

Chapter 2 Analytic Functions, (on to the calculus!)

Functions and Mappings

We are all familiar with the idea that a function defined on a domain S is a rule that assigns each member of the domain to some member of another set called the range. For the case of complex numbers the idea is the same. A function $f(z)$ assigns to the complex number z some other complex value w . That is $w = f(z)$.

We know that we can represent a complex number z using two real numbers and the imaginary unit $z = x + iy$

It turns out that we can do the same thing for a complex valued function. Let $w = u + iv$ and $z = x + iy$

Then $w = f(z)$ becomes $u + iv = f(x + iy)$

$$f(z) = u(x, y) + i(v(x, y))$$

That is the result which comes from the function has both a real part and an imaginary part and the function itself depends on both x and y this means that the real part u depends on x and y and so does v . This means we can always find a representation of $f(z) = u(x, y) + iv(x, y)$. That is, we can separate f into a real and imaginary part.

As an example consider $f(z) = z^2$ Let's find the corresponding $u(x, y)$ and $v(x, y)$

$$(x + iy)(x + iy) = x^2 + ixy + ixy + (iy)^2$$

$$u(x, y) = x^2 - y^2 = x^2 + 2ixy - y^2$$

$$v(x, y) = 2xy = \underbrace{x^2 - y^2}_{u(x, y)} + i \underbrace{(2xy)}_{v(x, y)}$$

We can also define functions of complex values that return only real numbers. For example $f(z) = f(x + iy) = x^2 + y^2$ in this case we have just set $v(x, y) = 0$

$$f(z) = |z|^2$$

We also know that we can represent complex numbers in exponential form. So what about $f(z) = z^2$?

$$z = re^{i\theta}$$

$$z^2 = (re^{i\theta})^2 = r^2 e^{i2\theta}$$

$$f(z) = r^2 e^{i2\theta}$$

So $f(z) = r^2 e^{i2\theta}$

Note that we can put this into polar form too!

$$\begin{aligned} r^2 e^{i2\theta} &= r^2 (\cos(2\theta) + i\sin(2\theta)) \\ &= \underbrace{r^2 \cos(2\theta)}_{u(r, \theta)} + i \underbrace{r^2 \sin(2\theta)}_{v(r, \theta)} \end{aligned}$$

In general we can also always write $f(z) = u(r, \theta) + iv(r, \theta)$ as well.

Chapter 2 Analytic Functions, (on to the calculus!)

Multi-Valued functions

Some functions return more than one value for a given number, while not functions we call them multi-valued functions

For example consider $f(z) = \sqrt[3]{z} = \sqrt[3]{r}e^{i(\theta_z + \frac{2k\pi}{3})}$ we know this gives us 3 values as an answer. If we wanted to make this single-valued we could simply require that the answer be the principal real root just like we do with $y = \sqrt{x}$ when we are in the real number system.

What about the graph of a complex function? We know that with the real numbers we can often graph a function to get a visual sense for how the function behaves. This is easy because the real numbers lie on a line and can be graphed in a plane. Here however our independent variable lives in the plane and so does our result making it difficult to draw.

The best way to visually understand a complex function is with the idea of a map. We are taking values in the complex plane to values in another complex plane and we can sometimes understand the function by looking at how it takes one region of the plane to another. Let me just show you an example...

Suppose we have $w = f(z)$ we can think of two planes, the original z plane and the resulting w plane

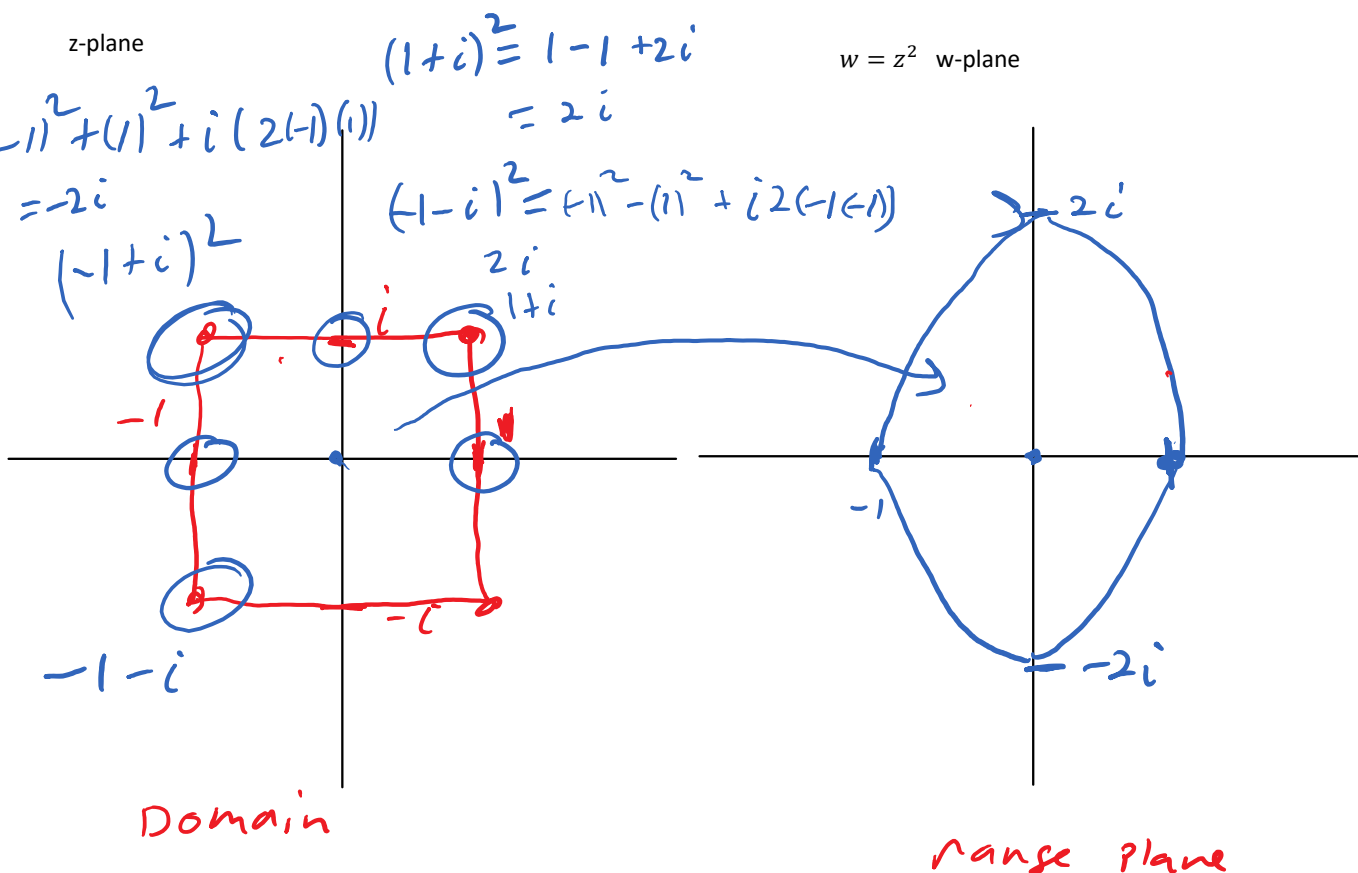
$$f(z) = x^2 - y^2 + i2xy$$

Let's play with $w = z^2$ and see what it does to a square region of the plane. Let's take a square of side length 2 for fun.



z -plane

$w = z^2$ w -plane



Chapter 2 Analytic Functions, (on to the calculus!)

Let's look at in line bits

We have $w = f(z)$ where $w = u(x, y) + iv(x, y) = \underline{x^2 - y^2 + i2xy}$. What happens to the line $z = x + \underline{1i}$

$$(x+i)^2 = x^2 - 1^2 + i2x(1) = x^2 - 1 + i2x$$

Since $u = (x^2 - 1)$ and $v = 2x$ we have parametric equations which we could plot. $u(x, y) = x^2 - 1$

$$v(x, y) = 2x$$

$$u = x^2 - 1$$

$$v = 2x$$

$$x = \frac{v}{2}$$

$$u = \left(\frac{v}{2}\right)^2 - 1$$

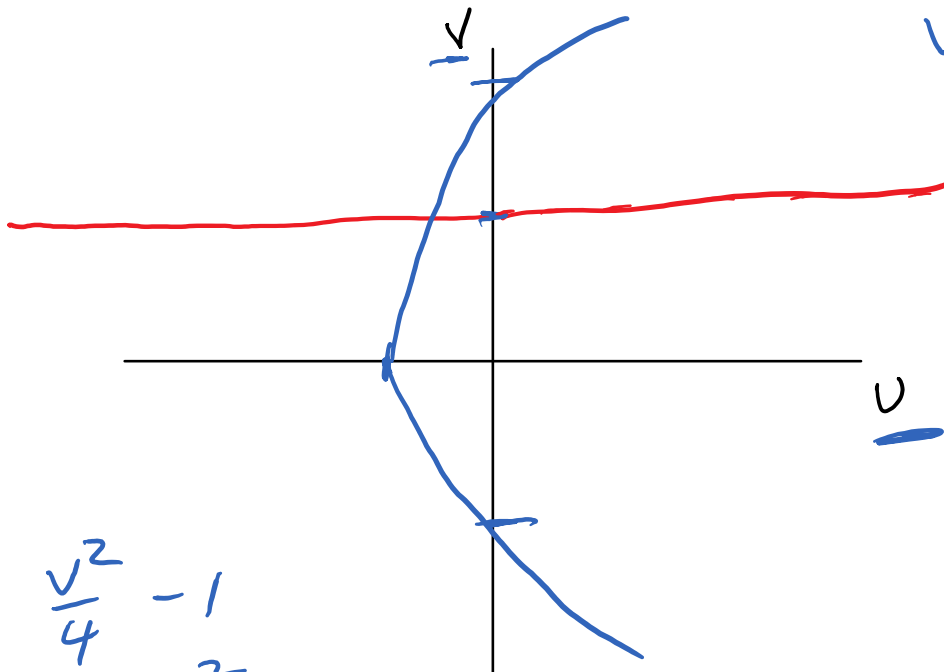
$$u = \frac{v^2}{4} - 1$$

$$0 = \frac{v^2}{4} - 1$$

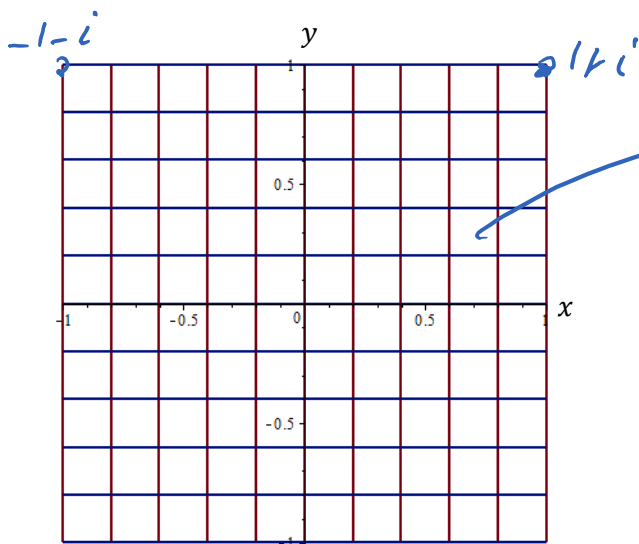
$$\frac{v^2}{4} = 1$$

$$v^2 = 4$$

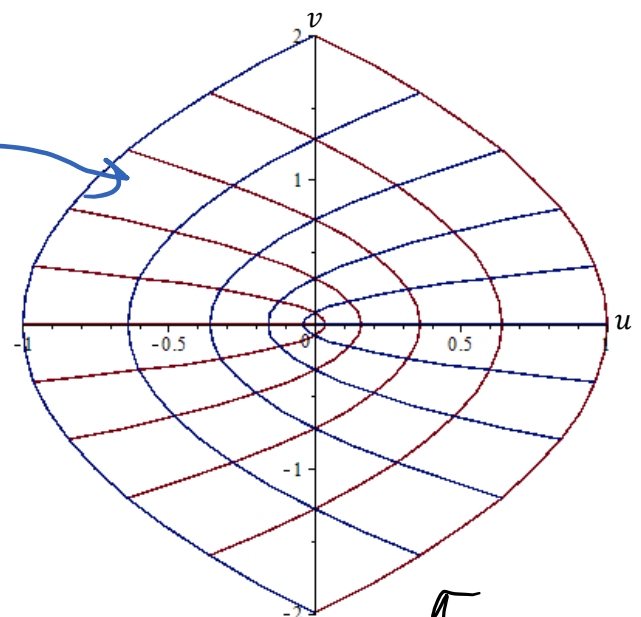
$$v = \pm 2$$



We can see that this is mapping lines to sideways parabolas! In fact here is a plot from a computer of our square region.



z - plane



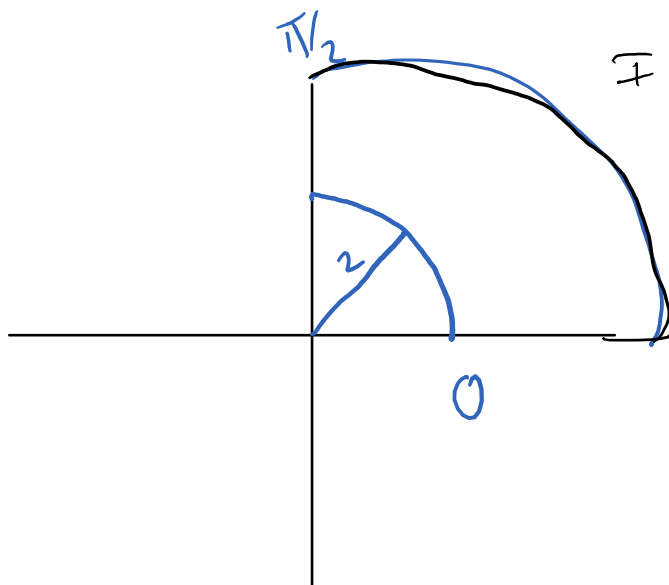
$w = z^2$ plane

Chapter 2 Analytic Functions, (on to the calculus!)

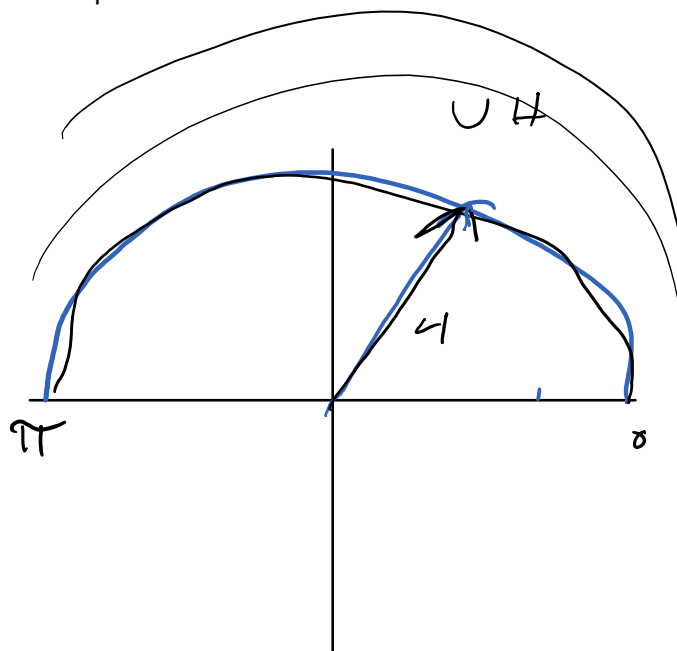
Let's see what happens to a different type of region, what if we take a circle of radius r defined by $z = re^{i\theta} = r(\cos(\theta) + i\sin(\theta))$?

Then $w = (re^{i\theta})^2 = r^2 e^{2i\theta} = r^2(\cos(2\theta) + i\sin(2\theta))$. From this we can see that the radius was squared and the period doubled.

Let's examine this by looking at the part of the circle in the first quadrant and see what happens to it.



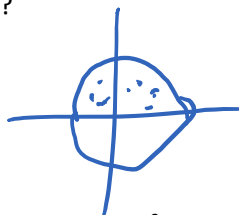
z - plane



w - plane

From this we can conclude that our map $w = z^2$ takes the first quadrant to the upper half plane as r can take on any value we wish.

What does $w = z^2$ do to the upper half plane? Is it a single valued function? What happens to the negative and positive real axes in the z plane?



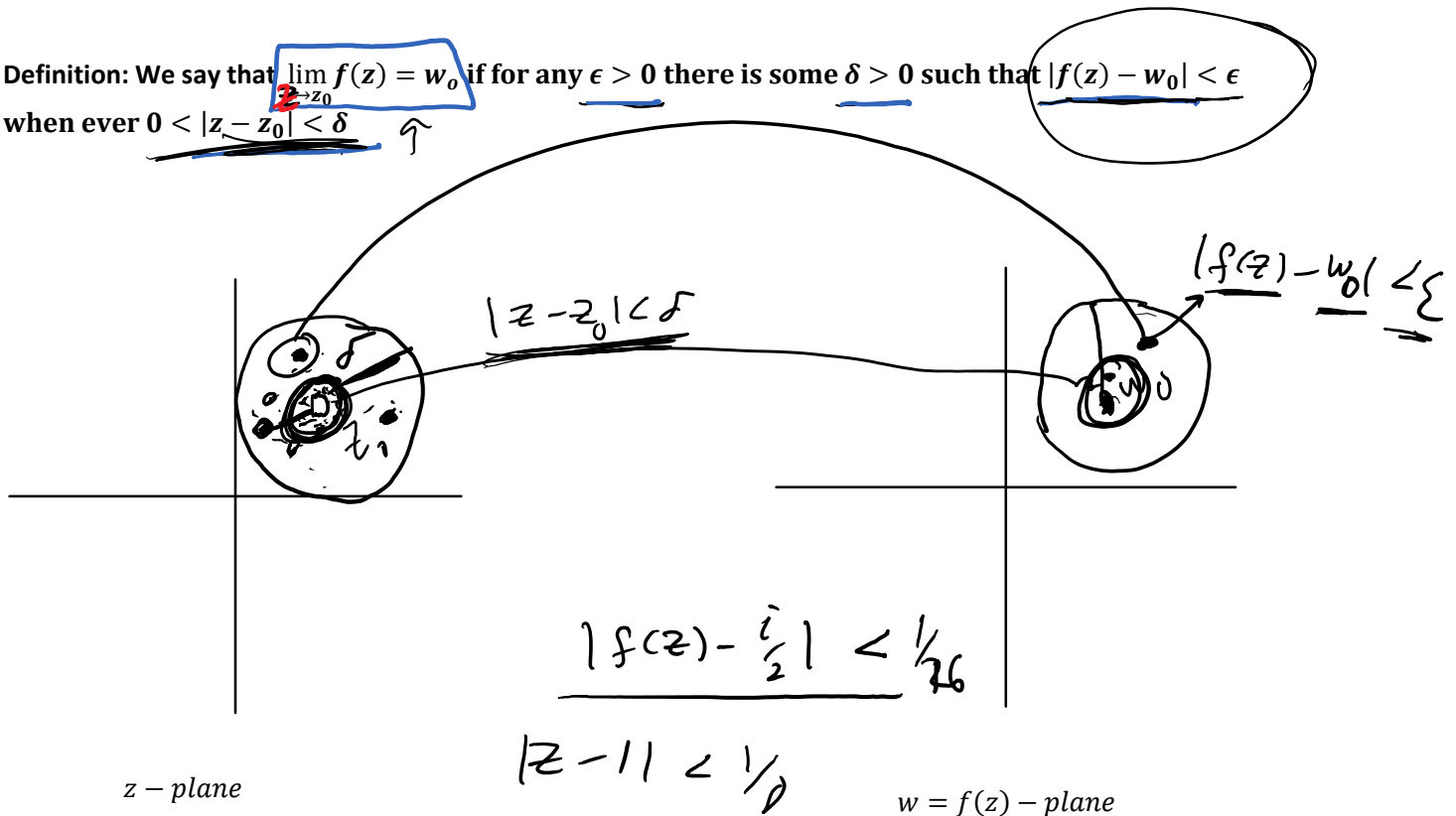
Through this idea of a mapping, and what happens to a specific region we can get a sense of what these complex functions are doing in a graphical sense.

Chapter 2 Analytic Functions, (on to the calculus!)

Complex Limits

Now that we have a sense of a complex function it is natural to ask about limits of complex functions.

Definition: We say that $\lim_{z \rightarrow z_0} f(z) = w_0$ if for any $\epsilon > 0$ there is some $\delta > 0$ such that $|f(z) - w_0| < \epsilon$ when ever $0 < |z - z_0| < \delta$



In words, we are saying that the point $w = f(z)$ can be made arbitrarily close to w_0 if we choose a point close enough to z_0 but not z_0 itself. That is, I could give you any tiny epsilon neighborhood around w_0 and if you can find a δ neighborhood around z_0 such that for any z value in that neighborhood the image is in the epsilon neighborhood in the w plane.

Let's do an example of an $\epsilon - \delta$ proof that $\lim_{z \rightarrow 1} \frac{i\bar{z}}{2} = \frac{i}{2}$. The trick is to start with $|f(z) - \frac{i}{2}|$ and make a series of inequalities until we end up with a multiple of $|z - 1|$.

$f(z) = \frac{i\bar{z}}{2}$
 $w_0 = \frac{i}{2}$

$\lim_{z \rightarrow 1} \frac{i\bar{z}}{2} = \frac{i}{2}$
 $\frac{i\bar{z}}{2} = i(x - iy) = ix + y$

$$\Rightarrow |f(z) - \frac{i}{2}| = \left| \frac{i\bar{z}}{2} - \frac{i}{2} \right| = \left| \frac{i}{2} \right| |\bar{z} - 1| = \frac{|\bar{z} - 1|}{2}$$

$$|\bar{z} - 1| = |x - iy - 1| = |(x-1) - iy| = \sqrt{(x-1)^2 + y^2}$$

$$|z - 1| = |x + iy - 1| = |(x-1) + iy| = \sqrt{(x-1)^2 + y^2}$$

$$\Rightarrow \frac{|z - 1|}{2} < \epsilon$$

$$\delta = 2\varepsilon$$

$$\boxed{|z-1| \leq 2\varepsilon}$$

$$\underline{|f(z) - i/2|} = \frac{|2\delta|}{2} \leq \frac{2\varepsilon}{2} = \varepsilon$$

$$\downarrow$$

$$\frac{|z-1|}{2} = \varepsilon$$

$$\frac{\delta}{2} = \varepsilon$$

Chapter 2 Analytic Functions, (on to the calculus!)

Let's do a homework example, we will prove that $\lim_{z \rightarrow 0} \frac{\bar{z}^2}{z} = 0$

$\downarrow \text{limit} = 0$

$\uparrow \frac{\bar{z}^2}{z}$

$\xrightarrow{\text{By estimates or inequalities}}$

$|f(z) - w_0| \rightarrow |z - 0| = |z| < \delta$

$\nearrow z_0$

$$\left| \frac{\bar{z}^2}{z} - 0 \right| = \left| \frac{\bar{z}^2}{z} \right| = \frac{|\bar{z}|^2}{|z|} = \frac{|\bar{z}|^2}{|z|} = \frac{|z|^2}{|z|} = |z| = \varepsilon$$

$\downarrow$
 $\delta = \varepsilon$

as long as $|z - 0| < \delta = \varepsilon \Rightarrow \left| \frac{\bar{z}^2}{z} - 0 \right| < \varepsilon$

Chapter 2 Analytic Functions, (on to the calculus!)

These $\epsilon - \delta$ proofs can get messy so we would like to have some limit theorems like we do in real number calculus to make life easier. Lucky for us there are some!

Limit Theorems:

Theorem 1

Let $f(z) = u(x, y) + iv(x, y)$, $z = x + iy$ and $z_0 = x_0 + iy_0$, $w_0 = u_0 + iv_0$.

If $\lim_{(x,y) \rightarrow (x_0,y_0)} u(x, y) = u_0$ and $\lim_{(x,y) \rightarrow (x_0,y_0)} v(x, y) = v_0$ then $\lim_{z \rightarrow z_0} f(z) = w_0$

Also,

If $\lim_{z \rightarrow z_0} f(z) = w_0$ Then $\lim_{(x,y) \rightarrow (x_0,y_0)} u(x, y) = u_0$ and $\lim_{(x,y) \rightarrow (x_0,y_0)} v(x, y) = v_0$

With this theorem it is easy to show when a limit does not exist. Let's do an example closely related to your homework. Let's show that $\lim_{z \rightarrow 0} \left(\frac{z}{\bar{z}} \right)$ Does not exist. The trick will be to approach from two different directions Let's go along the real axis then along the imaginary axis.

$$\lim_{z \rightarrow 0} \frac{z}{\bar{z}} = \lim_{z \rightarrow 0} \frac{x + iy}{x - iy}$$

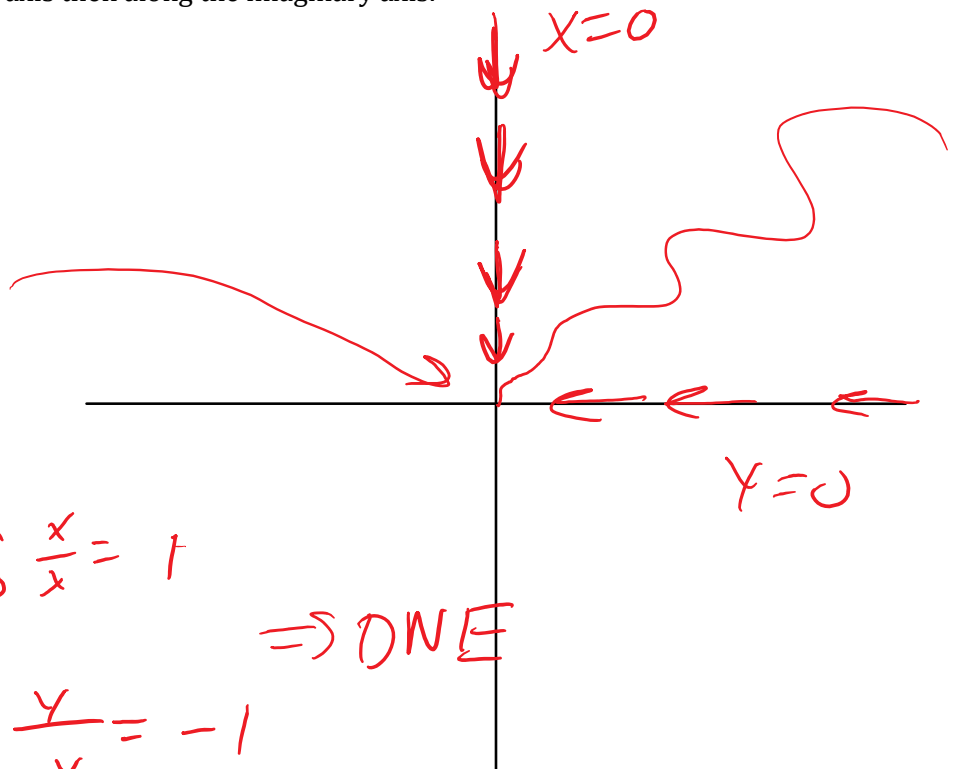
$$= \lim_{x,y \rightarrow (0,0)} \frac{x + iy}{x - iy}$$

Along the horizontal

$$\lim_{x \rightarrow 0} \frac{x + i(0)}{x - i(0)} = \lim_{x \rightarrow 0} \frac{x}{x} = 1$$

$$\lim_{y \rightarrow 0} \frac{0 + iy}{0 - iy} = \lim_{y \rightarrow 0} \frac{y}{-y} = -1$$

$\Rightarrow$ DNE



This is the tricky thing about this theorem, in general it is not easy to show that a two variable limit like $\lim_{(x,y) \rightarrow (x_0,y_0)} u(x, y)$ exists because there are infinite paths of approach. However, if we can find two paths where the limits disagree we can say the limit DNE!

Chapter 2 Analytic Functions, (on to the calculus!)

Theorem 2 (very useful!)

Suppose that $\lim_{z \rightarrow z_0} f(z) = w_0$ and $\lim_{z \rightarrow z_0} F(z) = W_0$

Then the following are true,

$$1. \lim_{z \rightarrow z_0} (f(z) + F(z)) = \lim_{z \rightarrow z_0} f(z) + \lim_{z \rightarrow z_0} F(z) = w_0 + W_0$$

$$2. \lim_{z \rightarrow z_0} (f(z)F(z)) = \lim_{z \rightarrow z_0} f(z) \lim_{z \rightarrow z_0} F(z) = w_0 W_0$$

$$3. \lim_{z \rightarrow z_0} \left(\frac{f(z)}{F(z)} \right) = \frac{\lim_{z \rightarrow z_0} f(z)}{\lim_{z \rightarrow z_0} F(z)} = \frac{w_0}{W_0}$$

$W_0 \neq 0$

We also get a couple limits for free to use.

$$\lim_{z \rightarrow z_0} c = c$$

$$\lim_{z \rightarrow z_0} z = z_0$$

Let's use these limits and our limits to find out what $\lim_{z \rightarrow z_0} z^n$ and then $\lim_{z \rightarrow z_0} a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \dots + a_n z^n$ are.

$$\begin{aligned} \lim_{z \rightarrow z_0} z^n &= \lim_{z \rightarrow z_0} \underbrace{z \cdot z \cdot z \cdot \dots \cdot z}_{n \text{ times}} = \underbrace{\lim_{z \rightarrow z_0} z \cdot \lim_{z \rightarrow z_0} z \cdot \dots \cdot \lim_{z \rightarrow z_0} z}_{n \text{ times}} \\ &= z_0^n \end{aligned}$$

$$\Rightarrow \boxed{\lim_{z \rightarrow z_0} z^n = z_0^n}$$

$$\begin{aligned} \lim_{z \rightarrow z_0} a_0 + a_1 z + a_2 z^2 + \dots &= \lim_{z \rightarrow z_0} a_0 + \lim_{z \rightarrow z_0} a_1 z + \dots + \lim_{z \rightarrow z_0} a_n z^n \\ &= \lim_{z \rightarrow z_0} a_0 + \lim_{z \rightarrow z_0} a_1 \lim_{z \rightarrow z_0} z + \dots + \lim_{z \rightarrow z_0} a_n \lim_{z \rightarrow z_0} z^n \\ &= a_0 + a_1 z_0 + \dots + a_n z_0^n \end{aligned}$$

$$\begin{aligned} P(z) &= a_0 + a_1 z + \dots + a_n z^n \\ \lim_{z \rightarrow z_0} P(z) &= P(z_0) \end{aligned}$$

$$= \begin{matrix} \downarrow \\ a_0 \end{matrix} + \begin{matrix} \downarrow \\ a_1 z_0 \end{matrix} + \dots + a_n z_0^n \quad \left| \begin{array}{l} P(z) = a_0 + a_1 z + \dots + a_n z^n \\ \lim_{z \rightarrow z_0} P(z) = P(z_0) \end{array} \right.$$

Chapter 2 Analytic Functions, (on to the calculus!)

Continuity of a complex function.

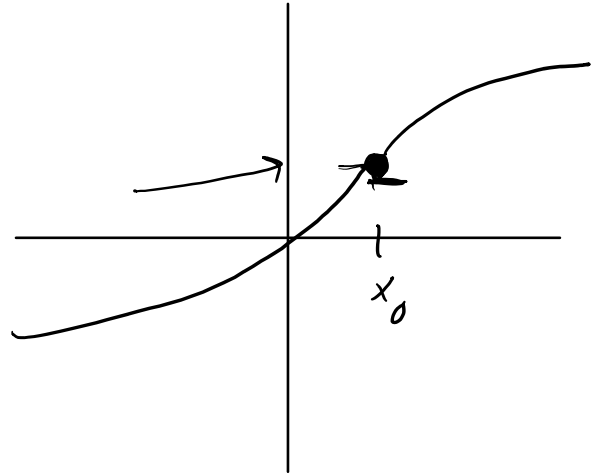
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You might recall the definition of a continuous function for real valued functions.

$f(x)$ is continuous at a point x_0 if and only if

1. $\lim_{x \rightarrow x_0} f(x)$ exists
2. $f(x_0)$ exists
3. $\lim_{x \rightarrow x_0} f(x) = f(x_0)$

These conditions just say that a continuous function at x_0 attains it's limit at that point for it's y value.



The definition for a complex function $f(z)$ is the same.

Definition:

$f(z)$ is continuous at a point z_0 if and only if

1. $\lim_{z \rightarrow z_0} f(z)$ exists
2. $f(z_0)$ exists
3. $\lim_{z \rightarrow z_0} f(z) = f(z_0)$

As you might expect, since we can split complex functions into a real and imaginary part $f(z) = u(x, y) + iv(x, y)$, our function f will only be continuous at $z_0 = x_0 + iy_0$ if and only if the component functions u, v are continuous at (x_0, y_0) .

This follows from Theorem 1 about limits!

One real benefit of this definition is that it makes taking the limit of a continuous function very easy, after all by the definition if $f(z)$ is continuous at z_0 then:

$$\lim_{z \rightarrow z_0} f(z) = \underline{f(z_0)}$$

This means we just plug in the point to our function and calculate the limit! So if we know a function is continuous at a point we want a limit we can get it by evaluating the function. That said here are some functions we know are continuous in the whole complex plane!

$$f(z) = c$$

$$f(z) = a_0 + a_1 z + a_2 z^2 + \dots + a_n z^n$$

$$f(z) = \sin(z)$$

$$f(z) = \cos(z)$$

Chapter 2 Analytic Functions, (on to the calculus!)

Since we love continuous functions it would be nice to know what combinations of continuous functions are continuous.

1. A sum of continuous functions is continuous
2. A composition of continuous functions continuous
3. A difference of continuous functions is continuous
4. A quotient of continuous functions is continuous wherever the denominator is non-zero
5. A composition of functions is continuous

Let's use this idea to find a limit.

Example: find $\lim_{z \rightarrow 1+i} \frac{z^2+z}{z+iz}$ *on t*

$$= \frac{(i+1)^2 + (i+1)}{(i+1) + i(i+1)}$$

A useful theorem in complex analysis is that

If $f(z)$ is continuous at z_0 and $f(z_0) \neq 0$ Then there is some neighborhood around z_0 for which $f(z) \neq 0$ for all z in that neighborhood.

The proof is actually easy:

Since f is continuous at z_0 the limit exists and thus satisfies the limit definition. That means we could choose $\epsilon = \frac{|f(z_0)|}{2}$ and we know that there is some $\delta > 0$ such that when $|z - z_0| < \delta$ Then $|f(z) - f(z_0)| < \epsilon = \frac{|f(z_0)|}{2}$ for all z in the delta disk. So if there were a z value where $f(z) = 0$ in the delta disk it would give the statement

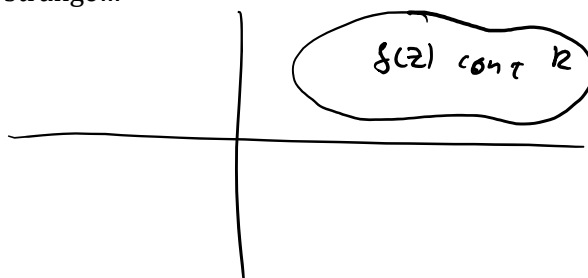
$$|f(z) - f(z_0)| = |0 - f(z_0)| = |f(z_0)| < \frac{|f(z_0)|}{2}$$

Which is clearly not possible!

Another theorem that will come in handy is that

If $f(z)$ is continuous at each point of a region closed and bounded region R then there exists an $M \geq 0$ such that $|f(z)| \leq M$

This tells that if f is continuous in a close bounded region it does not blow up at any pont in that region. However, the idea of infinity in the complex plain is a bit strange...

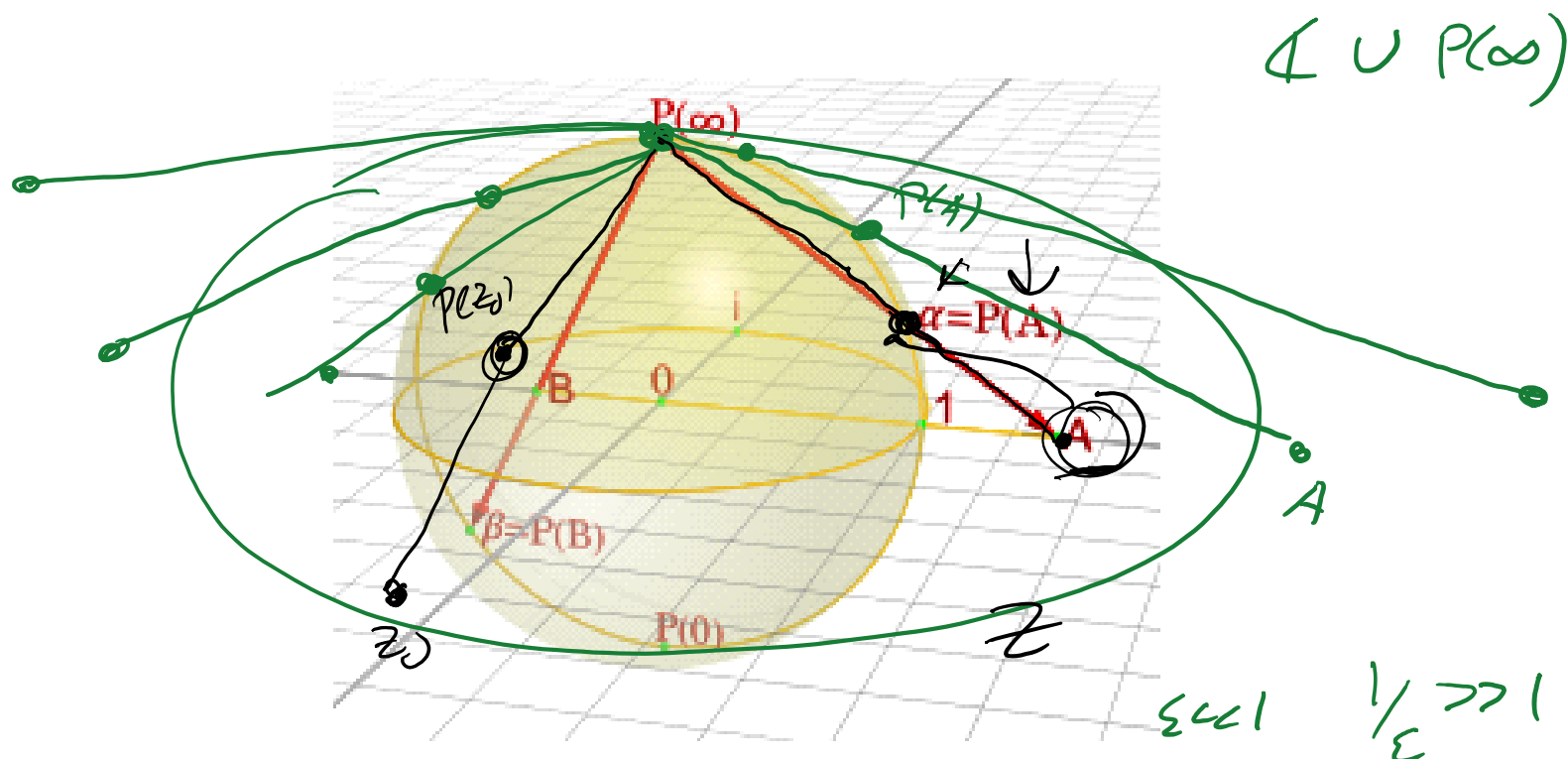


Chapter 2 Analytic Functions, (on to the calculus!)

Infinite Limits in the complex plane

In the real number system infinity makes a bit more sense because we can order the numbers, however we don't have ordering in the complex plane. We might think we could define infinity as infinitely far away from zero but to determine that we would need to take the magnitude of our complex numbers, that gives us a real number. What we really want is some way to talk about a complex infinite limit. It turns out there is a way to do this but it is a bit strange.

Here is how we add infinity to the complex plane, we call the combination the extended complex plane.



Notice that $P(\infty)$ is right above the origin. Consider a neighborhood around the origin of radius $\frac{1}{\epsilon}$ and let epsilon be small, thus $\frac{1}{\epsilon}$ is big. If we think about the points outside the circle $|z| > \frac{1}{\epsilon}$ by our map above these points will be in a small neighborhood of $P(\infty)$. Thus as epsilon gets smaller and smaller $\frac{1}{\epsilon}$ gets larger and larger and the points for which $|f(z)| > \frac{1}{\epsilon}$ have intersections with the sphere closer and closer to $P(\infty)$.

With this idea we can prove the following theorems.

$$1. \lim_{z \rightarrow z_0} f(z) = \infty \text{ if } \lim_{z \rightarrow z_0} \left(\frac{1}{f(z)} \right) = 0$$

$$2. \lim_{z \rightarrow \infty} f(z) = w_0 \text{ if } \lim_{z \rightarrow 0} f\left(\frac{1}{z}\right) = w_0$$

$$3. \lim_{z \rightarrow \infty} f(z) = \infty \text{ if } \lim_{z \rightarrow 0} \left(\frac{1}{f\left(\frac{1}{z}\right)} \right) = 0$$

Chapter 2 Analytic Functions, (on to the calculus!)

Proofs:

$$1. \lim_{z \rightarrow z_0} f(z) = \infty \quad \text{if} \quad \lim_{z \rightarrow z_0} \left(\frac{1}{f(z)} \right) = 0$$

$$2. \lim_{z \rightarrow \infty} f(z) = w_0 \quad \text{if} \quad \lim_{z \rightarrow 0} f\left(\frac{1}{z}\right) = w_0$$

$$3. \lim_{z \rightarrow \infty} f(z) = \infty \quad \text{if} \quad \lim_{z \rightarrow z_0} \left(\frac{1}{f\left(\frac{1}{z}\right)} \right) = 0$$

Chapter 2 Analytic Functions, (on to the calculus!)

Let's do a Homework Example:

1. Show that when

$$\rightarrow T(z) = \frac{az + b}{cz + d}, \quad ad - bc \neq 0$$

that

- (a) $\lim_{z \rightarrow \infty} T(z) = \infty$ if $c = 0$. $\leftarrow$ TH 3
 (b) $\lim_{z \rightarrow \infty} T(z) = \frac{a}{c}$, if $c \neq 0$. $\leftarrow$ TH 2
 (c) $\lim_{z \rightarrow -\frac{d}{c}} T(z) = \infty$, if $c \neq 0$. $\leftarrow$ TH 1

$$c=0 \quad T(z) = \frac{az + b}{0(z) + d} = \frac{az + b}{d} = \frac{a}{d}z + \frac{b}{d}$$

$$\rightarrow \lim_{z \rightarrow \infty} \frac{az + b}{d} = \infty$$

$$\lim_{z \rightarrow \infty} f(z) = \infty \text{ if } \lim_{z \rightarrow 0} \frac{1}{f(z)} = 0$$

$$f(1/z) = \frac{a(1/z) + b}{d}$$

$$1/f(z) = \frac{d}{a(1/z) + b}$$

$$\lim_{z \rightarrow 0} \frac{z d}{z(a(1/z) + b)} = \lim_{z \rightarrow 0} \frac{z d}{a + b z} = \frac{0 d}{a + b(0)}$$

$$= \frac{0}{a} = \boxed{0}$$