

Contour integrals involving branch cuts.

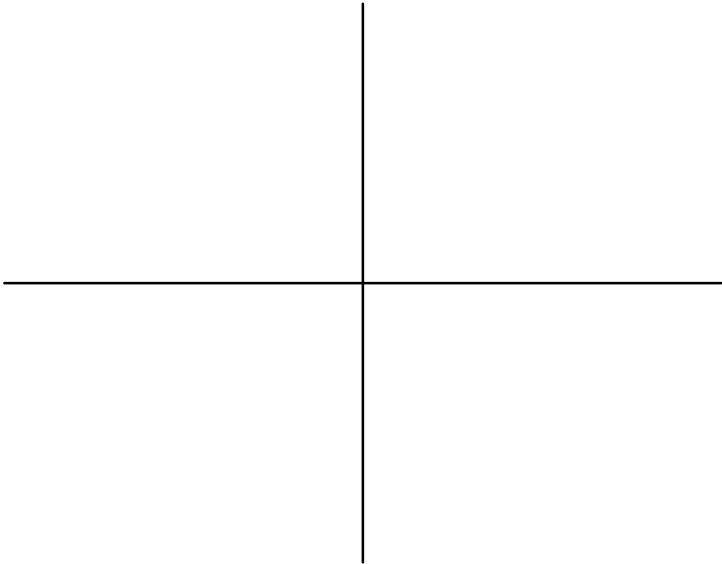
It will happen that sometimes we will need to integrate a multivalued function, if we are to do so we will need to make it single valued by taking a branch cut. As it turns out, if our path contains a point in the branch cut we can still preform the integration!

We know that as long as $f(z)$ is peicewise continuous on a contour C then then

$$\int_C f(z)dz = \int_a^b f(z(t))z'(t)dt$$

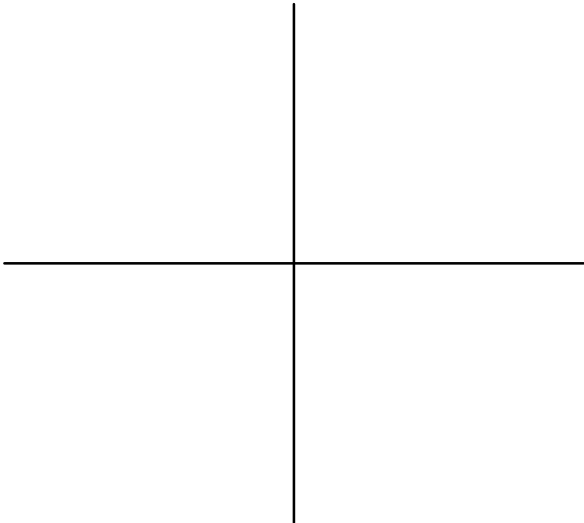
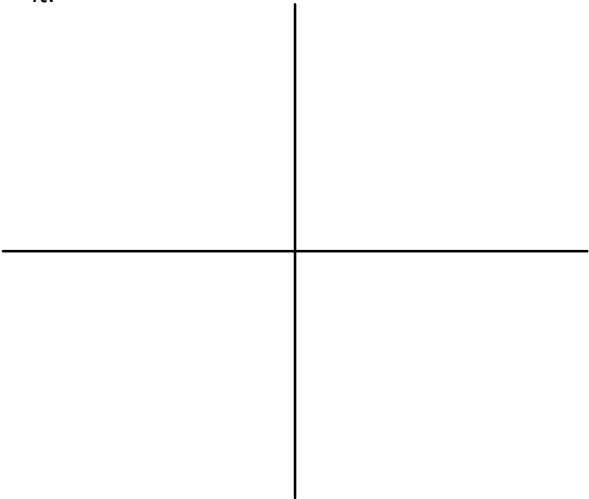
Where $z(t)$ is a parametrization of C for $a \leq t \leq b$

In analogy to integration in real numbers....



In the same way as long as $f(z)$ is locally continuous at $t = a$ $t = b$ (No blow ups) we can still integrate.

However Contours must say within the branch cut, that is you can start at the branch cut and end there but you can't cross it.



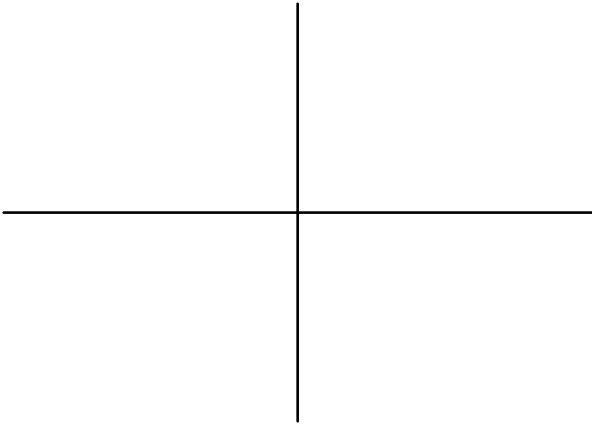
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Let's do an example.....

Compute the integral

$$\int_{C_i} z^{1/2} dz$$

where $z^{1/2}$ is taken on the branch $0 < \arg(z) < 2\pi$ and $r > 0$ where C_1 is any contour from -3 to 3 lying in the domain except the point 3 that approaches 3 from above the real axis, while C_2 approaches 3 from below the real axis.



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Bounds for Moduli of Contour Integrals

There are times when we need bounds on contour integrals to see if a specific integral satisfies a useful theorem or to simply calculate the value of the integral.

A Lemma we need:

$$\left| \int_a^b w(t) dt \right| \leq \int_a^b |w(t)| dt$$

To see why this is true Let's let

$$\int_a^b w(t) dt = r_0 e^{i\theta_0}$$

Now $\left| \int_a^b w(t) dt \right| = |r_0 e^{i\theta}| = r_0$

$Re \left(\int_a^b w(t) dt \right) = Re(r_0 e^{i\theta}) = r_0$

Further

$Re \left(\int_a^b w(t) dt \right) = \int_a^b Re(w(t)) dt$

We also know that $Re(w(t)) \leq |w(z)|$

Let's put it all together and get our result!

$$r_0 = \left| \int_a^b w(t) dt \right| = Re \left(\int_a^b w(t) dt \right) = \int_a^b Re(w(t)) dt \leq \int_a^b |w(t)| dt$$

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Theorem: Let C denote a contour of Length L , and suppose that a function $f(z)$ is piecewise continuous on C . If M is a nonnegative constant such that

$$|f(z)| \leq M$$

For all points z on C at which $f(z)$ is defined, then

$$\left| \int_C f(z) dz \right| \leq ML$$

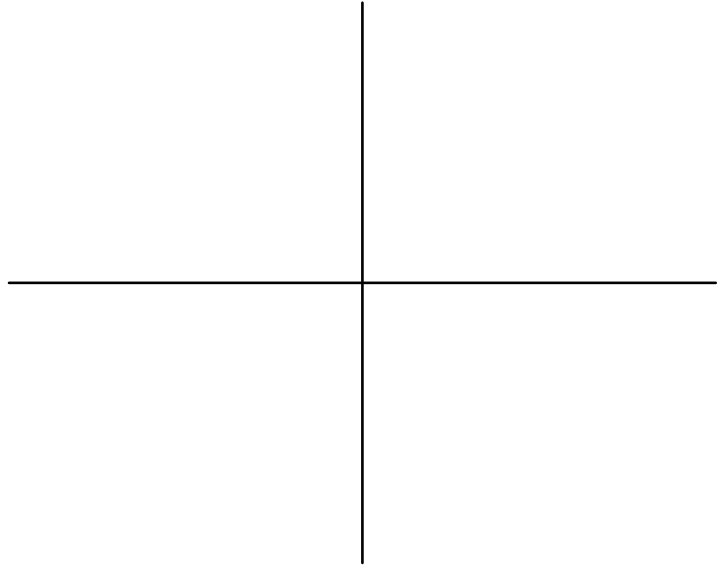
This is easy to obtain with our lemma and the fact that $L = \int_a^b |z'(t)| dt \dots$

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When is this useful? Later on we will learn about Cauchy's theorem which will easily allow us to calculate closed contour integrals.. This theorem can then be used to calculate real integrals!

Example: $\int_{-\infty}^{\infty} \frac{1}{x^2+1} dx$

Consider $\int_C \frac{1}{z^2+1} dz$



We will learn the tools to do the contour integral later...

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Antiderivatives

Some contour integrals do not depend on the path taken between the two end points.

It turns out if $f(z)$ has an antiderivative $F(z)$ everywhere in some domain D , note this means That $F(z)$ is analytic in D , then the value of a contour between two points does not matter!

In fact we get the following theorem.

Theorem: Suppose that a function $f(z)$ is continuous in a domain D . If any one of the following statements are true, then so are all of the others

- 1. $f(z)$ has an antiderivative $F(z)$ Throughout D
- 2. the integrals of $f(z)$ along contours lying entirely in D and extending from any fixed point z_1 to any fixed point z_2 all have the same value, that is

$$\int_{z_1}^{z_2} f(z) dz = F(z_2) - F(z_1)$$

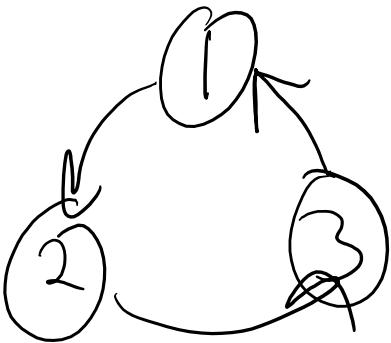
- 3. the integrals of $f(z)$ around closed contours lying entirely in D are all 0.

7. Using the antiderivative theorem evaluate the integrals

- (a) $\int_i^{i/2} e^{\pi z} dz$
- (b) $\int_0^{\pi+2i} \cos\left(\frac{z}{2}\right) dz$
- (c) $\int_{1+i}^{2+i} (z-2)^3 dz$

Proof!

To prove this we will need to show that 1 implies 2 and 2 implies 3 and 3 implies 1



1 implies 2

Suppose that $f(z)$ has antiderivative $F(z)$ everywhere in D then

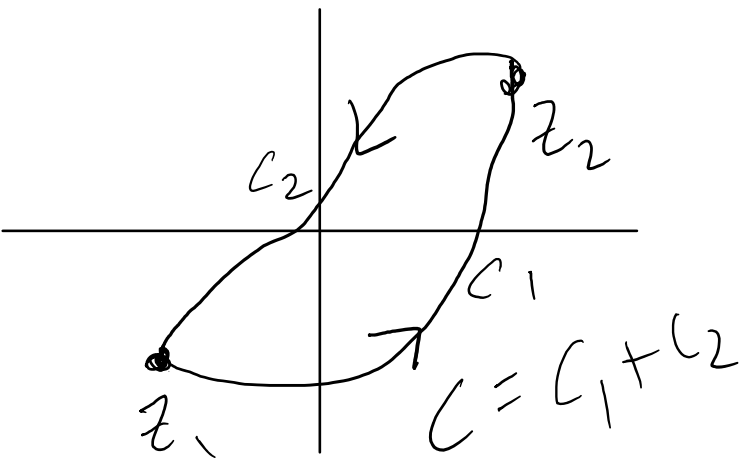
$$\frac{d}{dt}F(z(t)) = F'(z(t))z'(t) = f(z(t))z'(t)$$

Now we have a complex valued function of a real variable, this means the FTC applies as in section 42.

$$\int_c f(z)dz = \int_a^b f(z(t))z'(t)dt = F(z(b)) - F(z(a)) \quad \text{And } z(a) = z_1, z_b = z_2 \quad \text{so}$$

2 implies 3

Now suppose $\int_c f(z)dz = F(z_1) - F(z_2)$



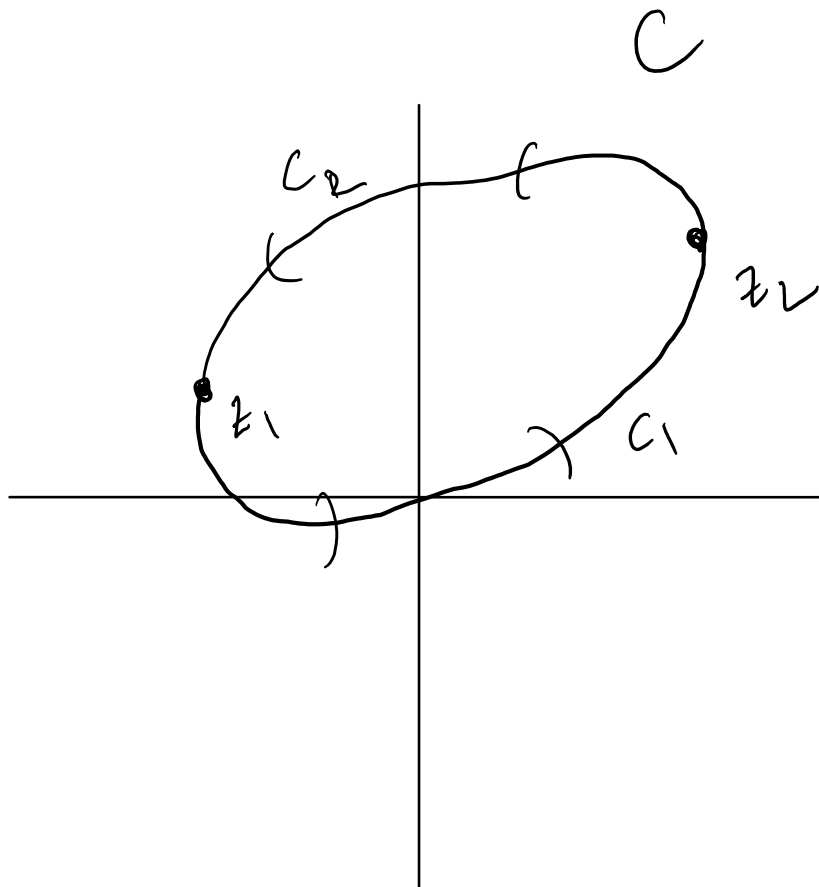
$$\int_c f(z)dz = \int_{c_1} f(z)dz + \int_{c_2} f(z)dz = F(z_2) - F(z_1) - (F(z_1) - F(z_2)) = 0$$

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To Prove 3 implies 1 we will need to show that 3 implies 2 first as we will require path independence for the proof.

Assuming 3 we may say that if C is a closed contour then $\int_C f(z) = 0$

Suppose C is looks like the following:



We know that

$$\int_C f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz$$

We also know that

$$-\int_C f(z)dz = \int_{-C} f(z)dz$$

Then:

$$\int_C f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz = 0 \Rightarrow \int_{C_1} f(z)dz = -\int_{C_2} f(z)dz = \int_{-C_2} f(z)dz$$

so

$$\int_{C_1} f(z)dz = \int_{-C_2} f(z)dz$$

In this way the two contour integrals over different paths are the same!

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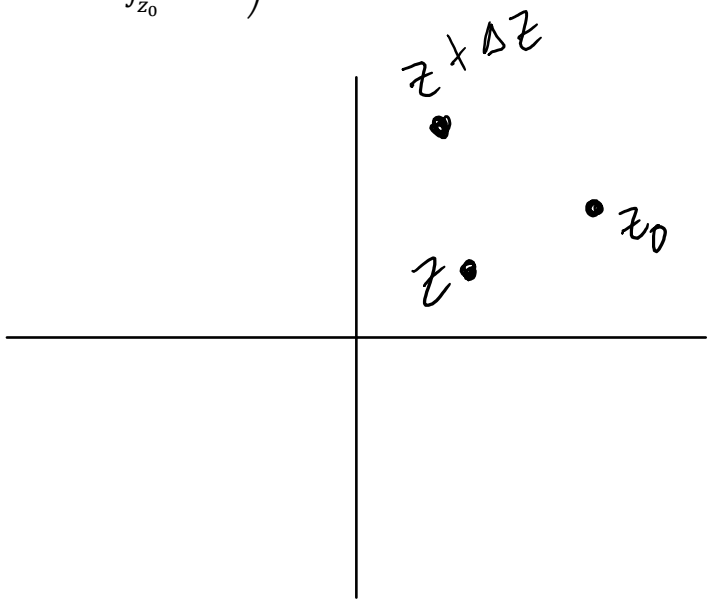
3 implies 1

We want to show that if the contour integrals around closed contours are always zero then $f(z)$ has an antiderivative $F(z)$ everywhere in D

The Definition of a derivative would then require that $\lim_{\Delta z \rightarrow 0} \frac{F(z+\Delta z)-F(z)}{\Delta z} = f(z)$

So here is the trick, We define $F(z) = \int_{z_0}^z f(s)ds$ Then:

$$\frac{F(z + \Delta z) - F(z)}{\Delta z} = \frac{1}{\Delta z} \left(\int_{z_0}^{z+\Delta z} f(s)ds - \int_{z_0}^z f(s)ds \right)$$



Because of Independence of path we can say:

$$\frac{1}{\Delta z} \left(\int_{z_0}^{z+\Delta z} f(s)ds - \int_{z_0}^z f(s)ds \right) = \frac{1}{\Delta z} \left(\int_z^{z+\Delta z} f(s)ds \right)$$

We want to show that $\lim_{\Delta z \rightarrow 0} \frac{F(z+\Delta z)-F(z)}{\Delta z} = f(z)$ Which is equivalent to showing that $\lim_{\Delta z \rightarrow 0} \frac{F(z+\Delta z)-F(z)}{\Delta z} - f(z) = 0$

Or with our integral representation that

$$\lim_{\Delta z \rightarrow 0} \frac{1}{\Delta z} \left(\int_z^{z+\Delta z} f(s)ds \right) - f(z) = 0$$

Now $\int_z^{z+\Delta z} f(s)ds$ is path independent so we can choose any contour we want to do it, let's choose a straight line.

In which case $\int_z^{z+\Delta z} ds = \Delta z$

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Then I can write

$$f(z) = f(z) \frac{1}{\Delta z} \int_z^{z+\Delta z} ds = \frac{1}{\Delta z} \int_z^{z+\Delta z} f(z) ds$$

And then

$$\lim_{\Delta z \rightarrow 0} \frac{1}{\Delta z} \left(\int_z^{z+\Delta z} f(s) ds \right) - f(z) = \lim_{\Delta z \rightarrow 0} \frac{1}{\Delta z} \left(\int_z^{z+\Delta z} f(s) - f(z) ds \right)$$

Now bounding the integral we can say:

$$\left| \frac{1}{\Delta z} \left(\int_z^{z+\Delta z} f(s) - f(z) ds \right) \right| \leq \frac{1}{|\Delta z|} \left(\int_z^{z+\Delta z} |f(s) - f(z)| ds \right)$$

Now $f(z)$ is continuous at z thus by the definition of continuity

Whenever $|s - z| \leq \delta$ then $|f(s) - f(z)| \leq \epsilon$ for any $\epsilon > 0$ and $|s - z| \leq |\Delta z|$ So what ever δ is required for the given ϵ for $|f(s) - f(z)| \leq \epsilon$ we can choose $\Delta z = \delta$ in which case...

Let $\epsilon > 0$ and let δ be such that $|f(s) - f(z)| \leq \epsilon$ and then set $|\Delta z - 0| = |\Delta z| = \delta$ then we may say that :

$$\frac{F(z + \Delta z) - F(z)}{\Delta z} \leq \frac{1}{|\Delta z|} \left(\int_z^{z+\Delta z} |f(s) - f(z)| ds \right)$$

$$\left| \frac{F(z + \Delta z) - F(z)}{\Delta z} - f(z) - 0 \right| \leq \frac{1}{|\Delta z|} \left(\int_z^{z+\Delta z} |f(s) - f(z)| ds \right) \leq \frac{1}{|\Delta z|} \left(\int_z^{z+\Delta z} \epsilon ds \right) = \frac{1}{|\Delta z|} \epsilon |\Delta z| = \epsilon$$

Thus whenever $|\Delta z - 0| \leq \delta$ we have that $\left| \frac{F(z+\Delta z)-F(z)}{\Delta z} - f(z) - 0 \right| \leq \epsilon$ for any $\epsilon > 0$ Which satisfies the definition of a limit and shows that

$$\lim_{\Delta z \rightarrow 0} \frac{F(z+\Delta z)-F(z)}{\Delta z} - f(z) = 0 \Rightarrow F'(z) = f(z) \text{ PHEW!}$$