

Sec 15.1-15.4

Behold! The 1-d wave equation!

$$\rho_0(x) \frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x} \left(T(x, t) \frac{\partial u}{\partial x} \right) + \rho_0(x) Q(x, t)$$

We get some variations of this equations making a few assumptions, if the tension does not change much in the string then $T(x, t) = \text{constant}$ so we could have:

$$\rho_0(x) \frac{\partial^2 u}{\partial t^2} = \left(T_0 \frac{\partial^2 u}{\partial x^2} \right) + \rho_0(x) Q(x, t)$$

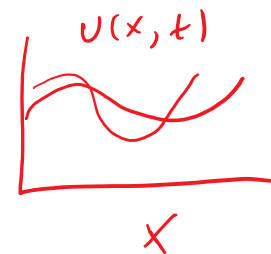
If there are no body forces or we let a sagged state of the string be it's equilibrium position we can ditch $Q(x, t)$ and obtain:

$$\rho_0(x) \frac{\partial^2 u}{\partial t^2} = T_0 \frac{\partial^2 u}{\partial x^2}$$

If we divide by $\rho_0(x)$ we obtain the most commonly written form of this PDE

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

$$\text{where } c^2 = \frac{T_0}{\rho_0(x)}$$



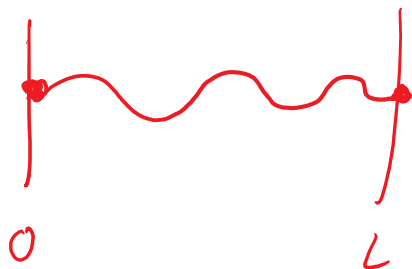
$u(x, t)$ vertical displacement

What kinds of boundary and initial conditions can we have?

Common Boundary Conditions

Fixed ends: $u(0, t) = 0$ or $u(L, t) = 0$

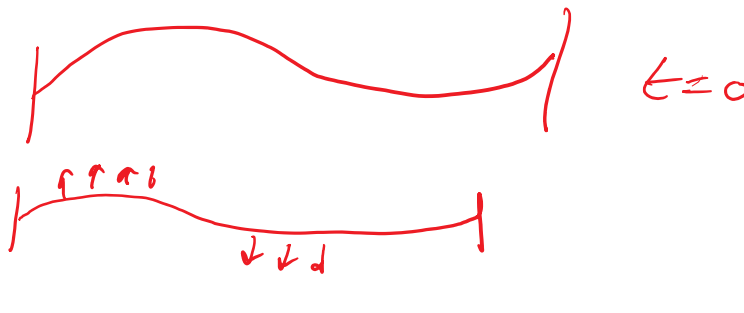
Forced ends: $u(0, t) = f(t)$ or $u(L, t) = f(t)$

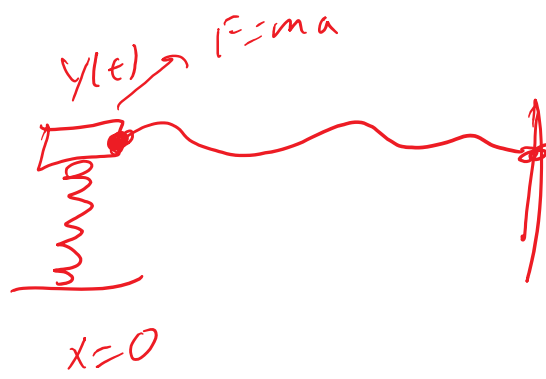


Common Initial Conditions

Initial shape of string: $u(x, 0) = f(x)$

Initial velocity of string: $\frac{\partial u}{\partial t}(x, 0) = g(x)$





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Solving the wave equation

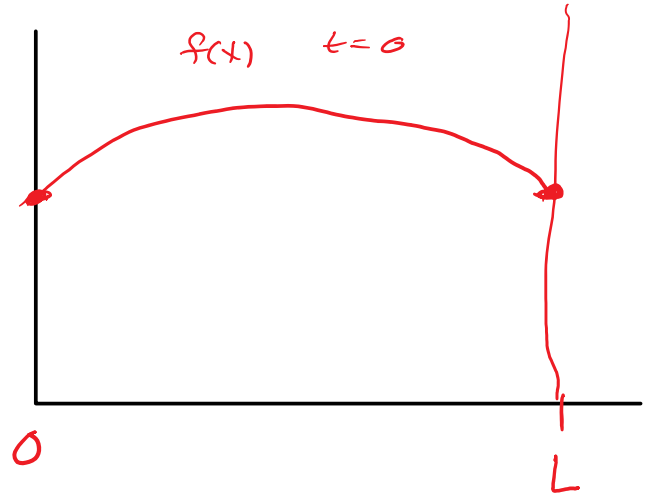
We are going to solve the wave equation assuming a uniform density $\rho_0(x) = \rho_0 = \text{constant}$ and constant tension T_0 and with two fixed ends a general initial shape and general initial velocity.

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \text{where } c^2 = \frac{T_0}{\rho_0(x)}$$

$$\underline{u(0, t) = 0} \quad \text{and} \quad \underline{u(L, t) = 0}$$

$$\text{Initial shape of string: } \underline{u(x, 0) = f(x)}$$

$$\text{Initial velocity of string: } \underline{\frac{\partial u}{\partial t}(x, 0) = g(x)}$$



We begin the usual way by separating variables we assume: $u(x, t) = \phi(x)h(t)$

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

$$\frac{\partial^2}{\partial t^2} (\phi(x) h(t)) = c^2 \frac{\partial^2}{\partial x^2} \phi(x) h(t)$$

$$\frac{\phi(x) \frac{\partial^2 h}{\partial t^2}}{\phi(x) h(t)} = c^2 \frac{h(t) \frac{\partial^2 \phi}{\partial x^2}}{\phi(x) h(t)}$$

$$\frac{1}{h(t)} \frac{\partial^2 h}{\partial t^2} = c^2 \frac{1}{\phi} \frac{\partial^2 \phi}{\partial x^2}$$

$$\frac{1}{c^2 h(t)} \frac{\partial^2 h}{\partial t^2} = \frac{1}{\phi} \frac{\partial^2 \phi}{\partial x^2} = \lambda$$

$$\frac{1}{c^2 h(t)} \frac{\partial^2 h}{\partial t^2} = \lambda$$

$$\frac{1}{\phi} \frac{\partial^2 \phi}{\partial x^2} = \lambda$$

$\lambda < 0$

$$\frac{\partial^2 \phi}{\partial x^2} + \lambda \phi = 0$$

$$r^2 + \lambda = 0$$

$$r^2 = -\lambda$$

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Think about what we might want our solution to be, we choose $\lambda > 0$ and attach a minus sign, that is we expect to have two oscillatory solutions here. It does turn out that for $\lambda = 0$ and $\lambda < 0$ we get only trivial solutions.

$$\frac{1}{c^2} \frac{1}{h} \frac{d^2 h}{dt^2} = \frac{1}{\phi} \frac{d^2 \phi}{dx^2} = -\lambda$$

Let's take a look at our equation for ϕ

$$\frac{1}{\phi} \frac{d^2 \phi}{dx^2} = -\lambda$$

$\phi(0) = 0$ and $\phi(L) = 0$

$U(0,t) = 0$
 $\phi(0)h(t) = 0$
 $\Rightarrow \phi(0) = 0$

$U(L,t) = 0$
 $\phi(L)h(t) = 0$
 $\phi(L) = 0$

Any guesses for how this will play out?

$$\frac{d^2 \phi}{dx^2} + \lambda \phi = 0$$

$$r^2 + \lambda = 0$$

$$r^2 = -\lambda$$

$$r = \pm \sqrt{-\lambda} i$$

$$\phi(x) = C_1 \cos(\sqrt{\lambda} x) + C_2 \sin(\sqrt{\lambda} x)$$

$$\phi(0) = 0 = C_1 \cos(0) + C_2 \sin(0) \Rightarrow C_1 = 0$$

$$\phi(L) = 0 = C_2 \sin(\sqrt{\lambda} L) = 0$$

$$\sqrt{\lambda} L = n\pi \rightarrow \lambda = \left(\frac{n\pi}{L}\right)^2$$

$$\phi(x) = C \sin\left(\frac{n\pi}{L} x\right)$$

Now let's look at our time dependent ODE for which we have no homogeneous boundary conditions.

$$\frac{1}{c^2} \frac{1}{h} \frac{d^2 h}{dt^2} = -\lambda$$

$$\frac{d^2 h}{dt^2} = -\lambda c^2 h$$

$$\frac{d^2 h}{dt^2} + \lambda c^2 h = 0$$

$$r^2 + \lambda c^2 = 0$$

$$r^2 = -\lambda c^2$$

$$\lambda > 0$$

$$r = \pm \sqrt{-\lambda c^2} = \pm \sqrt{-\lambda} c = \pm \sqrt{\lambda} c i$$

$$\Rightarrow h(t) = C_1 \cos(\sqrt{\lambda} c t) + C_2 \sin(\sqrt{\lambda} c t)$$

$$h(t) = C_1 \cos\left(\frac{n\pi c}{L} t\right) + C_2 \sin\left(\frac{n\pi c}{L} t\right)$$

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So we have $\phi_n(x) = c_2 \sin\left(\frac{n\pi x}{L}\right)$ and $h_n(t) = c_3 \cos\left(\frac{cn\pi}{L}t\right) + c_4 \sin\left(\frac{cn\pi}{L}t\right)$

Thus taking the product of the solutions

$$u_n(x, t) = c_2 \sin\left(\frac{n\pi x}{L}\right) \left(c_3 \cos\left(\frac{cn\pi}{L}t\right) + c_4 \sin\left(\frac{cn\pi}{L}t\right) \right) = A_n \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{cn\pi}{L}t\right) + B_n \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{cn\pi}{L}t\right)$$

Applying super position

$$u(x, t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{cn\pi}{L}t\right) + B_n \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{cn\pi}{L}t\right)$$

Now we can apply our initial conditions. $u(x, 0) = f(x)$ $\frac{\partial u}{\partial t}(x, 0) = g(x)$

$$u(x, 0) = f(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right) \cos(0) + B_n \sin\left(\frac{n\pi x}{L}\right) \sin(0)$$

$$f(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right)$$

$$A_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi}{L}x\right) dx$$

$$\frac{\partial u}{\partial t} = \sum_{n=1}^{\infty} -A_n \sin\left(\frac{n\pi x}{L}\right) \frac{cn\pi}{L} \sin\left(\frac{cn\pi}{L}t\right) + B_n \sin\left(\frac{n\pi x}{L}\right) \frac{cn\pi}{L} \cos\left(\frac{cn\pi}{L}t\right)$$

$$\frac{\partial u}{\partial t} \Big|_{t=0} = \sum_{n=1}^{\infty} -A_n \sin\left(\frac{n\pi x}{L}\right) \frac{cn\pi}{L} \sin(0) + B_n \sin\left(\frac{n\pi x}{L}\right) \frac{cn\pi}{L} \cos(0) = g(x)$$

$$\Rightarrow g(x) = \sum_{n=1}^{\infty} B_n \left(\frac{cn\pi}{L}\right) \sin\left(\frac{n\pi x}{L}\right) \Rightarrow B_n \left(\frac{cn\pi}{L}\right) = \frac{2}{L} \int_0^L g(x) \sin\left(\frac{n\pi}{L}x\right) dx$$

$$B_n = \frac{2}{cn\pi} \int_0^L g(x) \sin\left(\frac{n\pi}{L}x\right) dx$$

$$B_n = \frac{2}{cn\pi} \int_0^L g(x) \sin\left(\frac{n\pi}{L}x\right) dx$$

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So we have initial condition equations:

$$f(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right)$$

We then find our A_n and B_n in any of the usual ways.....

$$g(x) = \sum_{n=1}^{\infty} B_n \frac{n\pi c}{L} \sin\left(\frac{n\pi x}{L}\right)$$

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So we have our solution:

$$u(x, t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right) \left[A_n \cos\left(\frac{n\pi ct}{L}\right) + B_n \sin\left(\frac{n\pi ct}{L}\right) \right]$$

Where

$$A_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

$$B_n = \frac{2}{n\pi c} \int_0^L g(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

You might recall that it is possible to write

$$A_n \cos(\omega t) + B_n \sin(\omega t) = \sqrt{A_n^2 + B_n^2} \sin(\omega t - \phi)$$

For us, $\omega = \frac{n\pi c}{L}$

Note though that the amplitude of the wave is given by $\sqrt{A_n^2 + B_n^2}$. If we are vibrating a string it might be apparent to you that the larger the amplitude the louder the sound.

Consider $B_n = \frac{2}{n\pi c} \int_0^L g(x) \sin\left(\frac{n\pi x}{L}\right) dx$

For what value of n would this number be the largest?

Scaling by $\frac{1}{n}$

$$n=1$$

$$\frac{2}{n\pi c}$$

So then when B_n is at its largest we would have our largest amplitude $\sqrt{A_n^2 + B_n^2}$ and for a vibrating string the sound that the corresponding frequency, $\frac{n\pi c}{L}$ produces would be the most audible. We call this the fundamental frequency!

One could then adjust which frequency is heard by adjusting c or L . Recall that $c = \sqrt{\frac{T_0}{\rho_0}}$ so the fundamental frequency

is $\omega = \frac{\pi \sqrt{T_0}}{L \sqrt{\rho_0}}$

$$\frac{n\pi c}{L}$$

$$\frac{n\pi \sqrt{T_0}}{L \sqrt{\rho_0}}$$

$$\omega = \frac{\pi \sqrt{T_0}}{L \sqrt{\rho_0}}$$

Increase in tension = increase in frequency and a higher pitch

Longer lengths = decrease in frequency and a lower pitch

Higher Density = decrease in frequency and a lower pitch, note on a guitar the lowest pitch string is the thickest and the highest pitch string is the thinnest.

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$$\frac{1}{n}$$

$\sum \frac{1}{n}$ Harmonic Series

Of course our solution is the sum of terms:
$$\sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right) \left[A_n \cos\left(\frac{n\pi ct}{L}\right) + B_n \sin\left(\frac{n\pi ct}{L}\right) \right]$$

Each n corresponds to a specific frequency, we call these the normal modes of vibration and our solution is a superposition of all of them, it is just that the first mode has the largest amplitude.

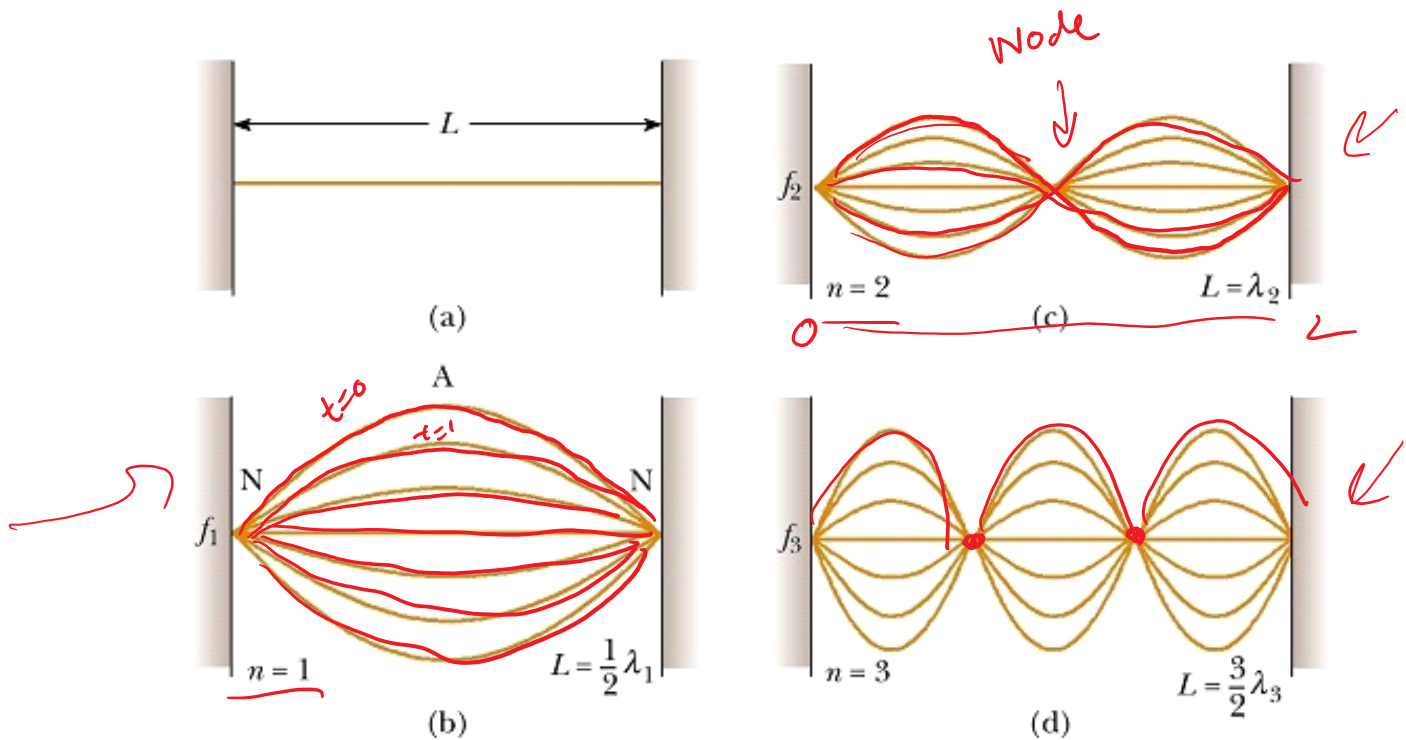


Figure 18.7 (a) A string of length L fixed at both ends. The normal modes of vibration form a harmonic series: (b) the fundamental, or first harmonic; (c) the second harmonic; (d) the third harmonic.

In the figure above each line is a normal mode plotted at a specific time, notice that the only thing that changes is the amplitude, also notice that there are places for $n > 1$ where the amplitude remains zero for all times t we call these nodes.

https://www.youtube.com/watch?v=6sgl7S_G-XI



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$$\frac{\partial u}{\partial t} = \frac{1}{2} [F'(x-ct)(-c) + F'(x+ct)(c)]$$

$$= \frac{1}{2} [-c F'(x-ct) + c F'(x+ct)]$$

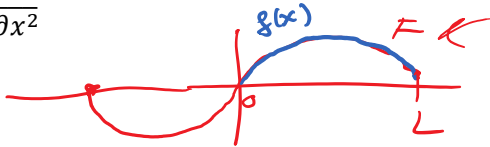
Homework Example: 15.4 #7

Suppose we have a vibrating string satisfying the following PDE problem.

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

$$u(0, t) = 0 \text{ and } u(L, t) = 0$$

$$\text{Initial shape of string: } u(x, 0) = f(x)$$



$$\text{Initial velocity of string: } \frac{\partial u}{\partial t}(x, 0) = 0$$

Show that: $u(x, t) = \frac{1}{2} [F(x-ct) + F(x+ct)]$ Where F is the odd extension of $f(x)$

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

$$d = vt$$

$$\frac{\partial^2}{\partial t^2} \frac{1}{2} [F(x-ct) + F(x+ct)] = c^2 \frac{\partial^2}{\partial x^2} \frac{1}{2} [F(x-ct) + F(x+ct)]$$

$$\frac{\partial}{\partial t} \left(\frac{\partial}{\partial t} F(x-ct) \right) = \frac{\partial}{\partial t} (F'(x-ct) \cdot (-c))$$

$$= F''(x-ct) \cdot c^2$$

$$\frac{d}{dx} \frac{d}{dx} (F(x-ct)) = \frac{d}{dx} F'(x-ct) \cdot 1$$

$$= F''(x-ct)$$

$$F = x^2 \quad F' = 2x$$

$$\frac{d}{dt} (x-ct)^2 = \frac{2(x-ct) \cdot (-c)}{F'}$$

$$F'(x-ct) = 2(x-ct)$$

$$\frac{1}{2} \left[\frac{\partial^2}{\partial t^2} F(x-ct) + \frac{\partial^2}{\partial t^2} F(x+ct) \right] = \frac{c^2}{2} \left[\frac{\partial^2}{\partial x^2} F(x-ct) + \frac{\partial^2}{\partial x^2} F(x+ct) \right]$$

$$\frac{1}{2} [F''(x-ct) c^2 + F''(x+ct) c^2] = \frac{c^2}{2} [F''(x-ct) + F''(x+ct)]$$

$$\frac{c^2}{2} [F''(x-ct) + F''(x+ct)] = \frac{c^2}{2} [F''(x-ct) + F''(x+ct)] \quad \checkmark$$

$$u(x, 0) = \frac{1}{2} (F(x-c(0)) + F(x+c(0))) = \frac{1}{2} (F(x) + F(x)) = F(x) = f(x) \quad \checkmark \text{ on } [0, L]$$

$$\frac{\partial u}{\partial t}(x, 0) = 0 = \frac{1}{2} (-c F'(x-0) + c F'(x+0))$$

$$= \frac{1}{2} [-c F'(x) + c F'(x)] = 0 \quad \checkmark$$

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A Damped String Homework Example 15.4 #3

Consider a damped string which satisfies..

$$\rho_0 \frac{\partial^2 u}{\partial t^2} = T_0 \frac{\partial^2 u}{\partial x^2} - \beta \frac{\partial u}{\partial t}$$

Handwritten notes: $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$, $\rho \frac{\partial^2 u}{\partial t^2} \approx m \vec{a} =$, $\frac{\partial u}{\partial t} > 0$, $\downarrow$ inc

a) Why is Beta > 0 to induce damping?

b) Determine the solution by separation of variables that satisfies the boundary conditions

$$u(0, t) = 0 \text{ and } u(L, t) = 0$$

$$u(x, 0) = f(x) \text{ and } \frac{\partial u}{\partial t}(x, 0) = g(x)$$

$$\text{We get to assume that } \beta^2 < \frac{4\pi^2 \rho_0}{L^2}$$

$$\rho_0 \frac{\partial^2 u}{\partial t^2} = T_0 \frac{\partial^2 u}{\partial x^2} - \beta \frac{\partial u}{\partial t}$$

$$u(x, t) = \phi(x) h(t)$$

$$\rho_0 \frac{\partial^2}{\partial t^2} \phi h = T_0 \frac{\partial^2}{\partial x^2} \phi h - \beta \frac{\partial}{\partial t} \phi h$$

$$\frac{\rho_0 \phi \frac{\partial^2 h}{\partial t^2}}{\phi h} = \frac{T_0 h \frac{\partial^2 \phi}{\partial x^2}}{\phi h} - \frac{\beta \phi \frac{\partial h}{\partial t}}{\phi h}$$

$$\frac{\rho_0}{h} \frac{\partial^2 h}{\partial t^2} = T_0 \frac{1}{\phi} \frac{\partial^2 \phi}{\partial x^2} - \beta \frac{1}{h} \frac{\partial h}{\partial t}$$

$$\frac{\rho_0}{h} \frac{\partial^2 h}{\partial t^2} + \beta \frac{1}{h} \frac{\partial h}{\partial t} = T_0 \frac{1}{\phi} \frac{\partial^2 \phi}{\partial x^2}$$

$$\frac{\rho_0}{h T_0} \frac{\partial^2 h}{\partial t^2} + \frac{\beta}{h T_0} \frac{\partial h}{\partial t} = \frac{1}{\phi} \frac{\partial^2 \phi}{\partial x^2} = -\lambda$$

$$\frac{\rho_0}{h T_0} \frac{\partial^2 h}{\partial t^2} + \frac{\beta}{h T_0} \frac{\partial h}{\partial t} = -\lambda$$

$$\frac{1}{\phi} \frac{\partial^2 \phi}{\partial x^2} = -\lambda$$

Handwritten notes: $\phi(0) = 0$, $\phi(L) = 0$, $\phi(x) = C \sin(\frac{n\pi}{L} x)$

λT_0

$$\frac{\rho_0}{h T_0} \frac{\partial^2 h}{\partial t^2} + \frac{\beta}{h T_0} \frac{\partial h}{\partial t} = -\lambda \quad \leftarrow$$

$$\rho_0 \frac{d^2 h}{dt^2} + \beta \frac{dh}{dt} = -\lambda T_0 h \quad \leftarrow$$

$$ax^2 + bx + c = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\rho_0 \frac{d^2 h}{dt^2} + \beta \frac{dh}{dt} + \lambda T_0 h = 0$$

$$\rho_0 r^2 + \beta r + \lambda T_0 = 0$$

$\begin{matrix} \text{''} & \text{''} & \text{''} \\ \underline{a} & \underline{b} & \underline{c} \end{matrix}$

$$r = \frac{-\beta \pm \sqrt{\beta^2 - 4\rho_0 \lambda T_0}}{2\rho_0}$$

$$\beta^2 < 4\rho_0 \lambda T_0$$

(1)

Imaginary roots
complex

$$r = \frac{-\beta \pm \frac{\sqrt{\beta^2 - 4\rho_0 \lambda T_0}}{2\rho_0} i}{2\rho_0} = \frac{-\beta}{2\rho_0} \pm \alpha i$$

$$\Rightarrow h(t) = e^{-\frac{\beta}{2\rho_0} t} (c_1 \cos(\alpha_n t) + c_2 \sin(\alpha_n t))$$

$$\phi(x) = C \sin \frac{n\pi}{L} x$$

$$u_n(x, t) = h(t) \phi(x) = e^{-\frac{\beta}{2\rho_0} t} (c_1 \cos(\alpha_n t) + c_2 \sin(\alpha_n t)) C \sin\left(\frac{n\pi}{L} x\right)$$

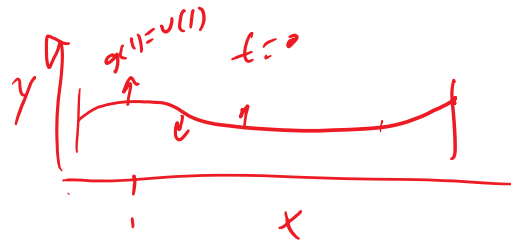
$$u(x, t) = \sum_{n=1}^{\infty} e^{-\frac{\beta}{2\rho_0} t} (A_n \cos(\alpha_n t) + B_n \sin(\alpha_n t)) \sin\left(\frac{n\pi}{L} x\right)$$

$$u(x, 0) = f(x) = \sum_{n=1}^{\infty} (A_n \cos(0) + B_n \sin(0)) \sin\left(\frac{n\pi}{L} x\right)$$

$$f(x) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi}{L} x$$

$$\Rightarrow A_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi}{L} x\right) dx$$

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$$\frac{\partial u}{\partial t}(x,0) = g(x)$$

$$u(x,t) = \sum_{n=1}^{\infty} e^{-\beta/2\rho_0 t} (A_n \cos(\alpha_n t) + B_n \sin(\alpha_n t)) \sin\left(\frac{n\pi}{L} x\right)$$

$$\frac{\partial u}{\partial t} = \sum_{n=1}^{\infty} \left[\frac{-\beta}{2\rho_0} e^{-\beta/2\rho_0 t} (A_n \cos(\alpha_n t) + B_n \sin(\alpha_n t)) + e^{-\beta/2\rho_0 t} (-A_n \alpha_n \sin(\alpha_n t) + B_n \alpha_n \cos(\alpha_n t)) \right] \sin\left(\frac{n\pi}{L} x\right)$$

$$\left. \frac{\partial u}{\partial t} \right|_{t=0} = g(x) = \sum_{n=1}^{\infty} \left[\frac{-\beta}{2\rho_0} (A_n \cos(0) + B_n \sin(0)) + (-A_n \alpha_n \sin(0) + B_n \alpha_n \cos(0)) \right] \sin\left(\frac{n\pi}{L} x\right)$$

$$= \sum_{n=1}^{\infty} \left(\frac{-\beta}{2\rho_0} A_n + B_n \alpha_n \right) \sin\left(\frac{n\pi}{L} x\right) = g(x)$$

$\sum \tilde{B}_n \sin \frac{n\pi}{L} x = f(x)$
 $\tilde{B}_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi}{L} x$

$$\tilde{B}_n = \frac{-\beta}{2\rho_0} A_n + B_n \alpha_n = \frac{2}{L} \int_0^L g(x) \sin\left(\frac{n\pi}{L} x\right) dx$$

$$\alpha_n B_n = \frac{2}{L} \int_0^L g(x) \sin \frac{n\pi}{L} x + \frac{\beta}{2\rho_0} A_n \quad \rightarrow \quad \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi}{L} x$$

$$B_n = \frac{1}{\alpha_n} \frac{2}{L} \int_0^L g(x) \sin \frac{n\pi}{L} x + \frac{1}{\alpha_n} \frac{\beta}{2\rho_0} \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi}{L} x\right)$$

$$B_n = \frac{2}{\alpha_n L} \left[\int_0^L \left(g(x) + \frac{\beta}{2\rho_0} f(x) \right) \sin\left(\frac{n\pi}{L} x\right) dx \right]$$