{2} \left(e^{\frac{i n \pi x}{L}} + e^{-\frac{i n \pi x}{L}} \right) + \frac{i B_n}{2} \left(e^{\frac{i n \pi x}{L}} - e^{-\frac{i n \pi x}{L}} \right) \right] \\ &= A_0 + \sum_{n=1}^{\infty} \left(\frac{A_n - i B_n}{2} \right) e^{\frac{i n \pi x}{L}} + \frac{A_n + i B_n}{2} e^{-\frac{i n \pi x}{L}} \end{aligned}$$

$\frac{1}{i} = -i$
 $\frac{2i \sin \theta}{2i} = \sin \theta$

Fourier Transform (16.1-16.2)

So we now have:

$$f(x) = A_0 + \frac{1}{2} \sum_{n=1}^{\infty} (A_n - iB_n) e^{\frac{in\pi}{L}x} + \frac{1}{2} \sum_{n=1}^{\infty} (A_n + iB_n) e^{-\frac{in\pi}{L}x}$$

Swapping $n = -n$ and reindexing the first sum we get

$$f(x) = A_0 + \frac{1}{2} \sum_{n=-1}^{-\infty} (A_{-n} - iB_{-n}) e^{\frac{in\pi}{L}x} + \frac{1}{2} \sum_{n=1}^{\infty} (A_n + iB_n) e^{-\frac{in\pi}{L}x}$$

$A_n \quad -B_n$

Note that $A_n = A_{-n}$ since cosine is even

$$A_n = \frac{1}{L} \int_{-L}^L f(x) \cos\left(\frac{n\pi}{L}x\right) dx$$

$$A_{-n} = \frac{1}{L} \int_{-L}^L f(x) \cos\left(\frac{-n\pi}{L}x\right) dx$$

And $B_{-n} = -B_n$ since sine is odd.

$$B_n = \frac{1}{L} \int_{-L}^L f(x) \sin\left(\frac{n\pi}{L}x\right) dx$$

$$B_{-n} = \frac{1}{L} \int_{-L}^L f(x) \sin\left(\frac{-n\pi}{L}x\right) dx$$

Applying these and combining the terms we get

$$= -\frac{1}{L} \int_{-L}^L f(x) \sin\left(\frac{n\pi}{L}x\right) dx = -B_n$$

$$f(x) = A_0 + \frac{1}{2} \sum_{n=-1}^{-\infty} (A_n + iB_n) e^{-\frac{in\pi}{L}x} + \frac{1}{2} \sum_{n=1}^{\infty} (A_n + iB_n) e^{-\frac{in\pi}{L}x}$$

$$= \sum_{n=-\infty}^{\infty} \frac{(A_n + iB_n)}{2} e^{-\frac{in\pi}{L}x}$$

$\downarrow c_n$

$$n=0 \quad \frac{A_0 + iB_0}{2} e^0$$

Fourier Transform (16.1-16.2)

So we can say:

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{-\frac{in\pi}{L}x}$$

$$c_0 = A_0$$

$$c_n = \frac{1}{2} (A_n + iB_n) = \frac{1}{2L} \int_{-L}^L f(x) \left(\cos\left(\frac{n\pi}{L}x\right) + i \sin\left(\frac{n\pi}{L}x\right) \right) dx$$

A_n B_n

Or just that

$$c_n = \frac{1}{2L} \int_{-L}^L f(x) e^{\frac{in\pi}{L}x} dx$$

This is what we call the complex form for a Fourier Series.

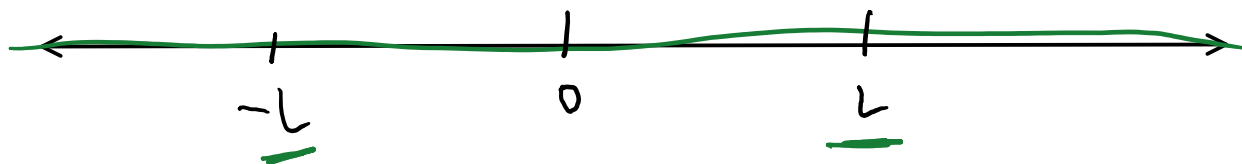
Now we are after the complex coefficients $c(\omega)$ for our initial condition

$$f(x) = \int_{-\infty}^{\infty} c(\omega) e^{-i\omega x} d\omega$$

Let's start by Looking at this on an interval from -L to L in which case

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{-\frac{in\pi}{L}x}$$

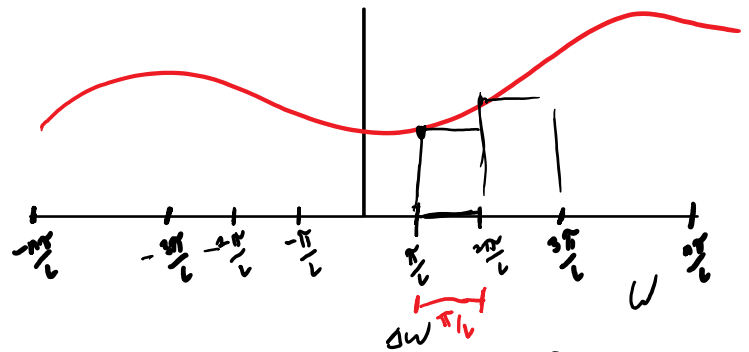
Since we are on the whole real line we might expect that we can get what we want in some limit as $L \rightarrow \infty$



Fourier Transform (16.1-16.2)

So

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{-\frac{i n \pi}{L} x}$$



Plugging in our formula for c_n $c_n = \frac{1}{2L} \int_{-L}^L f(x) e^{\frac{i n \pi}{L} x} dx$ Swapping s for x we get:

$$f(x) = \sum_{n=-\infty}^{\infty} \left(\frac{1}{2L} \int_{-L}^L f(s) e^{\frac{i n \pi}{L} s} ds \right) e^{-\frac{i n \pi}{L} x}$$

$$= \sum_{n=-\infty}^{\infty} \frac{1}{2L} \int_{-L}^L f(s) e^{\frac{i n \pi}{L} s} ds e^{-\frac{i n \pi}{L} x}$$

$$\frac{2}{2} \frac{n \pi}{L} = 2 \pi \frac{n}{2L}$$

We let $\omega = \frac{n \pi}{L} = 2 \pi \frac{n}{2L}$ then $\Delta \omega = \frac{\pi}{L}$ and $\frac{\Delta \omega}{2 \pi} = \frac{1}{2L}$ and we can say

$$\Delta \omega = \frac{\pi}{L}$$

$$\frac{\Delta \omega}{2 \pi} = \frac{1}{2L}$$

$$f(x) \approx \sum_{n=-\infty}^{\infty} \left(\frac{1}{2L} \int_{-L}^L f(s) e^{\frac{i n \pi}{L} s} ds \right) e^{-\frac{i n \pi}{L} x} = \sum_{n=-\infty}^{\infty} \left(\frac{\Delta \omega}{2 \pi} \int_{-L}^L f(s) e^{i \omega s} ds \right) e^{-i \omega x}$$

$$\Delta \omega = \frac{2 \pi}{2L}$$

$$\frac{\pi}{L} = \frac{2 \pi}{2L}$$

Again we make the argument that this approximation is more accurate the smaller $\Delta \omega$ notice that since $\Delta \omega = \frac{\pi}{L}$

$\Delta \omega \rightarrow 0$ as $L \rightarrow \infty$

So we say:

$$f(x) = \lim_{L \rightarrow \infty} \sum_{n=-\infty}^{\infty} \frac{1}{2 \pi} \int_{-L}^L f(s) e^{i \omega s} ds e^{-i \omega x} \Delta \omega$$

In which case our sum becomes an integral in omega and we get:

$$f(x) = \int_{-\infty}^{\infty} \left(\frac{1}{2 \pi} \int_{-\infty}^{\infty} f(s) e^{i \omega s} ds \right) e^{-i \omega x} d \omega$$

Comparing this to our initial condition

$$f(x) = \int_{-\infty}^{\infty} c(\omega) e^{-i \omega x} d \omega$$

We see that

$$c(\omega) = \frac{1}{2 \pi} \int_{-\infty}^{\infty} f(x) e^{i \omega x} dx$$

Where we swapped x back in for s

Fourier Transform (16.1-16.2)

This motivates the following definitions for the Fourier and Inverse Fourier Transforms!

The Fourier transformation of a function $f(x)$ is defined as:

$$\mathcal{F}[f(x)] = F(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{i\omega x} dx$$

The inverse Fourier transform of a function $F(\omega)$ is defined as:

$$\mathcal{F}^{-1}[F(\omega)] = f(x) = \int_{-\infty}^{\infty} F(\omega) e^{-i\omega x} d\omega$$

Let's examine our initial condition equation:

$$f(x) = \int_{-\infty}^{\infty} c(\omega) e^{-i\omega x} d\omega$$

Looking at this we notice that this is the inverse Fourier Transform of $c(\omega)$! What does this say about $c(\omega)$?

$$c(\omega) = \widetilde{\mathcal{F}}(f(x)) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{i\omega x} dx$$

Just a quick note, There are about 8 different ways to define the Fourier transform, in fact Wikipedia does it with the signs in the exponentials switched and a 2π in the complex exponential, they are equivalent though just think about u substitutions, there are lots of ways to write the same integrals... We will stick with the above as it is convenient for the heat equation.

Fourier Transform (16.1-16.2)

Homework Example

Show that

$$u(x, t) = \int_0^{\infty} e^{-k\omega^2 t} (A(\omega) \cos(\omega x) + B(\omega) \sin(\omega x)) d\omega = \int_{-\infty}^{\infty} c(\omega) e^{-i\omega x} e^{-k\omega^2 t} d\omega$$

For some complex $c(\omega)$