

Chapter 1

What are complex numbers?

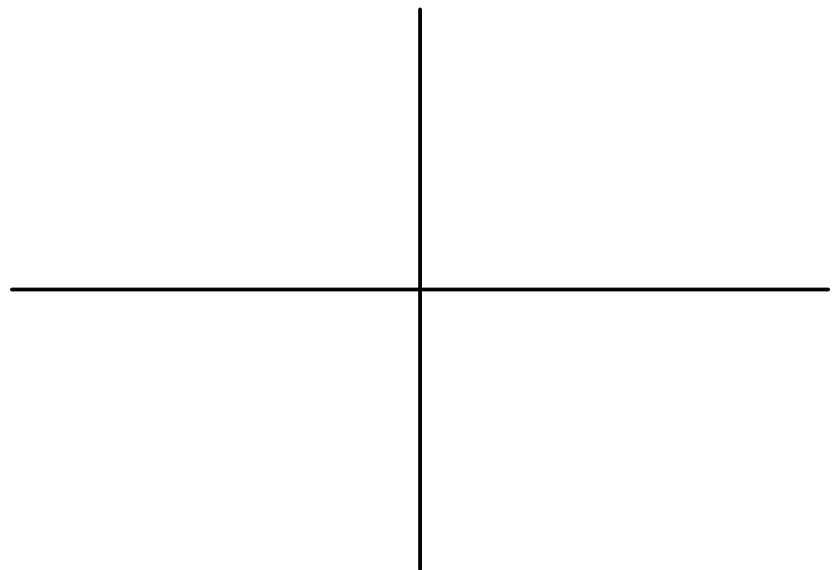
Believe it or not, the Greeks used to argue whether or not $\sqrt{2}$ was indeed a number. As we all know if we take a right triangle with legs of length 1 the hypotenuse has length $\sqrt{2}$, demonstrating its existence.

Once that was settled, a new class of number was added to their number system, that of the irrational numbers. As we will see the complex numbers are a further extension of the real numbers. That is, real numbers are complex but complex numbers are not real numbers.

Let's pretend we are mathematicians of the old days and try to answer the question, what is $\sqrt{-1}$?

So we have a new number system! We might ask, how do they add how do they multiply?

Given that we can get real numbers by multiplying imaginary ones, is there some framework we can create which combines the two types of numbers?



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Playing with new number systems and operations is what a lot of mathematicians do. In many cases new wacky number systems turn out to be useful, such is the case with the complex numbers. From electromagnetism to elastic wave propagation, complex numbers and the calculus associated with them prove to have REAL application, despite the unfortunate name of the "imaginary" number system. Just as real numbers store information about physical systems complex numbers can do the same.

Let's get familiar with the system and its rules.....

There are two basic ways we use to represent complex numbers

$$z = (x, y)$$

$$z = x + iy$$

We call $x = \text{Re}(z)$ and $y = \text{Im}(z)$

How do we add them?

How do we multiply them?

What about $z * 0$ then? Well $z * 0 = (x + iy)(0 + 0i)$

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Within the real numbers we have some special numbers, 0 (the additive Identity) and 1 (the multiplicative identity)

That's because $a + 0 = a$ and $a * 1 = a$

Associated with these special numbers are what we call inverses, the additive inverse $(-a)$ and the multiplicative inverse a^{-1}

Inverses take us to identities that is, $a + (-a) = 0$ and $a * a^{-1} = 1$. In the real numbers system we know the additive inverse of a number is just its negative and the multiplicative inverse the reciprocal I.E. $a^{-1} = \frac{1}{a}$. We might then ask what are the identities and inverses for the complex numbers ?

Addition is fairly straight forward

What about multiplication?

So the additive inverse of $z = (x, y)$ is just $-z = (-x, -y) = -x - iy$ and the multiplicative inverse of $z = (x, y)$ is

$$z^{-1} = \left(\frac{x}{x^2 + y^2}, -\frac{y}{x^2 + y^2} \right) = \frac{x}{x^2 + y^2} - \left(\frac{y}{x^2 + y^2} \right) i$$

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We also have a lot of loved algebraic properties from the real numbers which we should investigate in the complex number system. Things like, associativity $(ab)c=a(bc)$ and commutativity $ab=ba$ etc.

Let's show it in general for addition.

It is also true for multiplication. Let's just do a quick numerical example. How about $z_1 = 1 + i, z_2 = 2 - i, z_3 = 2 + 2i$

Indeed for complex numbers we can say: $\forall$ (for all) $z_1, z_2, z_3 \in \mathbb{C}$ that the following properties hold.

For addition of complex numbers

$$0 + z_1 = z_1$$

$$z_1 + (-z_1) = 0$$

$$z_1 + z_2 = z_2 + z_1$$

$$(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$$

For multiplication of complex numbers

$$1(z_1) = z_1$$

$$z_1(z_1)^{-1} = 1$$

$$z_1 z_2 = z_2 z_1$$

$$(z_1 z_2) z_3 = z_1 (z_2 z_3)$$

In fact with this and the multiplicative inverse property we can show that if $z_1 z_2 = 0$ Then $z_1 = 0$ or $z_2 = 0$. To see why suppose $z_1 z_2 = 0$ and that $z_1 \neq 0$ then we can say: $z_1^{-1} * 0 = z_1^{-1}(z_1 z_2)$

So: $0 = z_1^{-1}(z_1 z_2) = (z_1^{-1} z_1) z_2 = 1 * z_2 = z_2$ thus $z_2 = 0$

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Division of complex numbers and the complex conjugate.

How might we calculate something like $\frac{z_1}{z_2}$? Well, $\frac{z_1}{z_2} = z_1 z_2^{-1}$ That is if $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$

$$z_1 z_2^{-1} = (x_1 + iy_1) \left(\frac{x_2}{x_2^2 + y_2^2} - \left(\frac{y_2}{x_2^2 + y_2^2} \right) i \right)$$

While this will work it is kind of complicated looking it turns out that the above formula is equivalent to

$$\frac{z_1}{z_2} = \frac{(x_1 + iy_1)}{(x_2 + iy_2)} * \frac{(x_2 - iy_2)}{(x_2 - iy_2)}$$

Not that we simply multiplied the top and bottom by $x_2 - iy_2$ This number is called the complex conjugate of $z_2 = x_2 + iy_2$ and we denote it by $\bar{z}_2 = x_2 - iy_2$ We will talk more about this in a bit.

Let's just do a quick example: $\frac{1+2i}{3+4i}$

Just a couple other quick things

$$\frac{z_1 + z_2}{z_3} = (z_1 + z_2) z_3^{-1} = z_1 z_3^{-1} + z_2 z_3^{-1} = \frac{z_1}{z_3} + \frac{z_2}{z_3}$$

$$(z_1 z_2)^{-1} = z_1^{-1} z_2^{-1} \text{ since } (z_1 z_2)(z_1^{-1} z_2^{-1}) = (z_1 z_1^{-1})(z_2 z_2^{-1}) = (1)(1) = 1$$

$$\left(\frac{1}{z_1} \right) \left(\frac{1}{z_2} \right) = z_1^{-1} z_2^{-1} = (z_1 z_2)^{-1} = \frac{1}{z_1 z_2}$$

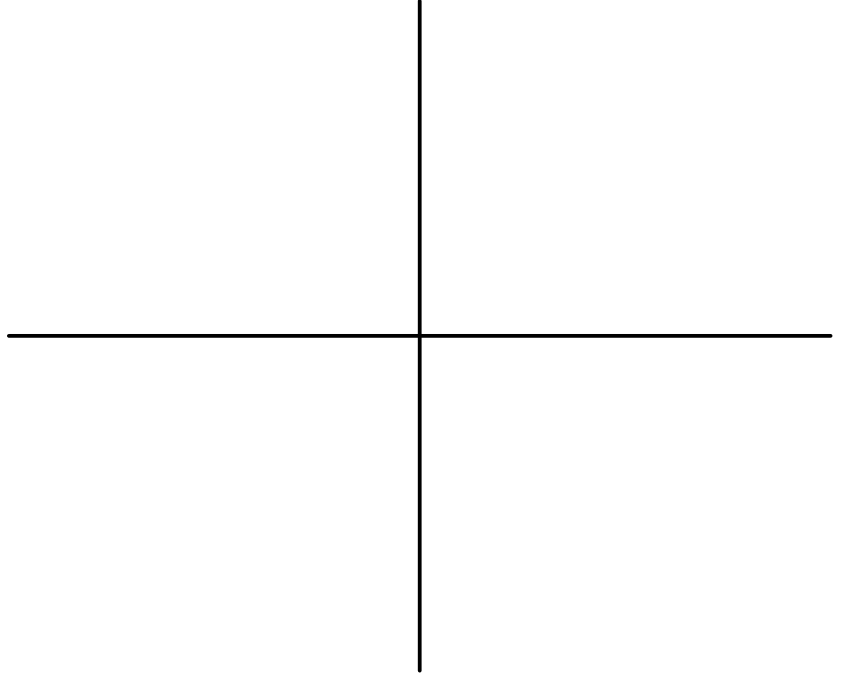
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Complex numbers as "vectors"

From the properties we have looked at it might be apparent to you that we add complex numbers in the same way that we add vectors. Indeed we have two components, a real part and an imaginary part which we plot in the complex plane much like we plot the x and y parts of a vector in the complex plane. In fact we often associate a vector with a complex number.

Let's get a sense of this with a picture, let's look at

$$z_1 = 1 + 2i, z_2 = 2 + 2i \text{ and } z_1 + z_2$$



With these associated vectors we can calculate their length in the usual way, $|\langle x, y \rangle| = \sqrt{x^2 + y^2}$ This leads to a natural definition of the absolute value of a complex number z some times called the modulus.

$$|z| = \sqrt{x^2 + y^2} \text{ for } z = x + iy$$

Geometrically speaking this represents the distance the number z is from the origin (0,0)

$$\text{Note that } |z_1 + z_2| = \sqrt{(x_1 + x_2)^2 + (y_1 + y_2)^2}$$

We also have the triangle inequalities

$$|z_1 + z_2| \leq |z_1| + |z_2| \text{ and } |z_1 + z_2| \geq |z_1| - |z_2|$$

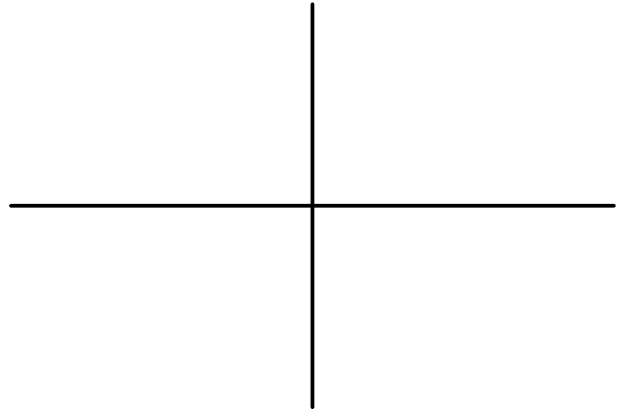
Another neat formula for the magnitude(modulus)(absolute value) of a complex number comes from multiplication of itself with its complex conjugate. $|z| = \sqrt{z\bar{z}}$ Let's verify this with $z = 3 + 4i$

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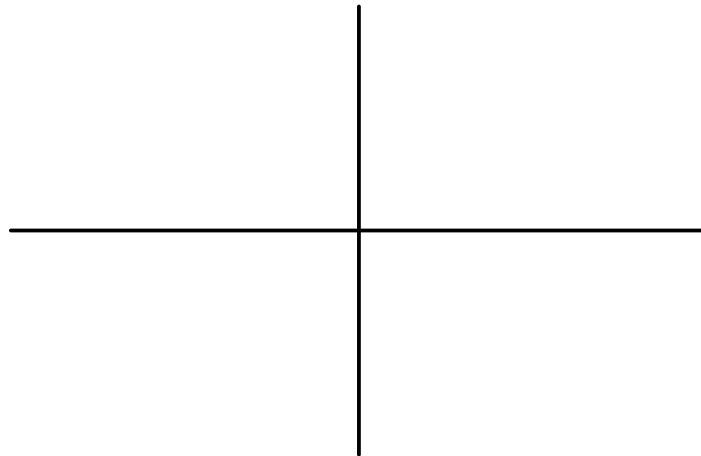
Circles and ellipses in the complex plane.

Using the idea of the modulus we can actually define circles and ellipses in the following way.

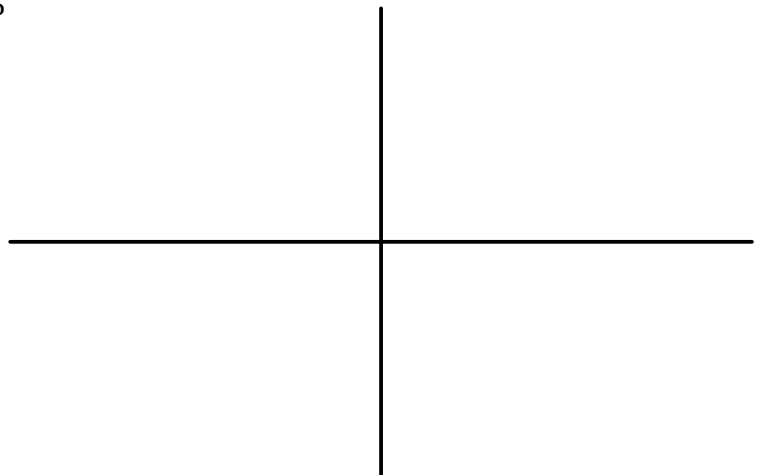
Consider the set of complex numbers z which satisfy $|z| < 2$



What about $|z - (3 + 2i)| < 1$?



What about the set which satisfies $|z - i| + |z + i| < 5$



We will use this idea often and define a disk of radius r centered at a point z_0 as:

$$D(r, z_0) = \{z: |z - z_0| \leq r\}$$

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Properties of the complex conjugate

We saw that the complex conjugate $\bar{z} = x - iy$ of a complex number $z = x + iy$ was useful for division and finding the modulus. Infact it will show up a number of times in this course so we should talk about a few of its properties.

$$1. \quad \bar{\bar{z}} = z$$

$$2. \quad |\bar{z}| = |z|$$

$$3. \quad \overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$$

$$4. \quad \overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$$

$$5. \quad \overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$$

$$6. \quad z\bar{z} = |z|^2$$

$$7. \quad \operatorname{Re}(z) = \frac{z + \bar{z}}{2}$$

$$8. \quad \operatorname{Im}(z) = \frac{z - \bar{z}}{2}$$

These properties also lead to the fact that.

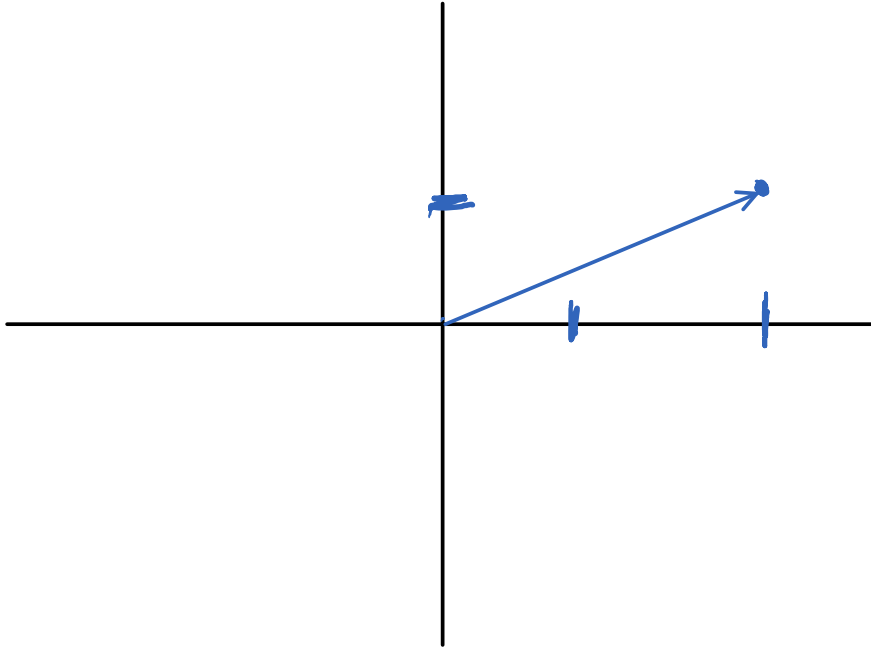
$$|z_1 z_2| = |z_1| |z_2|$$

$$\left| \frac{z_1}{z_2} \right| = \frac{|z_1|}{|z_2|}$$

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Other ways to represent complex numbers (Exponential Form)

Let's think back to the vector form of a complex number for example $z = 1 + 2i$ as (1,2)



So we can represent this as $(\sqrt{3} \cos(\theta), \sqrt{3} \sin(\theta))$ Going back to our other form of complex numbers this means that $z = 1 + 2i = \sqrt{3} \cos(\theta) + i\sqrt{3} \sin(\theta)$

In fact for any complex number with $z = x + iy$, we can write it as $z = r(\cos(\theta) + i\sin(\theta))$

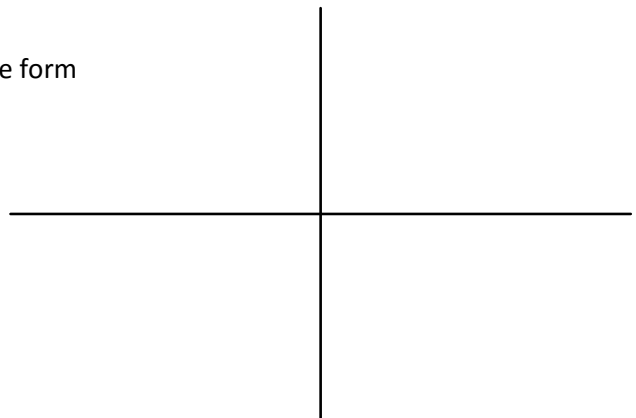
Here $r = |z| = \sqrt{x^2 + y^2}$ and $\theta = \arctan\left(\frac{y}{x}\right)$ We often refer to $\theta = \text{Arg}(z)$ Of course the value of θ that defines z is not unique. In practice we often use what we call the principal argument of z which is the value of θ which defines z and is in the region $-\pi \leq \theta \leq \pi$

That is for $z = r(\cos(\theta) + i\sin(\theta))$ the principal argument $\text{Arg}(z)$ lies between $-\pi \leq \theta \leq \pi$

The general argument of z is defined as

$$\arg(z) = \text{Arg}(z) + 2n\pi \text{ where } n = 0, \pm 1, \pm 2, \dots$$

Example: For $z = -1 - i$ find $\text{Arg}(z)$ and $\arg(z)$ and write it in the form $z = r(\cos(\theta) + i\sin(\theta))$



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I mentioned Exponential form, but where are the exponents? Well thanks to the all powerful Euler we know that

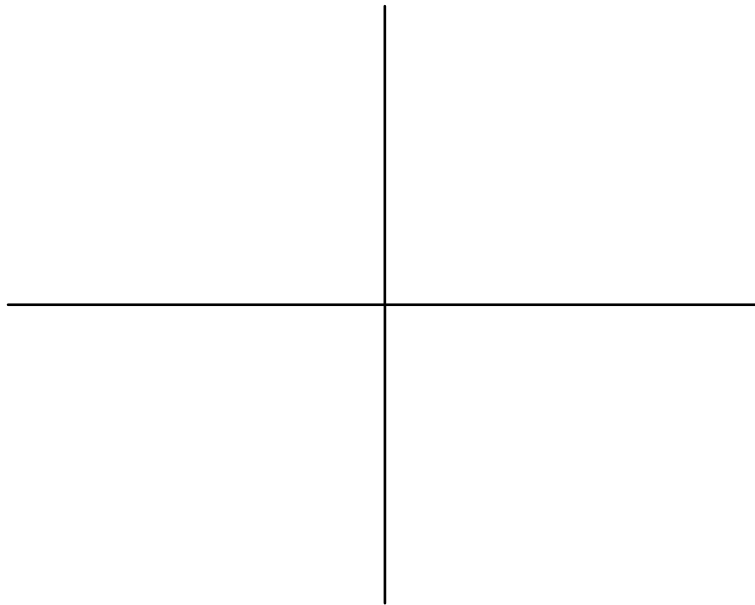
$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

Thus for our complex form of $z = r(\cos(\theta) + i\sin(\theta))$ we can write:

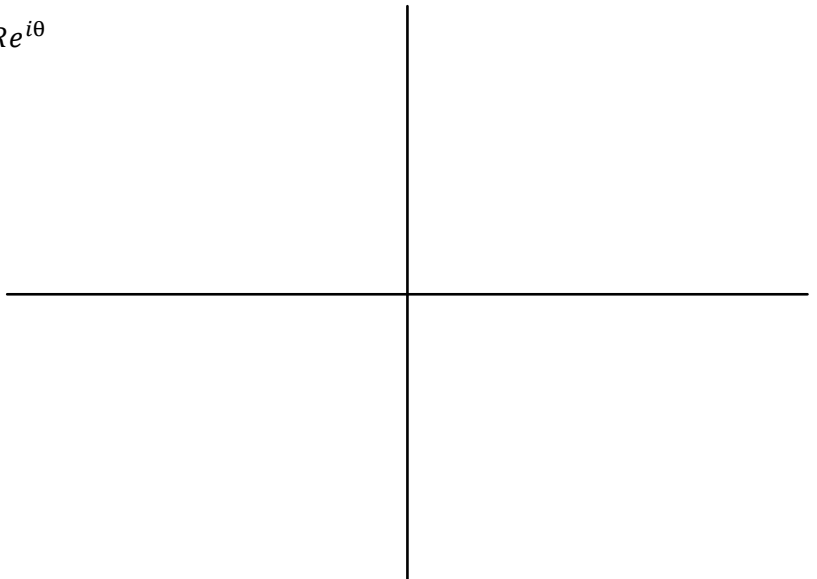
$$z = r(\cos(\theta) + i\sin(\theta)) = re^{i\theta}$$

This is the exponential form of a complex number. So for our number of $z = -1 - i = \sqrt{2}(\cos(-\frac{3\pi}{4}) + i\sin(-\frac{3\pi}{4})) = \sqrt{2}e^{-i\frac{3\pi}{4}}$

So we can write complex numbers in the form $z = Re^{i\theta}$ Let's think about what this looks like in the complex plane.



We can also shift the circle by any amount. $z = z_0 + Re^{i\theta}$



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Multiplication in exponential form and de Moivre's Formula

As it turns out the regular rules for exponents apply to our exponential form! This makes working with complex numbers quite nice.

Let $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$ then...

$$z_1 z_2 = (r_1 r_2) e^{i(\theta_1 + \theta_2)}$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}$$

Incidentally notice that this suggests $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$ and $\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$

What about a power of z ? Let's examine $z^n = (r e^{i\theta})^n$

$$z^n = r^n e^{in\theta}$$

If we take this back to the sine and cosine form with Euler, we get de Moivre's Formula

$$z^n = r^n (\cos(n\theta) + \sin(n\theta))$$

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We can also define the multiplicative inverse for the exponential form of a complex number $z^{-1} = \frac{1}{r}e^{-i\theta}$

Let's just verify it..

Notice this means that $\arg(z^{-1}) = -\arg(z)$

I mentioned that because of all this $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$, and this is true but, what about the principal argument $\text{Arg}(z_1 z_2) = \text{Arg}(z_1) + \text{Arg}(z_2)$

Recall that $\arg(z) = \text{Arg}(z) + 2n\pi$ so depending on the value of n the above equation may or may not hold, we will actually have to choose the right one. Let's illustrate with an example.

Let $z_1 = -1$ and $z_2 = i$. Then $\text{Arg}(z_1) = \pi$ and $\text{Arg}(z_2) = \frac{\pi}{2}$ Also $z_1 z_2 = -i$ so that $\text{Arg}(z_1 z_2) = -\frac{\pi}{2}$

But $\text{Arg}(z_1) + \text{Arg}(z_2) = \pi + \frac{\pi}{2} = \frac{3\pi}{2} \neq -\frac{\pi}{2}$

But if we use the general argument it does not matter since we can add any multiple of 2π

So we define

$$\arg(z_1 z_2) = \arg(z_1) + \arg(z_2) = \theta_1 + 2n_1\pi + \theta_2 + 2n_2\pi = \theta_1 + \theta_2 + 2(n_1 + n_2)\pi = \theta_1 + \theta_2 + 2n\pi$$

$n = 0, \pm 1, \pm 2 \dots$

From the above we can say $\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$ since..

$$\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1 z_2^{-1}) = \arg(z_1) + \arg(z_2^{-1}) = \arg(z_1) - \arg(z_2) = \theta_1 + 2n_1\pi - (\theta_2 + 2n_2\pi) = \theta_1 - \theta_2 + 2(n_1 - n_2)\pi$$

$= \theta_1 - \theta_2 + 2n\pi$

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Exponential form can be very useful for calculations too, Let's do a homework problem to see how.

5. 1.8.5: By writing the individual factors on the left in exponential form, performing the needed operations, and finally changing back to rectangular coordinates, show that

(a) $i(1 - \sqrt{3}i)(\sqrt{3} + i) = 2(1 + \sqrt{3}i)$

(b) $\frac{5i}{2+i} = 1 + 2i$

(c) $(-1 + i)^7 = -8(1 + i)$

(d) $(1 + \sqrt{3}i)^{-10} = 2^{-11}(-1 + \sqrt{3}i)$

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Roots of complex numbers

I bet you thought that $\sqrt[3]{1} = 1$, while that is true there are more solutions than that! We say $\sqrt[3]{1} = 1$ since $1^3 = 1$ but if there are any other values say z where $z^3 = 1$ we should include those too, just like we do with $\sqrt{1} = \pm 1$. We do however call 1 the principal root. In fact for any real number $\sqrt[n]{r}$ we define the principal root as the positive real root.

In fact $\sqrt[3]{1} = 1, e^{\frac{2\pi}{3}}, e^{\frac{4\pi}{3}}$. Let's just verify these roots..

$$(1)^3 =$$

$$\left(e^{\frac{2\pi}{3}}\right)^3 =$$

$$\left(e^{\frac{4\pi}{3}}\right)^3 =$$

Of course these are good up to any added multiple of 2π but at that point it becomes redundant.

Let's think about this more generally an n th root of a number is any number such that when we take it to the n th power we get the original number back. So for some complex number $z_0 = r_0 e^{i\theta_0}$ we want some number $z = r e^{i\theta}$ such that $z^n = (r e^{i\theta})^n = r_0 e^{i\theta_0}$. Let's see what this tells us.

$$\text{That is we want } n\theta = \theta_0 + 2k\pi \text{ and } r^n = r_0$$

Why the $2k\pi$? Well as long $n\theta$ is in the right place on the complex plane we will get the same answer from our exponential.

If we solve this for θ and r taking the positive principal root of r we get

$$\theta = \frac{\theta_0 + 2k\pi}{n} \quad r = \sqrt[n]{r_0}$$

Thus the n th roots of $z_0 = r_0 e^{i\theta_0}$ is given by: $z = \sqrt[n]{r_0} e^{\frac{i(\theta_0 + 2k\pi)}{n}} \quad k = 0, 1, 2, \dots, n-1$

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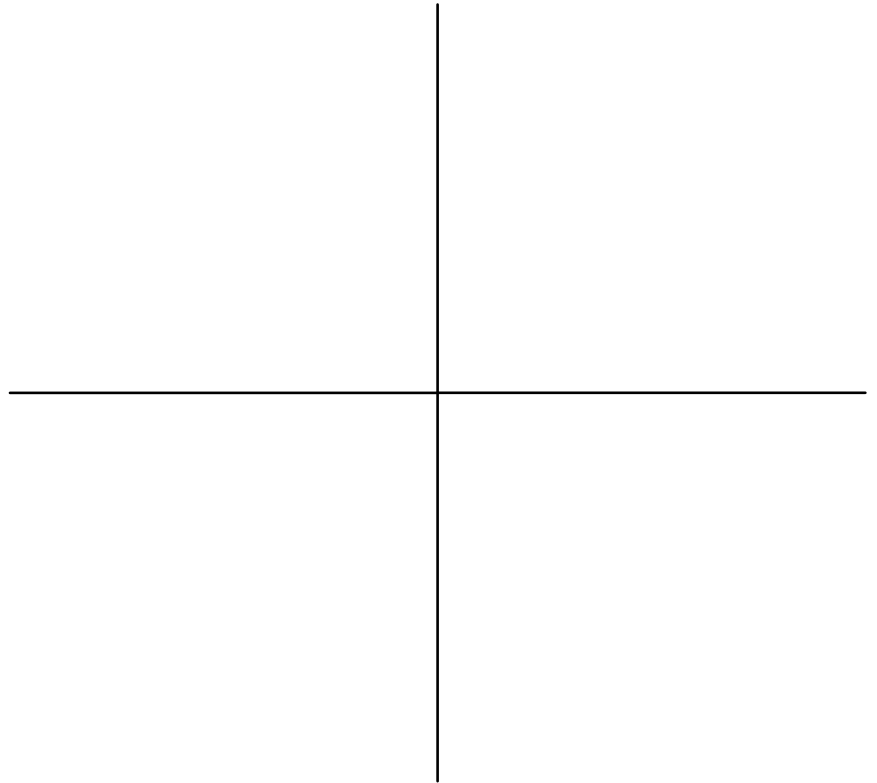
To see why we stop k at $n-1$ let's re-write our solution using rules for exponents.

$$z = \sqrt[n]{r_0} e^{\frac{i(\theta_0 + 2k\pi)}{n}} = \sqrt[n]{r_0} e^{\frac{i\theta_0}{n}} e^{\frac{2k\pi}{n}}$$

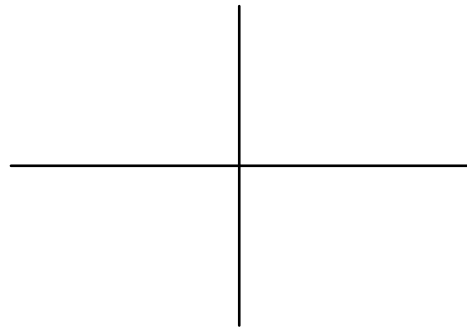
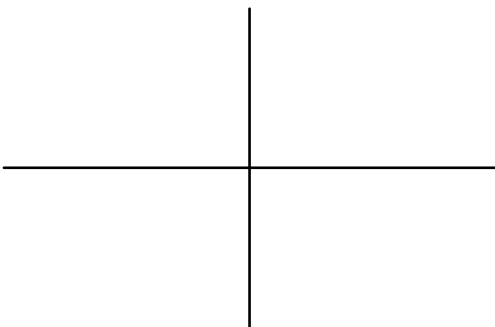
Note that when $k = n$ $e^{\frac{2k\pi}{n}} = e^{i2\pi}$ and we have come full circle and begin repeating ourselves.

To illustrate this let's draw our cube roots of 1 using our formula to find them and see when we hit an old root. In this case $z_0 = 1 = 1e^{i0}$ and $n = 3$ for third root so that our roots are given by

$$z = \sqrt[3]{1} e^{\frac{i(0+2k\pi)}{3}}$$



Amazingly these are the vertices of a regular polygon inscribed in the unit circle with the first vertex starting at the principal root of 1. In fact the same thing happens for the 4th roots, we get a square and so on.



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Example: Find the 4th roots of $z = 16 + 16i$

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Some definitions we will need.

An ϵ neighborhood of a point z_0 is defined as $N(\epsilon) = \{z: |z - z_0| < \epsilon\}$

Sometimes we will talk about a deleted neighborhood where we exclude the center point.

We will also need the idea of an open set and a closed set.

An open set does not include boundary points, for example

$$|z| < 1$$

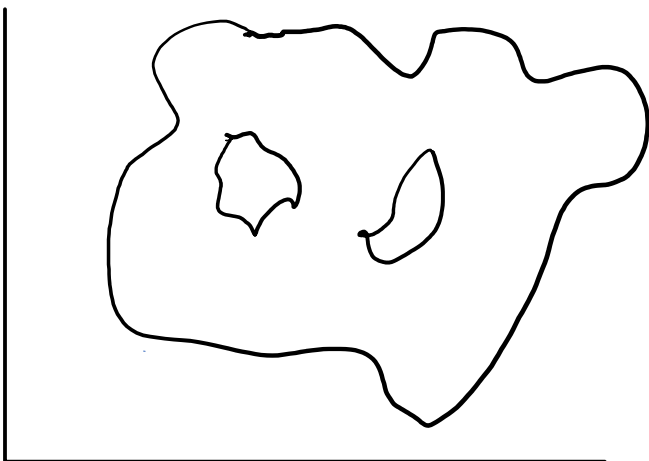
A closed set includes the boundary points, for example

$$|z| \leq 1$$

It is also possible for a set to be neither open nor closed, If a set is closed it must have all of the boundary points, if it is open it can contain none of them.

We will also use the idea of a *connected* set. A set is connected if we can find some polygonal line to connect any two points in the set with that line remaining in that set.

Connected



Not connected

