

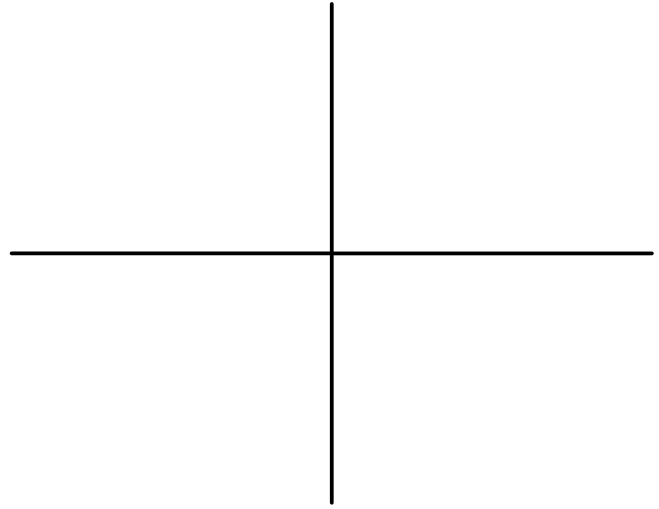
Chap 2 Sec 19-27

Derivatives!

If we are going to develop calculus for complex numbers we will certainly need derivatives! You might recall the definition of a derivative for a function of one real variable at a point x_0 .

$$\frac{df}{dx}(x_0) = \lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

That is the derivative measures the rate of change of change at the point x_0



The definition of a derivative takes on the same form for complex numbers.

Let f be a function whose domain of definition contains a neighborhood $|z - z_0| < \epsilon$. The derivative of f at the point z_0 is the limit:

$$\frac{df}{dz}(z_0) = \lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

We say the function f is differentiable at z_0 when this limit exists.

We often rewrite this definition by defining $\Delta z = z - z_0$ then our derivative definition limit takes the form

$$\frac{df}{dz}(z_0) = \lim_{\Delta z \rightarrow 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

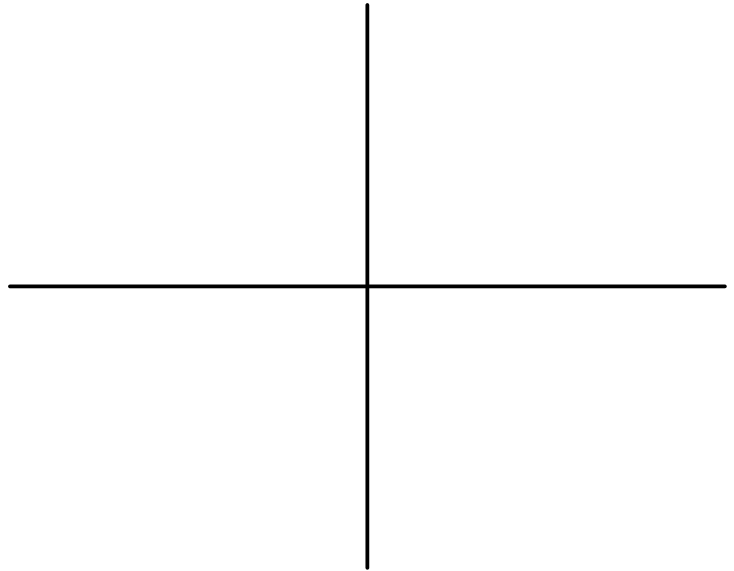
Yet another way to represent this is to think of $w = f(z)$ and define $\Delta w = f(z + \Delta z) - f(z_0)$ we can then write the derivative as

$$\frac{dw}{dz}(z_0) = \lim_{\Delta z \rightarrow 0} \frac{\Delta w}{\Delta z}$$

Chap 2 Sec 19-27

How do we interpret the complex derivative? This is not as simple as in the real case however we can think of in terms of scaling and rotations... For example if I multiply a complex number by $2i$ what does it do to that number?

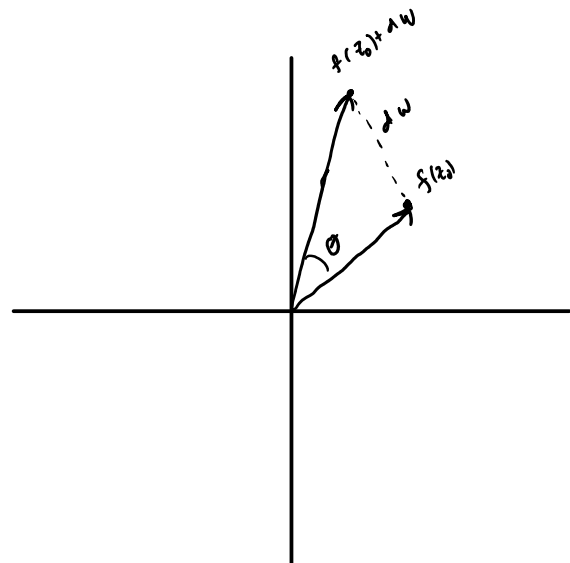
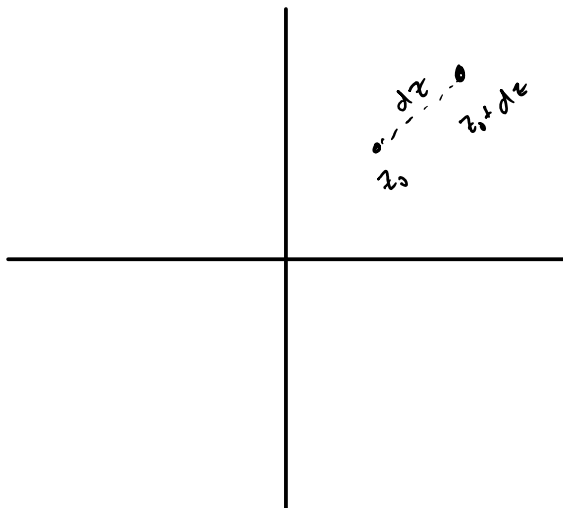
$$2iz = 2i(x + iy) = -2y + 2ix$$



So this simply rotates our number by 90 degrees and scales it by 2, In fact we can think of complex multiplication in terms of scaling and rotations. In this sense when we consider the derivative in terms of infinitesimal changes...

$$\frac{dw}{dz} = f'(z_0) \text{ means } dw = f'(z_0)dz$$

Then at $z = z_0$ for a change from z_0 to $z_0 + dz$ we get a change in $f(z_0)$ to $f(z_0) + dw$ by scaling and rotating dz by the amounts which correspond to the complex number $f'(z_0)$.



Chap 2 Sec 19-27

Let's calculate a derivative with the definition (Homework Example)

2. Apply the limit definition of the derivative to compute

$$\frac{dw}{dz} = -\frac{1}{z^2}, \quad \text{if } w(z) = \frac{1}{z}$$

when $z \neq 0$.

Chap 2 Sec 19-27

There will be functions which can't be differentiated anywhere. Let's illustrate this with a homework example

3. Show that $f'(z_0)$ does not exist at any point z_0

(a) $f(z) = \bar{z}$

Recall that we can approach our limit along any curve in the complex plane, let's go along the real axis and then along the imaginary axis.

Chap 2 Sec 19-27

One last example to think about.. Let's try to find the derivative of $f(z) = |z|$.

$$\Delta w = |z + \Delta z| - |z| = (z + \Delta z)\overline{(z + \Delta z)} - z\bar{z}$$

So

$$\begin{aligned}\lim_{\Delta z \rightarrow 0} \frac{\Delta w}{\Delta z} &= \lim_{\Delta z \rightarrow 0} \frac{(z + \Delta z)\overline{(z + \Delta z)} - z\bar{z}}{\Delta z} \\ &= \lim_{\Delta z \rightarrow 0} \bar{z} + \overline{\Delta z} + z \left(\frac{\overline{\Delta z}}{\Delta z} \right)\end{aligned}$$

Again let's approach along the real axis, in which case $\overline{\Delta z} = \Delta z$ and along the imaginary axis in which case $\overline{\Delta z} = -\Delta z$

If the limit exists then both limits must be the same which tells us $\bar{z} + z = \bar{z} - z$

This this limit can only exist if $z = 0$ and it is not differentiable at any other point!

This example illustrates 3 important things about derivatives of complex numbers.

1. A function $f(z) = u(x,y) + iv(x,y)$ can be differentiable at a point $z = (x,y)$ but at no other point close to it, ie in a neighborhood of that point.
2. $f(z) = |z|$ has real and imaginary parts of $u(x,y) = x^2 + y^2$ and $v(x,y) = 0$ both of which are differentiable on their own in the real numbers everywhere but the complex function is not.
3. Recall that if $u(x,y)$ and $v(x,y)$ are continuous in the plane then so is $f(z) = u + iv$ however $f(z)$ is not differentiable. This shows that continuity does not imply differentiability as with the real numbers.

Chap 2 Sec 19-27

Rules for Derivatives

Luckily many of the rules for derivatives of functions of real numbers apply to complex numbers, in fact all of our old favorites! If $f(z)$ is differentiable at z then we have

Power rule: $\frac{d}{dz} z^n = n z^{n-1}$

Sum Rule: $\frac{d}{dz} (f(z) + g(z)) = \frac{d}{dz} f(z) + \frac{d}{dz} g(z)$

Product rule: $\frac{d}{dz} f(z)g(z) = f(z) \frac{d}{dz} g(z) + g(z) \frac{d}{dz} f(z)$

Quotient rule: $\frac{d}{dz} \left[\frac{f(z)}{g(z)} \right] = \frac{\left(g(z) \frac{d}{dz} f(z) - f(z) \frac{d}{dz} g(z) \right)}{(g(z))^2}$

Chain Rule: $\frac{d}{dz} f(g(z)) = f'(g(z))g'(z)$

Chap 2 Sec 19-27

Cauchy-Riemann equations and sufficient conditions for differentiability

It's nice that we have the rules for derivatives on the previous page but they are only valid when $f(z)$ is differentiable, we might like to know when that is the case.

Recall that one way we can show a derivative does not exist is to check the horizontal and vertical approaches as $\Delta z \rightarrow 0$. With in this mind let's take a look at a complex function $f(x) = u(x, y) + iv(x, y)$ and consider the limit as $\Delta z \rightarrow 0$ or as $(\Delta x, \Delta y) \rightarrow (0, 0)$

$$\begin{aligned}\frac{df}{dz}(z_0) &= \lim_{\Delta z \rightarrow 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = \lim_{(\Delta x, \Delta y) \rightarrow (0, 0)} \frac{u(x + \Delta x, y + \Delta y) + iv(x + \Delta x, y + \Delta y) - u(x, y) - iv(x, y)}{\Delta x + i\Delta y} \\ &= \lim_{(\Delta x, \Delta y) \rightarrow (0, 0)} \frac{u(x + \Delta x, y + \Delta y) - u(x, y)}{\Delta x + i\Delta y} + i \frac{v(x + \Delta x, y + \Delta y) - v(x, y)}{\Delta x + i\Delta y}\end{aligned}$$

Coming along the real axis means $\Delta y = 0$

Coming along the imaginary axis means $\Delta x = 0$

Chap 2 Sec 19-27

So then if the derivative exists at the point (x, y) it must be that the limit does which means the values of those two limits need to be the same.

$$u_x(x, y) + iv_x(x, y) = v_y(x, y) - iu_y(x, y)$$

$$\text{Where } u_x = \frac{\partial u}{\partial x}, u_y = \frac{\partial u}{\partial y}, v_x = \frac{\partial v}{\partial x}, v_y = \frac{\partial v}{\partial y}$$

Equating the real and imaginary parts gives us the Cauchy-Riemann equations!

$$u_x = v_y, u_y = -v_x$$

This leads to the following theorem.

If $f(z) = u(x, y) + iv(x, y)$ and if we know that $f'(z)$ exists at a point (x, y) . Then the first-order partial derivatives of u, v must satisfy

$$u_x = v_y, u_y = -v_x$$

Further at the point (x, y) we can write the derivative as:

$$f'(z) = u_x + iv_x \quad \text{or} \quad f'(z) = v_y - iu_y$$

Note though that this theorem does not say that if $f(z)$ satisfies the C-R equations its derivative exists. It says if the derivative exists f will satisfy the C-R equations. However we can negate the theorem to find at what points a derivative does not exist... for example let's look at $f(z) = |z|$

Chap 2 Sec 19-27

Just as an example of a function which satisfies the C-R equations but is not differentiable consider: $f(z) =$

Chap 2 Sec 19-27

Sufficient conditions for differentiability.

Theorem: If $f(z) = u(x, y) + iv(x, y)$ is defined in some neighborhood of a point $z_0 = x_0 + iy_0$ and suppose that:

1. The first order partial derivatives of u and v exist everywhere in the neighborhood
2. Those first order partials satisfy the C-R equations $u_x = v_y$, $u_y = -v_x$ at (x_0, y_0) **and are continuous at (x_0, y_0)**

then $f'(z_0)$ exists and $f'(z_0) = u_x(x_0, y_0) + iv_x(x_0, y_0)$

Let's see where the function $f(z) = x^3 + i(1 - y)^3$ is differentiable.

What about $f(z) = e^x e^{iy}$?

Chap 2 Sec 19-27

C-R in polar coordinates

Assuming $z_0 \neq 0$ recall that $x = r\cos(\theta)$, $y = r\sin(\theta)$ and that we can write complex numbers in the forms: $z = r\cos(\theta) + ir\sin(\theta)$ and $z = re^{i\theta}$

From the chain rule we know that

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial r}$$

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial \theta}$$

$$\frac{\partial v}{\partial r} = \frac{\partial v}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial v}{\partial y} \frac{\partial y}{\partial r}$$

$$\frac{\partial v}{\partial \theta} = \frac{\partial v}{\partial x} \frac{\partial x}{\partial \theta} + \frac{\partial v}{\partial y} \frac{\partial y}{\partial \theta}$$

We can then calculate $u_r, u_\theta, v_r, v_\theta$

If we enforce the C-R equations for $u_x = v_y$, $u_y = -v_x$ for v_r, v_θ

We then find that $ru_r = v_\theta$ and $u_\theta = -rv_r$

Chap 2 Sec 19-27

Differentiability with a polar form

Theorem:

Let $f(r, \theta) = u(r, \theta) + iv(r, \theta)$ be defined throughout some neighborhood of a nonzero point $z_0 = r_0 e^{i\theta_0}$ and suppose that

1. The first order partial derivatives of the functions u and v with respect to r and θ exist everywhere in the neighborhood.
2. Those partial derivatives are continuous at (r_0, θ_0) and satisfy the polar C-R equations
 $ru_r = v_\theta$ and $u_\theta = -rv_r$

Then $f'(z_0)$ exists and $f'(z_0) = e^{-i\theta} (u_r(r_0, \theta_0) + iv_r(r_0, \theta_0))$

Let's Check for $f(z) = \frac{1}{z^2}$

Chap 2 Sec 19-27

Analytic Functions (the best kind) we also call these holomorphic or regular ...

A function $f(z)$ is said to be Analytic in an open set S if it has a derivative everywhere in that set. Note, this also implies it is continuous in that set.

We say that $f(z)$ is Analytic at a point z_0 if it is analytic in some neighborhood of z_0 . This also means that if $f(z)$ is analytic at a particular point then it must be analytic in some neighborhood of that point, this property will turn out to be quite useful.

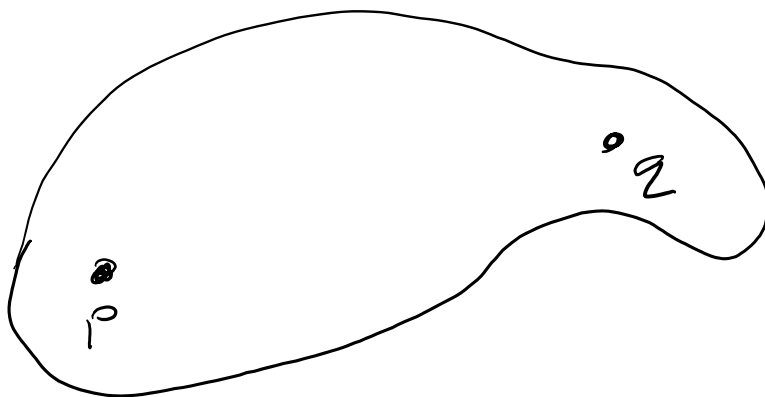
Example: Where is $f(z) = \frac{1}{z}$ analytic?

We also get that sums, products, compositions and quotients of analytic functions are themselves analytic provided we don't divide by zero anywhere.

We also get this neat theorem:

Theorem:

If $f(z)$ is analytic and $f'(z) = 0$ everywhere in a domain D , then $f(z)$ must be constant throughout D .



Chap 2 Sec 19-27

Harmonic Functions

A function $H(x, y)$ is said to be harmonic provided that the second partial derivatives are continuous and have sum zero that is.

$$H_{xx}(x, y) + H_{yy}(x, y) = 0$$

The above is known as Laplace's equation and is a partial differential equation which is important for everything from heat flow to fluid flow. This means harmonic functions are important! It turns out that if a function $f(z) = u(x, y) + iv(x, y)$ is analytic then both $u(x, y)$ and $v(x, y)$ are harmonic functions!

Let's just show this using the C-R equations since we know if f is analytic it must satisfy them.

If we have an analytic function $f(x) = u(x, y) + iv(x, y)$ we then refer to $u(x, y)$ and $v(x, y)$ as harmonic conjugates.

Chap 2 Sec 19-27

Homework Example

5. Determine if u is harmonic and if so find its harmonic conjugate v

(a) $u = 2x(1 - y)$