

Chapter 3 Sec 30-34

The exponential function

We have seen that we can represent a complex number z in the form $z = re^{i\theta}$ but what does it mean to say e^z where z is a complex number?

Let $z = x + iy$ then

$$e^z = e^{x+iy} = e^x e^{iy} = e^x (\cos(y) + i\sin(y)) \text{ Here we take } y \text{ in radians}$$

Note if z is real, then $e^z = e^{x+i0} = e^x$ so that we get back our old friend from the real numbers. Further since $e^x \neq 0$ this means that also $e^z \neq 0$

Suppose we assign $w = e^z$, w is itself a complex number and that means we can write it in exponential form $w = \rho e^{i\phi}$ Let's compare the two, Let $z = x + iy$ then.

$$\text{So } \rho = e^x \text{ and } \arg(e^z) = y + 2n\pi$$

What about $|e^z|$?

We also get some familiar properties of e^z we might remember from calculus.

1. $e^{z_1} e^{z_2} = e^{z_1+z_2}$
2. $\frac{e^{z_1}}{e^{z_2}} = e^{z_1-z_2}$
3. $(e^z)^n = e^{nz}$
4. $\frac{d}{dz} e^z = e^z$
5. e^z is entire (analytic everywhere in the complex plane)

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Let's prove some of them

$$e^{z_1}e^{z_2} = e^{z_1+z_2}$$

$$\frac{e^{z_1}}{e^{z_2}} = e^{z_1-z_2}$$

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Not everything is the same though, in the real numbers $e^x > 0$ Here however e^z can be negative. Just look at $e^{i\pi}$.

Another neat fact is that on the complex plane e^z is periodic! Recall a function is called P periodic if $F(z + P) = F(z)$

Consider then $F(z) = e^z$ and let's look at $F(z + 2\pi i) = e^{z+2\pi i}$

Homework example

7. Find all values of z such that

(a) $e^z = -2$

(b) $e^z = 1 + \sqrt{3}i$

(c) $\exp(2z - 1) = 1$

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The Logarithmic Function

If we have a complex exponential we would also like the logarithm. That is, we want to solve:

$$e^w = z \text{ for } w$$

Let $w = re^{i\theta}$ where θ is the principal argument, the one between $-\pi$ and π then

$$e^z = e^x e^{iy} = re^{i\theta}$$

We can then set the real and imaginary parts equal to each other.

$$\text{So, } w = \ln(r) + i(\theta + 2n\pi) \quad n = 0, \pm 1, \pm 2, \dots$$

Then if we write $\log(z) = \ln(r) + i(\theta + 2n\pi)$ we could see that $e^{\log(z)} = z$

What about the other way around if we define $\log(z) = \ln(r) + i(\theta + 2n\pi)$ is $\log(e^z) = z$?

Since we get $\log(e^z) = z + 2n\pi i$ we have to be careful and note that $\log(z)$ is a *multiple-valued function*

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Note that $\Theta + 2n\pi = \arg(z)$ and $r = |z|$ with this we can write the multiple-valued logarithmic function as

$$\log(z) = \ln |z| + i \arg(z)$$

We can also define the principal logarithm of $z = re^{i\theta}$ as $\text{Log}(z) = \ln(r) + i\Theta$ where Θ is the principal argument of z .

If we do this then $\text{Log}(e^z) = \ln |z| + i \text{Arg}(z) = \ln(e^x) + i\Theta = x + iy$ when $e^z = e^{x+iy} = e^x e^{iy}$

In this way we have a single valued function. We can then also write if we like:

$$\log(z) = \text{Log}(z) + 2n\pi i$$

Ok so we have a logarithm, what properties do we get? Unfortunately, as we will see not all of our favorites....

Let z_1, z_2 be any nonzero complex numbers...Then

$$1. \log(z_1 z_2) = \log(z_1) + \log(z_2)$$

$$2. \log\left(\frac{z_1}{z_2}\right) = \log(z_1) - \log(z_2)$$

$$3. \ln|z_1 z_2| = \ln|z_1| + \ln|z_2|$$

$$4. z^n = e^{n \log(z)}$$

$$5. z^{\frac{1}{n}} = e^{\frac{1}{n} \log(z)} = \exp\left[\left(\frac{1}{n}\right) \ln(r) + i\left(\frac{\Theta}{n} + \frac{2k\pi}{n}\right)\right] = r^{\frac{1}{n}} \exp\left[i\left(\frac{\Theta}{n} + \frac{2k\pi}{n}\right)\right]$$

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We can also say that if $\operatorname{Re}(z_1) > 0$ and $\operatorname{Re}(z_2) > 0$ then we do have the addition identity for the principal logarithm.

$$\operatorname{Log}(z_1 z_2) = \operatorname{Log}(z_1) + \operatorname{Log}(z_2) \quad \text{if } \operatorname{Re}(z_1) > 0 \quad \operatorname{Re}(z_2) > 0$$

The reason for this is that, when both our complex numbers are to the right of the imaginary axis their principal arguments both satisfy $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ that means that their sum will satisfy, $\pi < \theta_1 + \theta_2 < \pi$ so that when added the angle still sits in the principal argument range.

You may have noticed that one nice property that we have in real numbers, $\ln(x^n) = n \ln(x)$ was missing :(. In fact it is in general not true, let's illustrate with a homework problem.

10. (a) Show that $\log(i^2) = 2\log(i)$ when $\theta \in (\pi/4, 9\pi/4)$
(b) Show that $\log(i^2) \neq 2\log(i)$ when $\theta \in (3\pi/4, 11\pi/4)$

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With that pesky multivalued behavior of $\log$ taking away our beloved properties we might ask under what conditions might it be single valued...

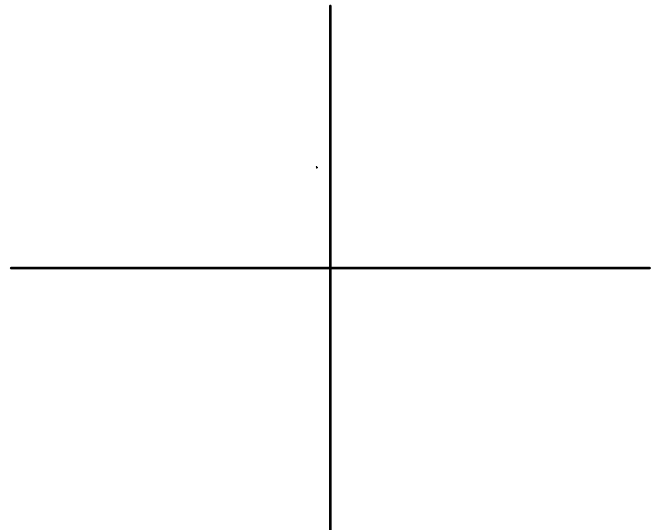
Branches and Derivatives of Logarithms

If $z = re^{i(\theta)}$ is a nonzero complex number we can get the exact same number by taking any integer multiple of 2π that is $z = re^{i\theta} = re^{i\theta}$ where $\theta = \theta + 2n\pi$

With this definition of θ we can write our multi-valued logarithm as $\log(z) = \ln(r) + i\theta$

We can make this multivalued if we restrict theta in the following way. Let α be any real number if we restrict θ to be between α and $\alpha + 2\pi$, $\alpha < \theta < \alpha + 2\pi$ then our $\log(z)$ function becomes single -valued

Note the strict inequality, that is because our logarithm is not continuous there . Not also that we could choose any alpha, and that as long as we are not on the ray defined at α our log function is single-valued and analytic. We call the α ray a **branch cut**. A branch cut is a curve we introduce into the complex plane to make a multi-valued function single valued on the remaining domain. $\log(z)$ on the restricted domain is referred to as a **branch** of the multivalued version of $\log(z)$. We will define this more later but let's first show $\log(z)$ is indeed analytic on our restricted domain.



Define $F(z) = \ln(r) + i\theta$ for $r > 0, \alpha < \theta < \alpha + 2\pi$

Then $F(z) = u(r, \theta) + iv(r, \theta)$ with $u(r, \theta) = \ln(r)$ $v(r, \theta) = \theta$ Checking the polar C-R equations $ru_r = v_\theta, u_\theta = -rv_r$

Since our first partials are continuous everywhere except at the α -ray and they satisfy the C-R equations we know that the derivative exists everywhere and infact is equal to

$$\frac{d}{dz}F(z) = \frac{d}{dz}\log(z) = e^{-i\theta}(u_r + iv_r) = e^{-i\theta}\left(\frac{1}{r} + 0i\right) = \frac{1}{re^{i\theta}} = \frac{1}{z} \quad \text{That is} \quad \frac{d}{dz}\log(z) = \frac{1}{z}$$

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Recall our homework example, they were in fact different branch cuts!

10. (a) Show that $\log(i^2) = 2\log(i)$ when $\theta \in (\pi/4, 9\pi/4)$
(b) Show that $\log(i^2) \neq 2\log(i)$ when $\theta \in (3\pi/4, 11\pi/4)$

This shows us that a function will behave differently on a different branch cut, infact once we make a branch cut we call the new single valued function a Branch of the multi-valued function.

Definition: A Branch of a multi-valued function $f(z)$ is a function $F(z)$ that is analytic in some domain at eac point z of which the value $F(z)$ is one of the values of f .

Definition: A Branch cut is a portion of a line or curve that is introduced in order to define a branch F of a multi-valued function f . Further, points on a branch cut are singular points(not analytic at z_0 but analytic on any neighbor hood of z_0).

Definition: A Branch Point is any point that is common to all branch cuts of $f(z)$.

