

Sec 15.1-15.2

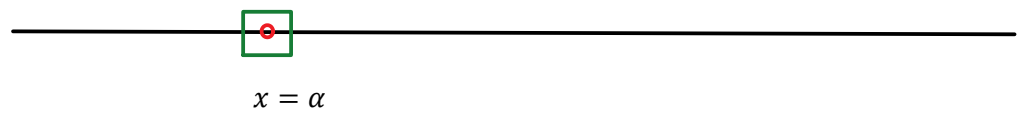
A vertically vibrating string.

Let's derive the PDE for a vertically vibrating string using Newton's second law, $\sum \vec{F} = m\vec{a}$

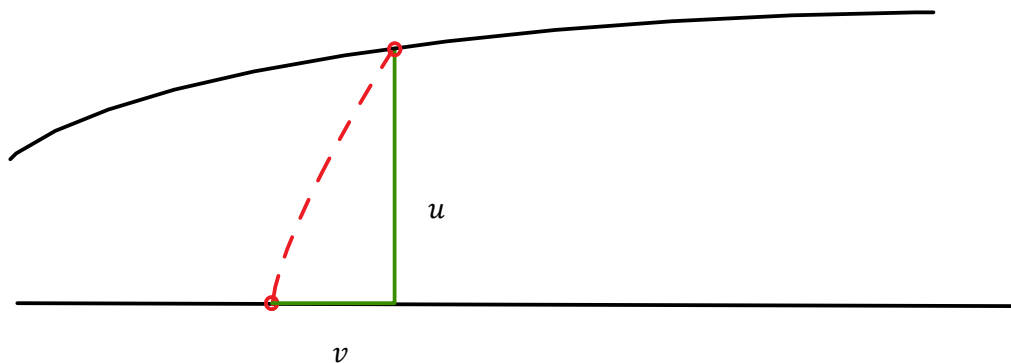
Suppose we have a string in its equilibrium position.

Let's single out a small particle on the string say at $x = \alpha$ and track the motion of that particle.

Equilibrium position.



Let's compare the equilibrium position to a perturbed position and name the Vertical displacement u and horizontal displacement v



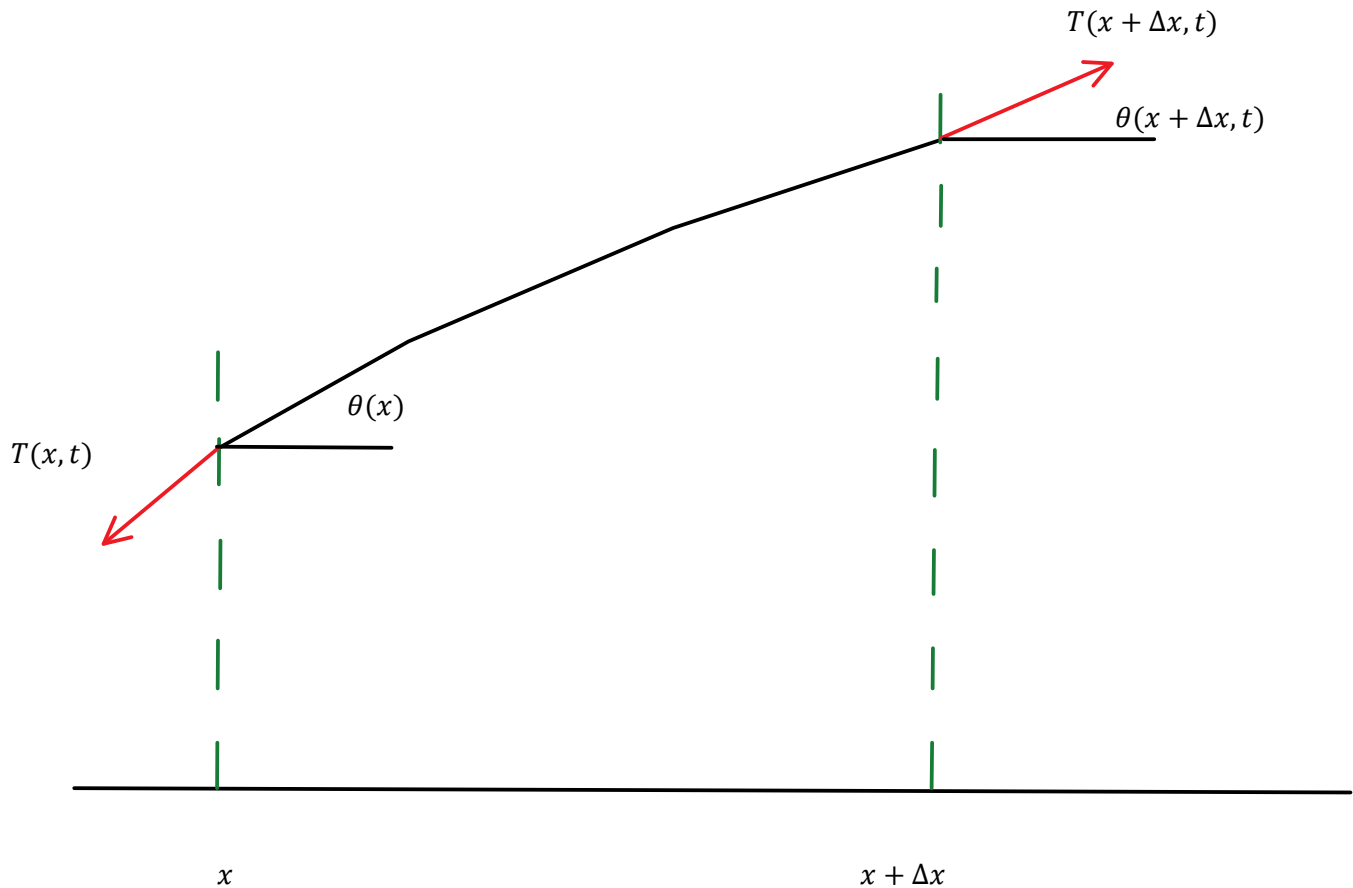
We notice that the vertical displacement of the point is much larger than the horizontal displacement for small enough perturbations. We will assume this is the case this will mean that our vertical displacement u will be a function of x and time only.

$$u = u(x, t)$$

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Now let's apply Newton's laws to a small section of the string of density $\rho_0(x)$ between x and $x + \Delta x$. If Δx is small enough we can simply use Newton's law for a point mass like the one we were tracking previously.

Let's consider the forces... We will have a tension T at both ends which will depend on both position x and time t . We will also have body forces such as gravity which we will wrap up into a term $Q(x, t)$ which will be the *body force per unit mass*



The mass of this section of string is the density times the length so: $m = \rho_0(x)\Delta x$

If we choose x to the right as positive then the sum of the forces is: $\sum \vec{F} = T(x + \Delta x, t) - T(x, t) + \rho_0(x)\Delta x Q(x, t)$

We claimed that the horizontal displacement was negligible thus we are only interested in finding an equation involving the y component of the displacement which we said was a function of x and time.

$u(x, t)$ Thus we will need only the vertical components of our forces to use with Newton's second law

$$F = ma \text{ where } a_y = \frac{\partial^2 u}{\partial t^2}$$



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So we have:

$$m = \rho_0(x)\Delta x \quad a_y = \frac{\partial^2 u}{\partial t^2} \quad \sum \vec{F}_y = T(x + \Delta x, t)\sin(\theta(x + \Delta x, t)) - T(x, t)\sin(\theta(x, t)) + \rho_0(x)\Delta x Q(x, t)$$

Thus:

$$\rho_0(x)\Delta x \frac{\partial^2 u}{\partial t^2} = T(x + \Delta x, t)\sin(\theta(x + \Delta x, t)) - T(x, t)\sin(\theta(x, t)) + \rho_0(x)\Delta x Q(x, t)$$

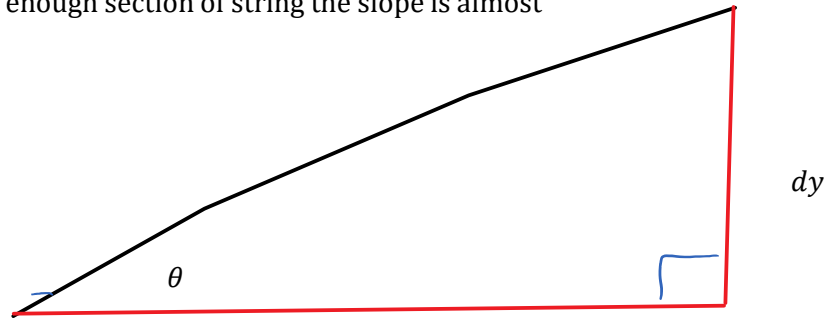
Now we want to get rid of Δx so we will divide by it and take the limit as $\Delta x \rightarrow 0$

So now we have

$$\rho_0(x) \frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x} (T(x, t) \sin(\theta(x, t))) + \rho_0(x) Q(x, t)$$

Now let's think about that $\sin(\theta(x, t))$ term.. For a small enough section of string the slope is almost constant

$$\text{slope} = \frac{dy}{dx} = \tan(\theta(x, t)) = \frac{\partial u}{\partial x}$$



and for small enough vertical displacements we have angles close to zero

$$\text{thus } \cos(\theta) \approx 1 \text{ and } \tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} \approx \sin(\theta) \quad dx$$

If we put this all together we can get an equation for u in terms of just t and x !

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Behold! The 1-d wave equation!

$$\rho_0(x) \frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x} \left(T(x, t) \frac{\partial u}{\partial x} \right) + \rho_0(x) Q(x, t)$$

We get some variations of this equations making a few assumptions, if the tension does not change much in the string then $T(x, t) = \text{constant}$ so we could have:

$$\rho_0(x) \frac{\partial^2 u}{\partial t^2} = \left(T_0 \frac{\partial^2 u}{\partial x^2} \right) + \rho_0(x) Q(x, t)$$

If there are no body forces or we let a sagged state of the string be it's equilibrium position we can ditch $Q(x, t)$ and obtain:

$$\rho_0(x) \frac{\partial^2 u}{\partial t^2} = T_0 \frac{\partial^2 u}{\partial x^2}$$

If we divide by $\rho_0(x)$ we obtain the most commonly written form of this PDE

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \text{where } c^2 = \frac{T_0}{\rho_0(x)}$$

HW Example 15.2 #1 a

Using $\rho_0(x) \frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x} \left(T_0 \frac{\partial u}{\partial x} \right) + \rho_0 Q(x, t)$ compute the sagged equilibrium position $u_E(x)$ if $Q(x, t) = -g$ With boundary conditions $u(0) = 0$ and $u(L) = 0$

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HW Example 15.2 #1 a

Using $\rho_0(x) \frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x} \left(T_0 \frac{\partial u}{\partial x} \right) + \rho_0 Q(x, t)$ compute the sagged equilibrium position $u_E(x)$ if $Q(x, t) = -g$ With boundary conditions $u(0) = 0$ and $u(L) = 0$

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15.2 #1 b) Show that if $u(x, t)$ is a solution to our equation in part a) $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} - g$ That the function $v(x, t) = u(x, t) - u_E(x)$ solves the simplified wave equation $\frac{\partial^2 v}{\partial t^2} = c^2 \frac{\partial^2 v}{\partial x^2}$ for v