

Sec 16.3-16.4

The Fourier transformation of a function $f(x)$ is defined as:

$$\mathcal{F}[f(x)] = F(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{i\omega x} dx$$

The inverse Fourier transform of a function $F(\omega)$ is defined as:

$$\mathcal{F}^{-1}[F(\omega)] = f(x) = \int_{-\infty}^{\infty} F(\omega) e^{-i\omega x} d\omega$$

We also saw that the solution to the heat equation on the whole real line

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} \qquad |u(\pm\infty, t)| < \infty \qquad u(x, 0) = f(x)$$

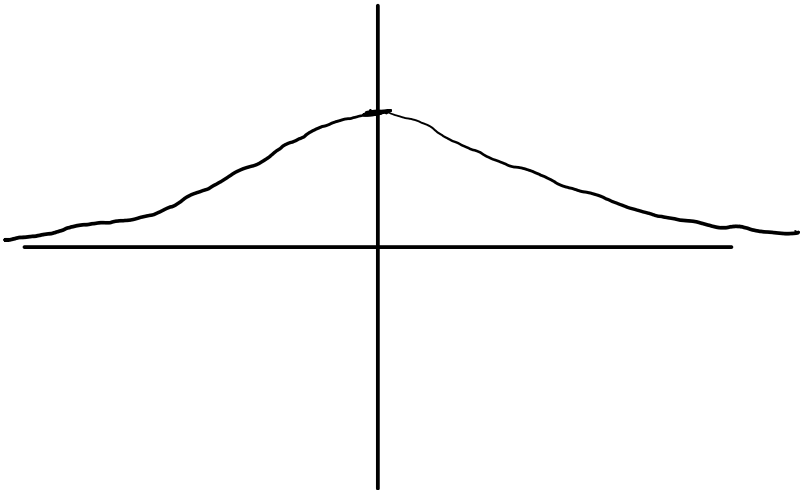
Was given by:

$$u(x, t) = \int_{-\infty}^{\infty} c(\omega) e^{-i\omega x} e^{-k\omega^2 t} d\omega \qquad \text{with} \qquad c(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{i\omega x} dx$$

It turned out that the coefficients $c(\omega)$ were just the Fourier Transform of the initial temperature profile $f(x)$

So It might be prudent to actually calculate a Fourier transform as an example of how one might determine $c(\omega)$
Now $f(x)$ can be pretty much anything but we will do one that is particularly important although it's transform is not for the faint of hart. (its really not that bad just messy.)

Suppose $f(x) = e^{-\alpha x^2}$



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So we have a solution to our infinite domain heat equation:

$$u(x,t) = \int_{-\infty}^{\infty} c(\omega) e^{-i\omega x} e^{-k\omega^2 t} d\omega$$

where:

$$f(x) = e^{-ax^2}$$

$$c(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{i\omega x} dx$$

Let's find our $c(\omega)$

$$c(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ax^2} e^{i\omega x} dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ax^2 + i\omega x} dx$$

Complete the square $-ax^2 + i\omega x$

$$-a \left(x^2 - \frac{i\omega}{a} x \right)$$

magic # take b term divide by 2 and square it

$$\left(\frac{i\omega}{2a} \right)^2 = -\frac{\omega^2}{4a^2}$$

$$= -a \left(x^2 - \frac{i\omega}{a} x + \frac{\omega^2}{4a^2} - \frac{\omega^2}{4a^2} \right) = -a \left[\left(x - \frac{i\omega}{2a} \right)^2 + \frac{\omega^2}{4a^2} \right]$$

$$= -a \left(x - \frac{i\omega}{2a} \right)^2 - \frac{\omega^2}{4a}$$

$$c(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-a \left(x - \frac{i\omega}{2a} \right)^2 - \frac{\omega^2}{4a}} dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-a \left(x - \frac{i\omega}{2a} \right)^2} e^{-\frac{\omega^2}{4a}} dx$$

$$= \frac{e^{-\frac{\omega^2}{4a}}}{2\pi} \int_{-\infty}^{\infty} e^{-a \left(x - \frac{i\omega}{2a} \right)^2} dx = \frac{e^{-\frac{\omega^2}{4a}}}{2\pi} \int_{-\infty}^{\infty} e^{-(\sqrt{a}x - \frac{i\omega}{2\sqrt{a}})^2} dx$$

$$u = \sqrt{a}x - \frac{i\omega}{2\sqrt{a}}$$

$$du = \sqrt{a} dx$$

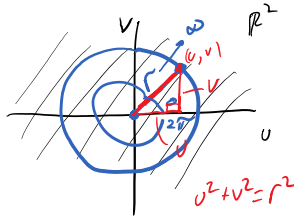
$$dx = \frac{1}{\sqrt{a}} du$$

$$= \frac{e^{-\frac{\omega^2}{4a}}}{2\pi} \int_{-\infty}^{\infty} e^{-u^2} \frac{1}{\sqrt{a}} du = \frac{e^{-\frac{\omega^2}{4a}}}{2\pi\sqrt{a}} \int_{-\infty}^{\infty} e^{-u^2} du$$

$$\frac{e^{-u^2/4a}}{2\pi\sqrt{a}} \int_{-\infty}^{\infty} e^{-v^2} dv$$

$$\int_{-\infty}^{\infty} e^{-v^2} dv = \sqrt{\pi}$$

$$\underline{I^2} = \left(\int_{-\infty}^{\infty} e^{-u^2} du \right) \left(\int_{-\infty}^{\infty} e^{-v^2} dv \right)$$



$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-u^2} e^{-v^2} du dv = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(u^2+v^2)} du dv$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(u^2+v^2)} du dv = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(u^2+v^2)} du dv$$

convert to polar what is $u^2+v^2=r^2$ $du dv = r dr d\theta$

$$I^2 = \int_0^{2\pi} \int_0^{\infty} e^{-r^2} r dr d\theta$$

$$s = r^2 \quad ds = 2r dr$$

$$\frac{ds}{2} = r dr$$

$$= \int_0^{2\pi} \int_0^{\infty} e^{-s} \frac{ds}{2} d\theta = \frac{1}{2} \int_0^{2\pi} \left[\int_0^{\infty} e^{-s} ds \right] d\theta$$

$$= \frac{1}{2} \int_0^{2\pi} -e^{-s} \Big|_0^{\infty} d\theta = \frac{1}{2} \int_0^{2\pi} \left[-\lim_{s \rightarrow \infty} e^{-s} + e^0 \right] d\theta$$

$$\frac{1}{2} \int_0^{2\pi} 1 d\theta = \frac{1}{2} \theta \Big|_0^{2\pi} = \frac{1}{2} (2\pi - 0) = \pi$$

$$I^2 = \pi \quad I = \sqrt{\pi} \quad I = \int_{-\infty}^{\infty} e^{-v^2} dv$$

$$\frac{e^{-w^2/4a}}{2\pi\sqrt{a}} \underbrace{\int_{-\infty}^{\infty} e^{-v^2} dv}_{=\sqrt{\pi}} = \frac{e^{-w^2/4a}}{2\pi\sqrt{a}} \sqrt{\pi} = e^{-w^2/4a} \underbrace{\frac{1}{2\sqrt{\pi}\sqrt{a}}}_{\hookrightarrow \frac{1}{\sqrt{4\pi a}}}$$

$$= \frac{1}{\sqrt{4\pi a}} e^{-w^2/4a} = \mathcal{F}[e^{-ax^2}] = F(w)$$

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It turns out we use Fourier Transforms to turn PDE's into ODE's! Let's do this with our heat equation.

$$\cancel{\mathcal{F}} \left[\frac{\partial u}{\partial t} \right] = k \cancel{\mathcal{F}} \left[\frac{\partial^2 u}{\partial x^2} \right]$$

The trick will be to take the Fourier Transform of both sides of this equation. Before we do that let's just name the Fourier Transform of $u(x, t)$ to be $\bar{U}(\omega, t)$ that is:

$$\mathcal{F}[u(x, t)] = \bar{U}(\omega, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} u(x, t) e^{i\omega x} dx$$

Now Let's see what happens when we take the Fourier Transform of both sides of our PDE.

The time derivative side is easy

$$\begin{aligned} \mathcal{F} \left[\frac{\partial u}{\partial t} \right] &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\partial u}{\partial t} e^{i\omega x} dx = \frac{\partial}{\partial t} \frac{1}{2\pi} \int_{-\infty}^{\infty} u(x, t) e^{i\omega x} dx \\ &= \frac{\partial}{\partial t} \bar{U}(\omega, t) = \frac{\partial \bar{U}}{\partial t} \end{aligned}$$

For the Spatial derivative side Let's first examine what happens when we take the Fourier Transform of $\frac{\partial u}{\partial x}$

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$$\mathcal{F}\left[\frac{\partial u}{\partial x}\right] = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\partial u}{\partial x} e^{i\omega x} dx$$

$$\int m dn \quad m \cdot n - \int n dm$$

Let's throw integration by parts at it.

$$m = e^{i\omega x} \quad dn = \frac{\partial u}{\partial x}$$

$$dm = i\omega e^{i\omega x} dx \quad n = u$$

$$|u(x, t)| < \infty$$

$$= \frac{1}{2\pi} \left[\cancel{u e^{i\omega x}} \Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} u i\omega e^{i\omega x} dx \right]$$

$$= -i\omega \underbrace{\frac{1}{2\pi} \int_{-\infty}^{\infty} u e^{i\omega x} dx}_= \mathcal{U}(\omega, t)$$

So $\mathcal{F}\left[\frac{\partial u}{\partial x}\right] = -i\omega \mathcal{F}[u]$

Any guesses for $\mathcal{F}\left[\frac{\partial^2 u}{\partial x^2}\right]$?

$$\begin{aligned} \mathcal{F}\left[\frac{\partial^2 u}{\partial x^2}\right] &= (-i\omega)^2 \mathcal{F}(u) = (-i\omega)^2 \bar{u} \\ &= -\omega^2 \bar{u} \end{aligned}$$

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Then if we take the Fourier transform of our heat equation on both sides we get:

$$\mathcal{F}\left[\frac{\partial u}{\partial t}\right] = \mathcal{F}\left[k \frac{\partial^2 u}{\partial x^2}\right]$$

$$\frac{\partial}{\partial t} \bar{U}(\omega, t) = k \underbrace{(-i\omega)^2}_{-\omega^2} \bar{U}(\omega, t)$$

An ODE! Let's solve this thing.

$$\frac{d\bar{U}}{dt} = -k\omega^2 \bar{U} \rightarrow \int \frac{d\bar{U}}{\bar{U}} = \int -k\omega^2 dt$$

$$\ln(\bar{U}) = -k\omega^2 t + C$$

$$e^{\ln(\bar{U})} = e^{-k\omega^2 t + C} \Rightarrow \bar{U} = e^{-k\omega^2 t + C} = e^{-k\omega^2 t} e^C$$

$$\Rightarrow \boxed{\bar{U} = C e^{-k\omega^2 t}}$$

So we have $\bar{U}(\omega, t) = c(\omega) e^{-k\omega^2 t}$ we can then simply apply the inverse Fourier Transform to obtain $u(x, t)$ That is

$$\mathcal{F}^{-1}[\bar{U}] = u$$

$$u(x, t) = \int_{-\infty}^{\infty} \bar{U}(\omega, t) e^{-i\omega x} d\omega = \int_{-\infty}^{\infty} c(\omega) e^{-k\omega^2 t} e^{-i\omega x} d\omega$$

Just as we had from separation of variables!

$$u(x, t) = \int_{-\infty}^{\infty} c(\omega) e^{-k\omega^2 t} e^{-i\omega x} d\omega$$

We could find $c(\omega)$ again using:

$$\boxed{c(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{i\omega x} dx}$$

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Shift Theorem

$\mathcal{F}[f(x)] = F(\omega)$ $\mathcal{F}[e^{i\omega\beta} F(\omega)] = \underline{\underline{f(x-\beta)}}$

If $F(\omega)$ is the Fourier Transform of $f(x)$ then the inverse fourier transform of $e^{i\omega\beta} F(\omega)$ is $f(x-\beta)$

We can show this by taking the forward transform of $f(x-\beta)$ which is $\frac{1}{2\pi} \int_{-\infty}^{\infty} f(x-\beta) e^{i\omega x} dx$ and making the substitution $u = x - \beta$

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} f(x-\beta) e^{i\omega x} dx$$

$$u = x - \beta \quad \downarrow$$

$$x = u + \beta \quad \rightarrow$$

$$du = dx$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(u) e^{i\omega(u+\beta)} du = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(u) e^{i\omega u} e^{i\omega\beta} du$$

$$= e^{i\omega\beta} \underbrace{\frac{1}{2\pi} \int_{-\infty}^{\infty} f(u) e^{i\omega u} du}_{\mathcal{F}(f(x)) = F(\omega)} = e^{i\omega\beta} F(\omega)$$

$\mathcal{F}[f(x)] = F(\omega)$

$\mathcal{F}(f(x-\beta)) = e^{i\beta\omega} F(\omega)$

$f(x) = x^2$
 ψ

$(x-2)^2 = f(x-2)$
 ψ

Note that this means

$$\int_{-\infty}^{\infty} \underbrace{e^{i\beta\omega} F(\omega)}_{f(x-\beta)} e^{-i\omega x} d\omega = \int_{-\infty}^{\infty} F(\omega) e^{-i\omega(x-\beta)} d\omega = f(x-\beta)$$

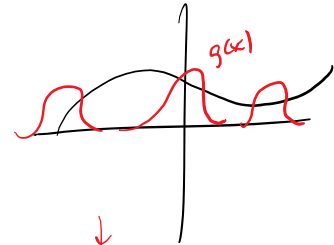
$$e^{i\beta\omega} e^{-i\omega x} = e^{i\beta\omega - i\omega x} = e^{-i\omega(x-\beta)}$$

We will use this idea to prove the next theorem.

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Another representation for our solution using the Convolution Theorem. First Let's define the convolution of two functions, $f * g = g * f$

$$f * g = \int_{-\infty}^{\infty} f(\bar{x}) g(x - \bar{x}) d\bar{x}$$



Let's ask the following question, What is the inverse Fourier transform $h(x)$ of a function $H(\omega) = F(\omega)G(\omega)$

Where:

$$F(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{i\omega x} dx$$

$$G(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} g(x) e^{i\omega x} dx$$

$$f(x) = \int_{-\infty}^{\infty} F(\omega) e^{-i\omega x} d\omega$$

$$g(x) = \int_{-\infty}^{\infty} G(\omega) e^{-i\omega x} d\omega$$

We start by just applying the inverse Fourier transform to $H(\omega)$

$$h(x) = \int_{-\infty}^{\infty} H(\omega) e^{-i\omega x} d\omega = \int_{-\infty}^{\infty} F(\omega) G(\omega) e^{-i\omega x} d\omega$$

$$h(x) = \int_{-\infty}^{\infty} F(\omega) G(\omega) e^{-i\omega x} d\omega = \int_{-\infty}^{\infty} F(\omega) \frac{1}{2\pi} \int_{-\infty}^{\infty} g(\bar{x}) e^{i\omega \bar{x}} d\bar{x} e^{-i\omega x} d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} g(\bar{x}) \int_{-\infty}^{\infty} F(\omega) e^{i\omega \bar{x} - i\omega x} d\omega d\bar{x}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} g(\bar{x}) \int_{-\infty}^{\infty} F(\omega) e^{i\omega \bar{x} - i\omega x} d\omega d\bar{x}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} g(\bar{x}) \left[\int_{-\infty}^{\infty} F(\omega) e^{-i\omega(x - \bar{x})} d\omega \right] d\bar{x}$$

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} g(\bar{x}) f(x - \bar{x}) d\bar{x} = \frac{1}{2\pi} [f * g]$$

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$$\mathcal{F}^{-1}(F(\omega)G(\omega))$$

So we have that if $H(\omega) = F(\omega)G(\omega)$ then $h(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} g(\bar{x})f(x - \bar{x})d\bar{x} = \frac{1}{2\pi} (g * f)$

In words we are just saying that inverse Fourier transform of a product of two Fourier transforms is $\frac{1}{2\pi}$ times the convolution of the product of the original functions.

What does this do for us with regard to the heat equation? Well recall that the solution was:

$$u(x, t) = \int_{-\infty}^{\infty} c(\omega) e^{-k\omega^2 t} e^{i\omega x} d\omega = \mathcal{F}^{-1}[c(\omega) e^{-k\omega^2 t}]$$

$\mathcal{F}(\omega) \quad G(\omega)$

Let's name $G(\omega) = c(\omega)$ and $F(\omega) = e^{-k\omega^2 t}$

Then by the convolution theorem $u(x, t)$ will be equal to the $\frac{1}{2\pi}$ times convolution of the inverse Fourier transforms of $G(\omega) = c(\omega)$ and $F(\omega) = e^{-k\omega^2 t}$

$$\mathcal{F}^{-1}(G(\omega)) = f(x)$$

We know that the inverse Fourier transform of $c(\omega)$ is just of $f(x)$ and we found out earlier that:

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\alpha x^2} e^{i\omega x} dx = \frac{1}{\sqrt{4\pi\alpha}} e^{-\frac{\omega^2}{4\alpha}}$$

If we take the Inverse Fourier transform of $F(\omega) = e^{-k\omega^2 t}$ it would read: $\int_{-\infty}^{\infty} e^{-k\omega^2 t} e^{-i\omega x} d\omega$ which we can figure out easily using the known integral above. Let $\alpha = kt$, $x = -\omega$

$$\mathcal{F}^{-1}[e^{-\alpha x^2}] = \frac{1}{\sqrt{4\pi\alpha}} e^{-\frac{\omega^2}{4\alpha}}$$

$$\mathcal{F}^{-1}(F(\omega)) = \mathcal{F}^{-1}(e^{-k\omega^2 t})$$

$$\mathcal{F}^{-1}[e^{-k\omega^2 t}] = \frac{1}{\sqrt{4\pi kt}} e^{-\frac{\omega^2}{4kt}}$$

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so we have that $\mathcal{F}^{-1}[c(\omega)] = f(x)$ and $\mathcal{F}^{-1}[e^{-k\omega^2 t}] = \sqrt{\frac{\pi}{kt}} e^{-\frac{x^2}{4kt}}$

What we want is $\mathcal{F}^{-1}[c(\omega)e^{-k\omega^2 t}]$

The convolution Theorem States that:

$$\mathcal{F}^{-1}[H(\omega)] = \mathcal{F}^{-1}[F(\omega)G(\omega)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} g(\bar{x}) f(x - \bar{x}) d\bar{x}$$

where $g(x) = \mathcal{F}^{-1}[F(\omega)]$ and $f(x) = \mathcal{F}^{-1}[G(\omega)]$

If we apply this to our solution

$$\bar{U}(\omega, t) = c(\omega)e^{-k\omega^2 t}$$

$$\underline{U(x,t)} = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) \frac{\sqrt{\pi}}{\sqrt{kt}} e^{-\frac{(x-\bar{x})^2}{4kt}} d\bar{x}$$

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So we have a solution to the heat equation on the whole real line for any initial condition $f(x)$

$$u(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\bar{x}) \sqrt{\frac{\pi}{kt}} e^{-\frac{(x-\bar{x})^2}{4kt}} d\bar{x}$$

Note that if I just bring the $\frac{1}{2\pi}$ into the integral and multiply it into $\sqrt{\frac{\pi}{kt}}$ the solution takes the form:

$$u(x,t) = \int_{-\infty}^{\infty} f(\bar{x}) \left[\frac{1}{\sqrt{4\pi kt}} e^{-\frac{(x-\bar{x})^2}{4kt}} \right] d\bar{x}$$

We often refer to the function $G(x,t,\bar{x}) = \frac{1}{\sqrt{4\pi kt}} e^{-\frac{(x-\bar{x})^2}{4kt}}$ as the heat kernel, or sometimes the influence function.

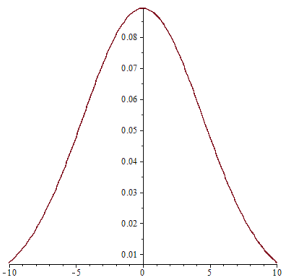
Think of it this way, at each time t G changes and then multiplies the initial temperature distribution "updating" it to the current time. It turns out that this G is specific for the heat equation, also known as the diffusion equation if we take u to represent a mass concentration. Let's take a maple break and watch what this does to an initial distribution!



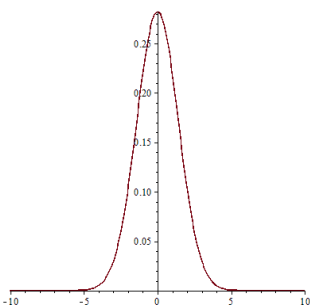
3150heatkern
real

Now it should be that at time $t = 0$ we recover our initial temperature distribution, clearly we can't just plug in $t=0$ as G would be singular there so let's instead look at the limit of G as $t \rightarrow 0^+$ and see what happens to it.

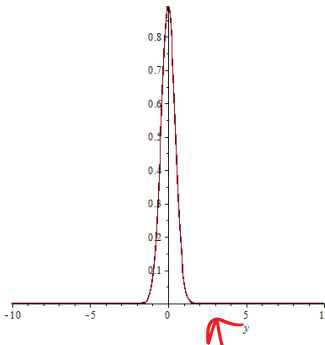
$$\frac{1}{\sqrt{4\pi kt}} e^{-\frac{(x-\bar{x})^2}{4kt}}$$



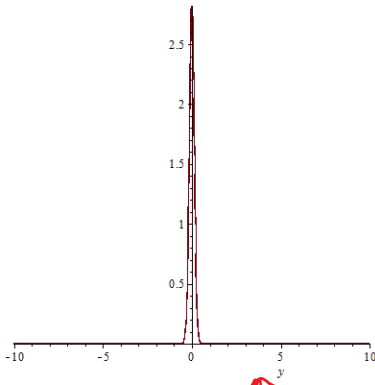
t=10



t=1



t=0.1



t=0.01

It turns out the thing that this is headed toward is what we call the Dirac delta function or delta distribution $\delta(x - \bar{x})$ it has the neat property that at the point $x = \bar{x}$ it has a value of infinity but is zero everywhere else. Not only that it turns out that

$$\int_{-\infty}^{\infty} \delta(x) dx = 1$$

$$\int_{-\infty}^{\infty} f(\bar{x}) \delta(x - \bar{x}) dx = f(x)$$

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Homework Example:

Show that the Fourier Transform is a linear Operator.