

## Sec 4.10

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Before we talk about inner product spaces it is useful to review the idea of a "basis", "orthogonal basis" and "orthogonal projection".

In the context of vectors a basis can be defined as a set of linearly independent vectors which span the space that they belong to.

$c_1 \langle 1, 0 \rangle + c_2 \langle 0, 1 \rangle = \langle c_1, c_2 \rangle$   
 Ex. The vectors  $\langle -1, 1 \rangle$  and  $\langle 0, 1 \rangle$  form a basis for  $\mathbb{R}^2$   
 $v_1 \quad v_2$

Linear Independence:  $c_1 = c_2 = 0$   
 $c_1 v_1 + c_2 v_2 = 0$

$c_1 \langle -1, 1 \rangle + c_2 \langle 0, 1 \rangle = \langle 0, 0 \rangle$   
 $\langle -c_1, c_1 \rangle + \langle 0, c_2 \rangle = \langle 0, 0 \rangle$   
 $\langle -c_1, c_1 + c_2 \rangle = \langle 0, 0 \rangle$

They "span"  $\mathbb{R}^2$   
 for any  $v$  in  $\mathbb{R}^2$   
 $v = c_1 v_1 + c_2 v_2$

$-c_1 = 0$   
 $c_1 + c_2 = 0$   
 $0 + c_2 = 0$   
 $c_2 = 0$

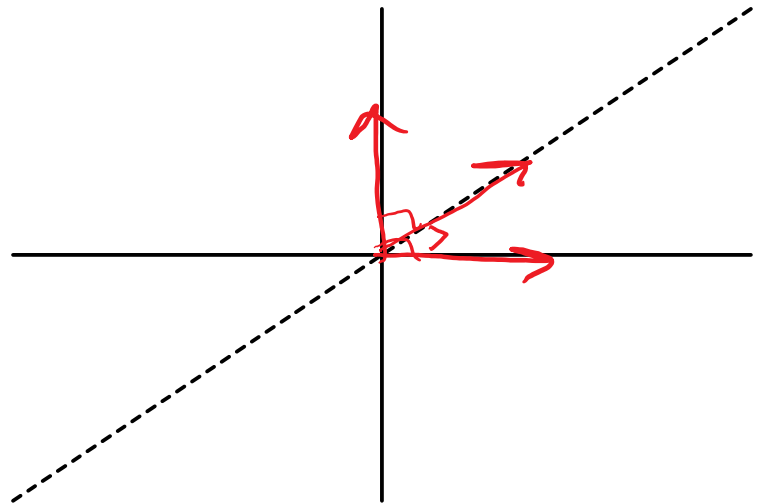
$v = \langle 1, 1 \rangle$   
 $v = c_1 \langle -1, 1 \rangle + c_2 \langle 0, 1 \rangle = \langle 1, 1 \rangle$   
 $-1 \langle -1, 1 \rangle + 2 \langle 0, 1 \rangle = \langle 1, 1 \rangle$

An orthogonal basis is a basis of vectors for a given space which are orthogonal.

Example:  $\langle 1, 0, 0 \rangle, \langle 0, 1, 0 \rangle, \langle 0, 0, 1 \rangle$  form an orthogonal basis for  $\mathbb{R}^3$

Let's show that they are orthogonal.

$\vec{a}_1 \cdot \vec{a}_2 = 0$   
 $\langle 1, 0, 0 \rangle \cdot \langle 0, 1, 0 \rangle$   
 $0 + 0 + 0 = 0$



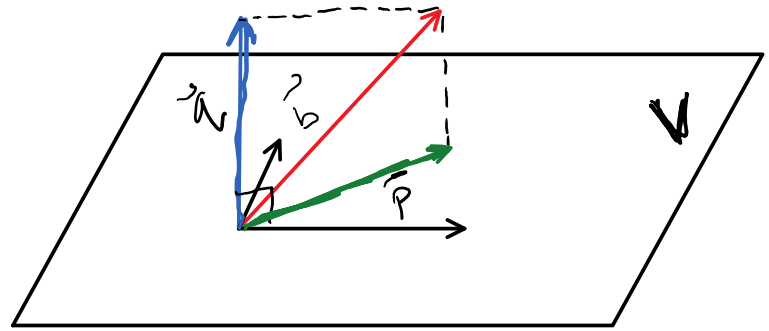
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Orthogonal Projection Problem:

Given a vector  $\vec{b}$  not in a subspace  $V$ , find vectors  $\vec{p}$  in  $V$  and  $\vec{q}$  orthogonal to  $V$  such that  $\vec{b} = \vec{p} + \vec{q}$ .

Further we wish to represent  $\vec{p}$  with the basis of  $V$   $\{\vec{a}_0, \vec{a}_1, \vec{a}_2, \dots, \vec{a}_n\}$  ideally an orthogonal basis.



Notice that  $\vec{p}$  is the closest vector to  $\vec{b}$  that lies in  $V$

Let's first suppose that we have an orthogonal basis for a subspace  $V$   $\{\vec{a}_0, \vec{a}_1, \vec{a}_2, \dots, \vec{a}_n\}$ . This means that  $\vec{a}_i \cdot \vec{a}_j = 0$  if  $i \neq j$ .

Recall that the projection of one vector onto another is defined by  $\text{comp}_{\vec{a}}(\vec{b}) = \frac{\vec{a} \cdot \vec{b}}{\|\vec{a}\|}$ .

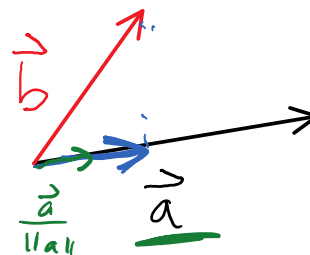
That is it is a measure of how much of the vector  $\vec{b}$  is in the direction of  $\vec{a}$ .

We could then take this amount and scale  $\vec{a}$  to make a vector in the same direction as  $\vec{a}$  but has length corresponding to the amount that  $\vec{b}$  is in the direction of  $\vec{a}$ .

How? Well take  $\vec{a}$  and divide it by its magnitude to create a unit vector in the same direction  $\frac{\vec{a}}{\|\vec{a}\|}$  then multiply it by the amount that  $\vec{b}$  is in the direction of  $\vec{a}$  namely the scalar we get from  $\text{comp}_{\vec{a}}(\vec{b})$ .

$$\vec{p} = \frac{\vec{a} \cdot \vec{b}}{\|\vec{a}\|} \frac{\vec{a}}{\|\vec{a}\|} = \frac{\vec{a} \cdot \vec{b}}{\vec{a} \cdot \vec{a}} \vec{a}$$

We then have

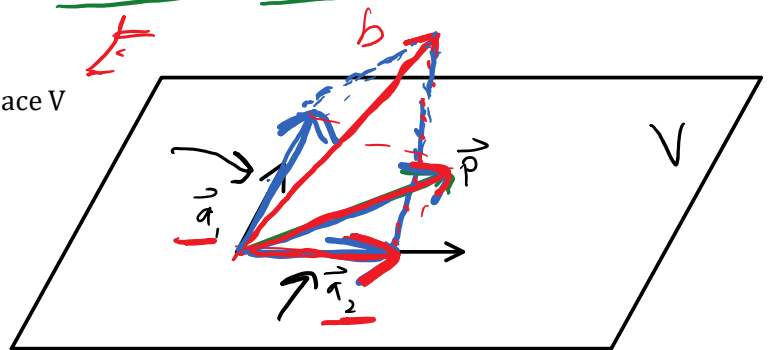


$$\vec{a} \cdot \vec{a} = \|\vec{a}\|^2$$

It turns out that if we do this to  $\vec{b}$  with each vector in our orthogonal basis  $\{\vec{a}_0, \vec{a}_1, \vec{a}_2, \dots, \vec{a}_n\}$  and add them up, we end up with the orthogonal projection of  $\vec{b}$  on to the subspace  $V$  in terms of the orthogonal basis of  $V$ .

The orthogonal projection of the vector  $\vec{b}$  on to the subspace  $V$  is given by:

$$\vec{p} = \frac{\vec{a}_1 \cdot \vec{b}}{\vec{a}_1 \cdot \vec{a}_1} \vec{a}_1 + \frac{\vec{a}_2 \cdot \vec{b}}{\vec{a}_2 \cdot \vec{a}_2} \vec{a}_2 + \dots + \frac{\vec{a}_n \cdot \vec{b}}{\vec{a}_n \cdot \vec{a}_n} \vec{a}_n$$



$$\frac{\vec{a}_1 \cdot \vec{b}}{\vec{a}_1 \cdot \vec{a}_1} \vec{a}_1 + \frac{\vec{a}_2 \cdot \vec{b}}{\vec{a}_2 \cdot \vec{a}_2} \vec{a}_2 = \vec{p}$$

$$c_1 \vec{a}_1 + c_2 \vec{a}_2 = \vec{p}$$

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It turns out that the vector  $\vec{p}$  that we found is the best approximation of  $\vec{b}$  using the basis of  $V$   $\{\vec{a}_0, \vec{a}_1, \vec{a}_2, \dots, \vec{a}_n\}$

While it may not seem particularly useful to approximate a known vector it does have its uses in inverse problems where we can't find an exact answer. For now though we might consider the case where  $\vec{b}$  lies in the subspace  $V$ . In this case  $\vec{p} = \vec{b}$  and our formula decomposes  $\vec{b}$  into a linear combo of our orthogonal basis. This idea becomes most useful for PDE's when we extend our notion of a "vector space" and a "dot product".

**Definition:** A vector space is a set  $V$  on which two operations  $+$  and  $\cdot$  are defined, called *vector addition* and *scalar multiplication*. **Note this  $\cdot$  is not the dot product here**

The operation  $+$  (vector addition) must satisfy the following conditions:

Closure: If  $\underline{u}$  and  $\underline{v}$  are any vectors in  $V$ , then the sum  $\underline{u + v}$  belongs to  $V$ .

(1) Commutative law: For all vectors  $\underline{u}$  and  $\underline{v}$  in  $V$ ,  $\underline{u + v = v + u}$

(2) Associative law: For all vectors  $\underline{u}, \underline{v}, \underline{w}$  in  $V$ ,  $\underline{u + (v + w) = (u + v) + w}$

(3) Additive identity: The set  $V$  contains an additive identity element, denoted by  $\underline{0}$ , such that for any vector  $\underline{v}$  in  $V$ ,  $\underline{0 + v = v}$  and  $\underline{v + 0 = v}$ .

(4) Additive inverses: For each vector  $\underline{v}$  in  $V$ , the equations  $\underline{v + x = 0}$  and  $\underline{x + v = 0}$  have a solution  $\underline{x}$  in  $V$ , called an additive inverse of  $\underline{v}$ , and denoted by  $\underline{-v}$ .

The operation  $\cdot$  (scalar multiplication) is defined between real numbers (or scalars) and vectors, and must satisfy the following conditions:

Closure: If  $\underline{v}$  is any vector in  $V$ , and  $c$  is any real number, then the product  $\underline{c \cdot v}$  belongs to  $V$ .

(5) Distributive law: For all real numbers  $c$  and all vectors  $\underline{u}, \underline{v}$  in  $V$ ,  $\underline{c \cdot (u + v) = c \cdot u + c \cdot v}$

(6) Distributive law: For all real numbers  $c, d$  and all vectors  $\underline{v}$  in  $V$ ,  $\underline{(c + d) \cdot v = c \cdot v + d \cdot v}$

(7) Associative law: For all real numbers  $c, d$  and all vectors  $\underline{v}$  in  $V$ ,  $\underline{c \cdot (d \cdot v) = (cd) \cdot v}$

(8) Unitary law: For all vectors  $\underline{v}$  in  $V$ ,  $\underline{1 \cdot v = v}$

Notice in the above definition  $V$  was just a set of general objects not necessarily vectors as we have used them.

In fact we can actually define a vector space of functions. For example let's take the set of all polynomials  $P$  on the real line and define a vector space. Let  $p(x), q(x) \in P$

Vector addition:  $(p + q)(x) = p(x) + q(x)$

Scalar multiplication:  $c \cdot f(x) = cf(x)$

Let's just check a few properties:

Is it closed under addition and scalar multiplication?

(1)

(2)

(3)

(4)

(5)

(6)

(7)

(8)

$$P(x) = \cancel{a_n x^n} + \dots + \cancel{a_0}$$

$$= P(x) = \cancel{a_n x^n} + \dots + \cancel{a_0}$$

0

$$P(x) = \underset{0}{a_n} x^n + \underset{0}{a_{n-1}} x^{n-1} + \dots + \underset{0}{a_0}$$

$$2P = 2x^2 + 4$$

$$P(x) = x^2 + 2, Q(x) = x^5 + 5x + 1, P + Q = x^5 + x^2 + 5x + 3$$

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What about a basis for  $P$ ? Well recall that a basis is a set of objects in our set which are linearly independent and span the set. For the set of all polynomials one basis is:

$$\{1, x, x^2, x^3, x^4, \dots\}$$

$\begin{matrix} \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\ \vec{a}_1 & \vec{a}_2 & \vec{a}_3 & \vec{a}_4 & \vec{a}_5 \end{matrix}$

$$p(x) = 5x^4 + 2x^2 + 1$$

Are they linearly independent?

$$c_1(1) + c_2(x) + c_3(x^2) + c_4(x^3) + \dots + c_n(x^n) = 0$$

$$c_1 = c_2 = \dots = c_n = 0$$

Do they span the set of all polynomials?

$$p(x) = 5\vec{a}_5 + 2\vec{a}_3 + 1\vec{a}_1 = 5x^4 + 2x^2 + 1$$

yes,  
polynomials

Ok so we have a basis for our vector space of functions. One natural question now is what does our "dot product" look like?

Well it turns out that the dot product is a special case of what we call an "inner product"

An **inner product** on a vector space  $V$  is a function that associates with each pair of vectors  $u$  and  $v$  in  $V$  a scalar number  $\langle u, v \rangle$  such that:

Dot Product

$$(1) \langle u, v \rangle = \langle v, u \rangle$$

$$(2) \langle u, v + w \rangle = \langle u, v \rangle + \langle u, w \rangle \rightarrow \vec{u} \cdot (\vec{v} + \vec{w})$$

$$(3) \langle cu, v \rangle = c \langle u, v \rangle \quad c\vec{u} \cdot \vec{v} = c(\vec{u} \cdot \vec{v})$$

$$(4) \langle u, u \rangle \geq 0 \text{ and } \langle u, u \rangle = 0 \text{ if and only if } u = 0$$

$$\langle x, y \rangle$$

$$\langle a, b \rangle$$

$$\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$$

$$\langle x, y \rangle \cdot \langle a, b \rangle$$

$$xa + yb$$

$$\langle a, b \rangle \cdot \langle x, y \rangle$$

$$ax + by$$

If we have a vector space  $V$  with together with an inner product we call this an inner product space. For function vector spaces our inner products are usually integrals of products of functions.

$$\vec{a} \cdot \vec{a} = \|\vec{a}\|^2$$



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An example of an inner product space: Let  $P^n[-1,1]$  be the set of continuous polynomials of degree at most  $n$  on the interval  $[-1,1]$ .

We define the inner product between two polynomials  $p(x), q(x) \in P^n[-1,1]$  to be:

$$\langle p(x), q(x) \rangle = \int_{-1}^1 p(x)q(x)dx$$

Let's verify that this is indeed an inner product:

$$\rightarrow (1) \langle u, v \rangle = \langle v, u \rangle$$

$$(2) \langle u, v+w \rangle = \langle u, v \rangle + \langle u, w \rangle$$

$$(3) \langle cu, v \rangle = c \langle u, v \rangle$$

$$(4) \langle u, u \rangle \geq 0 \text{ and } \langle u, u \rangle = 0 \text{ if and only if } u = 0$$

$$(1) \int_{-1}^1 u(x)v(x)dx = \int_{-1}^1 v(x)u(x)dx$$

$$(2) \int_{-1}^1 u(x)(v(x)+w(x))dx = \int_{-1}^1 u(x)v(x)dx + \int_{-1}^1 u(x)w(x)dx$$

$$= \int_{-1}^1 u(x)v(x) + u(x)w(x)dx$$

$$= \int_{-1}^1 u(x)(v(x)+w(x))dx$$

$$(3) \langle cu, v \rangle = \int_{-1}^1 cu(x)v(x)dx = c \int_{-1}^1 u(x)v(x)dx$$

$$(4) \int_{-1}^1 u(x)u(x)dx = \int_{-1}^1 u^2(x)dx \geq 0$$

In checking (4) we might think we can use the inner product to measure the "length" of a function in fact we can, we define the norm of a function  $f$  on our space as:

$$\langle f, f \rangle = \int_{-1}^1 f \cdot f dx = \int_{-1}^1 f^2 dx$$

$$\|f\| = \left( \int_{-1}^1 (f(x))^2 dx \right)^{\frac{1}{2}} = \sqrt{\langle f, f \rangle}$$

$$\vec{a} \cdot \vec{a} = \langle \vec{a}, \vec{a} \rangle$$

$$\sqrt{\vec{a} \cdot \vec{a}} = \|\vec{a}\|$$

$$\|\vec{a}\|^2$$

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As we stated before the dot product we are used to is simply a special case of an inner product we might also recall that for conventional vectors if their dot product is zero they are orthogonal.

We extend this notion to all inner product spaces and say that:

Two general vectors  $u$  and  $v$  are orthogonal if and only if  $\langle u, v \rangle = 0$

Let's see if our basis for the set of all polynomials on  $[-1, 1]$   $\{1, x, x^2, x^3, x^4, \dots\}$  is an orthogonal basis:

Let's try  $x^2$  and  $x$

$\swarrow$  even       $\searrow$  odd

$$\langle x^2, x \rangle = \int_{-1}^1 x^2 \cdot x \, dx = \int_{-1}^1 x^3 \, dx$$

$$= \frac{x^4}{4} \Big|_{-1}^1 = \frac{1^4}{4} - \frac{(-1)^4}{4} = \frac{1}{4} - \frac{1}{4} = 0$$

Let's try  $x^2$  and  $x^4$

$\swarrow$  even       $\searrow$  even

$$\langle x^2, x^4 \rangle = \int_{-1}^1 x^2 x^4 \, dx = \int_{-1}^1 x^6 \, dx = \frac{x^7}{7} \Big|_{-1}^1$$

$$= \frac{1^7}{7} - \frac{(-1)^7}{7} = \frac{1}{7} - \left(-\frac{1}{7}\right) = \frac{2}{7} \neq 0$$

$$\int_{-b}^b \text{even} \cdot \text{odd} = 0$$

$$\sin(nx)$$

$$\cos(nx)$$

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For reasons that will become apparent in a bit we prefer to have orthogonal bases. Thanks to two math-magicians, Gram and Schmidt, we have an algorithm for turning any given basis into an orthogonal basis.

Gram-Schmidt Orthogonalization:

Given any set of linearly independent vectors  $v_1, v_2, v_3 \dots v_n$  we can obtain a set of orthogonal vectors  $u_1, u_2, u_3, \dots u_n$  that span the same space with the following algorithm.

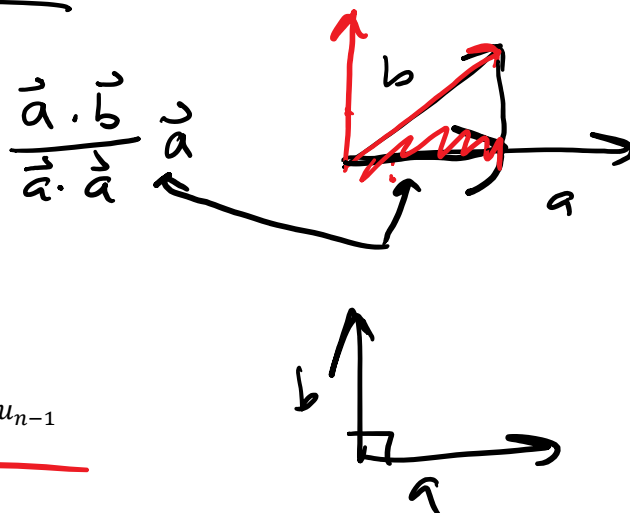
$$u_1 = v_1$$

$$u_2 = v_2 - \frac{\langle u_1, v_2 \rangle}{\langle u_1, u_1 \rangle} u_1$$

$$u_3 = v_3 - \frac{\langle u_1, v_3 \rangle}{\langle u_1, u_1 \rangle} u_1 - \frac{\langle u_2, v_3 \rangle}{\langle u_2, u_2 \rangle} u_2$$

$\vdots$

$$u_n = v_n - \frac{\langle u_1, v_n \rangle}{\langle u_1, u_1 \rangle} u_1 - \frac{\langle u_2, v_n \rangle}{\langle u_2, u_2 \rangle} u_2 - \dots - \frac{\langle u_{n-1}, v_n \rangle}{\langle u_{n-1}, u_{n-1} \rangle} u_{n-1}$$



In order to get a sense for why this works let's think of our conventional vectors. For the vectors we are used to the inner product becomes a dot product.

$$u_1 = v_1$$

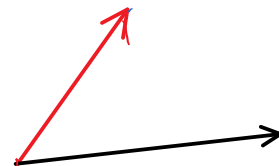
$$u_2 = v_2 - \frac{u_1 \cdot v_2}{u_1 \cdot u_1} u_1$$

$$u_3 = v_3 - \frac{u_1 \cdot v_3}{u_1 \cdot u_1} u_1 - \frac{u_2 \cdot v_3}{u_2 \cdot u_2} u_2$$

$\vdots$

$$u_n = v_n - \frac{u_1 \cdot v_n}{u_1 \cdot u_1} u_1 - \frac{u_2 \cdot v_n}{u_2 \cdot u_2} u_2 - \dots - \frac{u_{n-1} \cdot v_n}{u_{n-1} \cdot u_{n-1}} u_{n-1}$$

$$\vec{p} = \frac{\vec{a} \cdot \vec{b}}{||\vec{a}||} \frac{\vec{a}}{||\vec{a}||} = \frac{\vec{a} \cdot \vec{b}}{\vec{a} \cdot \vec{a}} \vec{a}$$



So for each new basis vector  $u_i$  we are just removing the bits that are parallel to  $v_i$  ! It is much the same with the general algorithm we are removing the bits that are not orthogonal.

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If we apply this algorithm to our set of basis vectors for  $P^n[-1,1]$   $\{1, x, x^2, x^3, x^4, \dots\}$

we obtain multiples of a famous set of polynomials called Legendre Polynomials of which the first 6 are:

Gram-Schmidt Orthogonalization:

Given any set of linearly independent vectors  $v_1, v_2, v_3, \dots, v_n$  we can obtain a set of orthogonal vectors  $u_1, u_2, u_3, \dots, u_n$  that span the same space with the following algorithm.

$$p_0 = 1$$

$$p_1 = x$$

$$p_2 = \frac{1}{2}(3x^2 - 1)$$

$$p_3 = \frac{1}{2}(5x^3 - 3x)$$

$$p_4 = \frac{1}{8}(35x^4 - 30x^2 + 3)$$

$$p_5 = \frac{1}{8}(63x^5 - 70x^3 + 15x)$$

$$u_1 = v_1$$

$$u_2 = v_2 - \frac{\langle u_1, v_2 \rangle}{\langle u_1, u_1 \rangle} u_1$$

$$u_3 = v_3 - \frac{\langle u_1, v_3 \rangle}{\langle u_1, u_1 \rangle} u_1 - \frac{\langle u_2, v_3 \rangle}{\langle u_2, u_2 \rangle} u_2$$

$$\vdots$$

$$u_n = v_n - \frac{\langle u_1, v_n \rangle}{\langle u_1, u_1 \rangle} u_1 - \frac{\langle u_2, v_n \rangle}{\langle u_2, u_2 \rangle} u_2 - \dots - \frac{\langle u_{n-1}, v_n \rangle}{\langle u_{n-1}, u_{n-1} \rangle} u_{n-1}$$

Let's calculate the first two, you get to do the rest in your homework. :) Note that you will get a multiple of the ones listed... Think of them as just scaled versions of the same basis vectors.

$$p_0 = 1$$

$$p_1 = x - \frac{\langle 1, x \rangle}{\langle 1, 1 \rangle} = x - \frac{\int_{-1}^1 1x \, dx}{\int_{-1}^1 1(1) \, dx}$$

$$= x - \frac{\frac{x^2}{2} \Big|_{-1}^1}{x \Big|_{-1}^1} = x - \frac{\frac{1}{2} - (-\frac{1}{2})}{1 - (-1)}$$

$$= x - \frac{1/2 - (-1/2)}{2} = x - 0$$

$\boxed{x}$

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Ok so what is the point of all of this math... Recall earlier that if we had a vector  $\vec{p}$  in a space  $V$  we could decompose it into a linear combo the orthogonal basis vectors for  $V$  with the formula

$$\vec{p} = \frac{\vec{a}_1 \cdot \vec{b}}{\vec{a}_1 \cdot \vec{a}_1} \vec{a}_1 + \frac{\vec{a}_2 \cdot \vec{b}}{\vec{a}_2 \cdot \vec{a}_2} \vec{a}_2 + \dots + \frac{\vec{a}_n \cdot \vec{b}}{\vec{a}_n \cdot \vec{a}_n} \vec{a}_n$$

It turns out we can do the same thing with functions! All we have to do is replace the dot product with the appropriate inner product that is for a given function  $f(x) \in C([a, b])$  (the set of continuous functions on the interval  $[a, b]$ ) if we have an orthogonal basis for  $C([a, b])$   $\{g_1(x), g_2(x), \dots\}$

We can write:

$$f(x) = \frac{\langle g_1, f \rangle}{\langle g_1, g_1 \rangle} g_1 + \frac{\langle g_2, f \rangle}{\langle g_2, g_2 \rangle} g_2 + \dots + \frac{\langle g_n, f \rangle}{\langle g_n, g_n \rangle} g_n + \dots$$

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!}$$

This means that we can represent a given function as a linear combination of basis functions!

One example that we will see in this course is the representation of a function using sine and cosines, or just one or the other. That is for the right kind of function  $f$  we will be able to write it as a linear combination of basis functions. For example we will see what we call a sine series for a function defined on an interval  $[0, L]$

$$f(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}x\right)$$

Where

$$B_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi}{L}x\right) dx = \frac{\int_0^L f(x) \sin\left(\frac{n\pi}{L}x\right) dx}{\int_0^L \sin^2\left(\frac{n\pi}{L}x\right) dx} = \frac{\langle \sin\left(\frac{n\pi}{L}x\right), f(x) \rangle}{\langle \sin\left(\frac{n\pi}{L}x\right), \sin\left(\frac{n\pi}{L}x\right) \rangle}$$

If we define the inner product

$$\langle f, g \rangle = \int_0^L f(x)g(x)dx \text{ we are simply saying:}$$

$$f(x) = \frac{\langle g_1, f \rangle}{\langle g_1, g_1 \rangle} g_1 + \frac{\langle g_2, f \rangle}{\langle g_2, g_2 \rangle} g_2 + \dots + \frac{\langle g_n, f \rangle}{\langle g_n, g_n \rangle} g_n + \dots$$

With  $g_n = \sin\left(\frac{n\pi x}{L}\right)$  which are indeed orthogonal that is:

$$\langle \sin\left(\frac{n\pi x}{L}\right), \sin\left(\frac{m\pi x}{L}\right) \rangle = \int_0^L \sin\left(\frac{n\pi}{L}x\right) \sin\left(\frac{m\pi}{L}x\right) dx = \begin{cases} 0 & \text{when } m \neq n \\ \frac{L}{2} & \text{when } m = n \end{cases}$$

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It turns out that an orthogonal basis for the set of piecewise smooth functions on the interval  $[-L, L]$  is:

$$\left\{ 1, \sin\left(\frac{\pi x}{L}\right), \sin\left(\frac{2\pi x}{L}\right), \dots, \cos\left(\frac{\pi x}{L}\right), \cos\left(\frac{2\pi x}{L}\right), \dots \right\}$$

This means if we have any piecewise smooth function on the interval  $[-L, L]$  we could represent it as a sum of a constant and sines and cosines with the appropriate coefficients determined by an inner product.

For the interval  $[-L, L]$  we would use the innerproduct  $\langle f, g \rangle = \int_{-L}^L f(x)g(x)dx$

That is we can write

$$\begin{aligned} f(x) &= \frac{\langle 1, f \rangle}{\langle 1, 1 \rangle} 1 + \frac{\langle \sin\left(\frac{\pi x}{L}\right), f \rangle}{\langle \sin\left(\frac{\pi x}{L}\right), \sin\left(\frac{\pi x}{L}\right) \rangle} \sin\left(\frac{\pi x}{L}\right) + \frac{\langle \sin\left(\frac{2\pi x}{L}\right), f \rangle}{\langle \sin\left(\frac{2\pi x}{L}\right), \sin\left(\frac{2\pi x}{L}\right) \rangle} \sin\left(\frac{2\pi x}{L}\right) + \dots \\ &+ \frac{\langle \cos\left(\frac{\pi x}{L}\right), f \rangle}{\langle \cos\left(\frac{\pi x}{L}\right), \cos\left(\frac{\pi x}{L}\right) \rangle} \cos\left(\frac{\pi x}{L}\right) + \frac{\langle \cos\left(\frac{2\pi x}{L}\right), f \rangle}{\langle \cos\left(\frac{2\pi x}{L}\right), \cos\left(\frac{2\pi x}{L}\right) \rangle} \cos\left(\frac{2\pi x}{L}\right) + \dots \end{aligned}$$

And

$$\frac{\langle 1, f \rangle}{\langle 1, 1 \rangle} = \frac{\int_{-L}^L f(x)dx}{\int_{-L}^L dx} = \frac{\int_{-L}^L f(x)dx}{2L} = \frac{1}{L} \int_{-L}^L f(x)dx = A_0$$

$$\frac{\langle \sin\left(\frac{n\pi x}{L}\right), f \rangle}{\langle \sin\left(\frac{n\pi x}{L}\right), \sin\left(\frac{n\pi x}{L}\right) \rangle} = \frac{\int_{-L}^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx}{\int_{-L}^L \sin^2\left(\frac{n\pi x}{L}\right) dx} = \frac{\int_{-L}^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx}{L} = \frac{1}{L} \int_{-L}^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx = B_n$$

$$\frac{\langle \cos\left(\frac{n\pi x}{L}\right), f \rangle}{\langle \cos\left(\frac{n\pi x}{L}\right), \cos\left(\frac{n\pi x}{L}\right) \rangle} = \frac{\int_{-L}^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx}{\int_{-L}^L \cos^2\left(\frac{n\pi x}{L}\right) dx} = \frac{\int_{-L}^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx}{L} = \frac{1}{L} \int_{-L}^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx = A_n$$

So we can then represent our sum in the more concise form of:

$$f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi}{L}x\right) + \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}x\right)$$

With this choice of basis functions we call this a Fourier series!

# Sec 4.10

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