

Chapter 3 Sec 35-40

Some of our Favorite Functions in the complex plane.

We have already looked at e^z and $\log(z)$ and seen how they are a bit different in the complex plane. Let's investigate some other functions we know from the real numbers in the same way.

The Power Function $f(z) = z^c$

$$z^2$$

$$z^{i+2}$$

We define the power function for complex z, c in the following way:

$$z^c = e^{c \log(z)}$$

$$\log(z) = \ln r + i\theta$$

$$z = re^{i\theta}$$

To see why this works let's take c as an integer, something we have dealt with.

Suppose $z = re^{i\theta}$ then $z^c = r^c e^{ic\theta}$ and with the definition $z^c = e^{c \log(z)} = (e^{\log(z)})^c = (re^{i\theta})^c = r^c e^{ic\theta}$

The fact that this definition coincides when c is an integer or $c = \frac{1}{n}$ is what motivates this definition of z^c for any complex c

We can also say that $\frac{1}{z^c} = \frac{1}{e^{c \log(z)}} = e^{-c \log(z)} = z^{-c}$

This function is however multi-valued but we can make it single valued by choosing some branch cut $\alpha < \theta < \alpha + 2\pi$. For $\log(z) = \ln(r) + i\theta$. If we do this we then can look at the derivative.

$$\frac{d}{dz} z^c = \frac{d}{dz} e^{c \log(z)} = \frac{c}{z} e^{c \log(z)}$$

$$\frac{d}{dz} e^z = e^z$$

$$\frac{d}{dz} e^{c \log(z)} = e^{c \log(z)} \cdot \frac{d}{dz} (c \log(z)) = e^{c \log(z)} \cdot \frac{c}{z}$$

You might think, wait, you said $\frac{d}{dz} z^c = cz^{c-1}$ well indeed it does, recall again that $z = e^{\log(z)}$ let's plug this into the result above and verify that they indeed are the same!

$$\frac{c}{z} e^{c \log(z)} = \frac{c}{z} (e^{\log(z)})^c$$

$$= \frac{c}{z} z^c = c z^{c-1}$$

We say that we take the Principal value (P.V.) of z^c when we use $\text{Log}(z)$ instead of $\log(z)$

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$$z^c = e^{c \log(z)}$$

The exponential function of base c

From our definition of the ~~power~~ ^{power} function we can define the exponential function of base c by simply switching z and c!

$$c^z = e^{z \log(c)}$$

Let's take the derivative as before:

$$\frac{d}{dz} c^z = \frac{d}{dz} e^{z \log(c)} = \log(c) e^{z \log(c)} = \log(c) c^z$$

Homework Example:

1. Compute the following (show work)

(a) $(1+i)^i$

(b) $(-1)^{\frac{1}{\pi}}$

$$z^c = e^{c \log(z)}$$

a) $(1+i)^i$

$$= e^{i \log(1+i)}$$

$$= e^{i \log(\sqrt{2} e^{i\pi/4})}$$

$$= e^{i(\ln(\sqrt{2}) + i(\pi/4 + 2n\pi))}$$

$$= e^{i \ln(\sqrt{2})} e^{-\pi/4 - 2n\pi}$$

$$(1+i)^i = \underbrace{e^{-\pi/4 - 2n\pi}}_r e^{i \ln(\sqrt{2})} = r e^{i\theta}$$

$$\log(z) = \ln|r| + i\theta$$

$$1+i = r e^{i\theta} \quad r = \sqrt{1^2 + 1^2} = \sqrt{2}$$

$$1+i = \sqrt{2} e^{i\pi/4}$$



$$\tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{1}{1}\right)$$

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Rules for exponents

$$z^c = e^{c \log(z)}$$

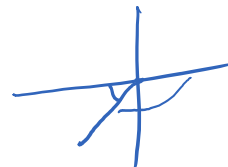
While we do have some familiar rules for exponents in the complex number system, namely:

1. $z^c z^d = z^{c+d}$
2. $\frac{z^c}{z^d} = z^{c-d}$
3. $(z^c)^n = z^{cn}$
4. $\frac{1}{z^c} = z^{-c}$ *can integer.*

$$-\pi < \theta \leq \pi$$

Others won't for certain numbers.

$$z_1 = \sqrt{2} e^{i\pi/4} \quad z_2 = \sqrt{2} e^{-i3\pi/4}$$



Let $z_1 = 1 + i$, $z_2 = -1 - i$ Let's show by taking the principal values that $(z_1 z_2)^i \neq z_1^i z_2^i$

$$\begin{aligned} (z_1 z_2)^i &= (\sqrt{2} e^{i\pi/4} \sqrt{2} e^{-i3\pi/4})^i \\ &= (2 e^{i(\pi/4 - 3\pi/4)})^i \\ &= (2 e^{-i\pi/2})^i \\ &= e^{i \log(2 e^{-i\pi/2})} \\ &= e^{i(\ln(2) + -\frac{\pi}{2}i)} \\ &= e^{i \ln(2) + \pi/2} \\ &= e^{\pi/2} e^{i \ln(2)} \end{aligned}$$

$$\begin{aligned} z_1^i z_2^i &= (\sqrt{2} e^{i\pi/4})^i (\sqrt{2} e^{-i3\pi/4})^i \\ &= e^{i \log(\sqrt{2} e^{i\pi/4})} e^{i \log(\sqrt{2} e^{-i3\pi/4})} \\ &= (e^{i(\ln(\sqrt{2}) + i\pi/4)}) (e^{i(\ln(\sqrt{2}) - i3\pi/4)}) \\ &= e^{i \ln(\sqrt{2}) - \pi/4} e^{i \ln(\sqrt{2}) + 3\pi/4} \\ &= e^{i \ln(\sqrt{2})} e^{i \ln(\sqrt{2})} e^{-\pi/4} e^{3\pi/4} \\ &= e^{i(\ln(\sqrt{2}) + \ln(\sqrt{2}))} e^{-\pi/4 + 3\pi/4} \\ &= e^{i \ln(2)} e^{\pi/2} \end{aligned}$$

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The Trigonometric Functions

We know that $e^{iz} = \cos(z) + i\sin(z)$ it turns out that we can use this to define the trig functions of a complex variable, that is

$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2}$$

$$\sin(z) = \frac{e^{iz} - e^{-iz}}{2i}$$

If z is a real number these formulas correspond exactly to the real versions of sine and cosine.

Let's just check that let $z = x$

$$\cos(x) = \frac{e^{ix} + e^{-ix}}{2}$$

$$\begin{aligned} \sin(x) &= \frac{\cos(x) + i\sin(x) - (\cos(x) - i\sin(x))}{2i} \\ &= \frac{\cancel{\cos(x)} + i\sin(x) - \cancel{\cos(x)} + i\sin(x)}{2i} \\ &= \frac{2i\sin(x)}{2i} = \sin(x) \end{aligned}$$

$$\begin{aligned} \cos(x) &= \frac{\cos(x) + i\sin(x) + \cos(-x) + i\sin(-x)}{2} \\ &= \frac{\cos(x) + i\sin(x) + \cos(x) - i\sin(x)}{2} = \frac{2\cos(x)}{2} = \cos(x) \end{aligned}$$

$$\frac{2i\sin(x)}{2i} = \sin(x)$$

$$\cos(x) = \frac{\cos(x) + i\sin(x) + \cos(x) - i\sin(x)}{2} = \frac{2\cos(x)}{2} = \cos(x)$$

Further since e^z is entire so are $\cos(z)$ and $\sin(z)$! Let's find their derivatives using the definition

$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2}$$

$$\sin(z) = \frac{e^{iz} - e^{-iz}}{2i}$$

$$\frac{d}{dz} \cos(z) = \frac{d}{dz} \frac{e^{iz} + e^{-iz}}{2} = \frac{\frac{d}{dz} e^{iz} + \frac{d}{dz} e^{-iz}}{2}$$

$$= \frac{ie^{iz} - ie^{-iz}}{2} = \frac{i(e^{iz} - e^{-iz})}{2}$$

$$= \frac{-ie^{iz} + ie^{-iz}}{2} = -\sin(z)$$

$$\frac{1}{i} = -i$$

$$\frac{d}{dz} \sin(z) = \cos(z)$$

$$\frac{d}{dz} \tan(z) = \sec^2 z$$

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We also get some of our favorite trig identities as well.

$$\begin{aligned} \rightarrow \sin(z_1 + z_2) &= \sin(z_1) \cos(z_2) + \cos(z_1) \sin(z_2) \\ \rightarrow \cos(z_1 + z_2) &= \cos(z_1) \cos(z_2) - \sin(z_1) \sin(z_2) \end{aligned}$$

We can use these to show

$$\sin(2z) = 2 \sin(z) \cos(z), \cos(2z) = \cos^2 z - \sin^2 z, \sin\left(z + \frac{\pi}{2}\right) = \cos(z), \sin\left(z - \frac{\pi}{2}\right) = -\cos(z)$$

The Pythagorean identity holds as well

$$\sin^2 z + \cos^2 z = 1$$

The functions also keep their 2π Periodicity as adding 2π amounts to multiplying by 1, Let's just show that.

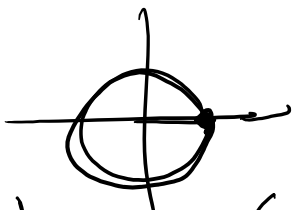
$$\sin(z) = \frac{e^{iz} - e^{-iz}}{2i}$$

$$\sin(z + 2\pi) = \frac{e^{i(z+2\pi)} - e^{-i(z+2\pi)}}{2i}$$

$$= \frac{e^{iz} e^{i2\pi} - e^{-iz} e^{-i2\pi}}{2i}$$

$$= \frac{e^{iz} (1) - e^{-iz} (1)}{2i}$$

$$= \frac{e^{iz} - e^{-iz}}{2i} = \sin(z)$$

$$\begin{aligned} e^{i2\pi} &= 1 \\ e^{-i2\pi} &= 1 \\ \cos(2\pi) + i \sin(2\pi) &= 1 + i \cdot 0 = 1 \end{aligned}$$


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One thing we have been doing is decomposing our complex functions into forms like $f(z) = \underline{u(x, y)} + i \underline{v(x, y)}$. Naturally we might ask, what is this decomposition for our trig functions.

$$\sin(z) = \sin(x) \cosh(y) + i \cos(x) \sinh(y)$$

$$\cos(z) = \cos(x) \cosh(y) + i \sin(x) \sinh(y)$$

$$\sin(z_1 + z_2) = \sin(z_1) \cos(z_2) + \sin(z_2) \cos(z_1)$$

Let's show this for $\sin(z)$ recall that $\cosh(y) = \frac{e^y + e^{-y}}{2}$, $\sinh(y) = \frac{e^y - e^{-y}}{2}$ and we will employ our additive identity with $\underline{z_1 = x}$ $\underline{z_2 = iy}$ noting that $\sin(iy) = i \sinh(y)$ and $\cos(iy) = \cosh(y)$

$$z_1 = x \quad z_2 = iy \quad z_1 + z_2 = x + iy = z \quad \frac{1}{i} = -i$$

$$\sin(iy) = \frac{e^{i(iy)} - e^{-i(iy)}}{2i} = \frac{e^{-y} - e^y}{2i} = i \frac{e^y - e^{-y}}{2}$$

$$\sin(z_1 + z_2) = \sin(x + iy) = \sin(x) \cos(iy) + \cos(x) \sin(iy)$$

$$\sin(z) = \sin(x) \cosh(y) + \cos(x) i (\sinh(y))$$

$$\sin(z) = \sin(x) \cosh(y) + i \cos(x) \sinh(y)$$

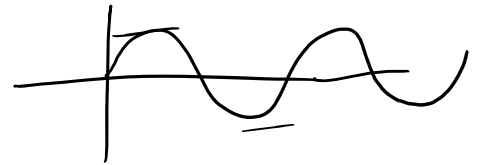
$$u(x, y) + i(v(x, y))$$

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As a result of these formulas we can show one major difference between sine and cosine on the real line vs the complex plane. On the real line we know that $|\sin(x)| \leq 1$ and $|\cos(x)| \leq 1$. However in the complex plane this is not true...

$$\sin(z) = \sin(x) \cosh(y) + i \cos(x) \sinh(y)$$

$$\cos(z) = \cos(x) \cosh(y) + i \sin(x) \sinh(y)$$



Let's take the moduli and show this for sine at least.

When I sat down to do this a nicer proof came to mind
Proof as opposed to the algebraic mess we avoided today...
We just need the Identities ① $\cos^2(x) + \sin^2(x) = 1$ ② $\cosh^2(y) - \sinh^2(y) = 1$

So:

$$|\sin(z)|^2 = \sin^2(x) \cosh^2(y) + \cos^2(x) \sinh^2(y)$$

$$\downarrow \text{①} \qquad \qquad \downarrow \text{②}$$

$$= 1 + \sinh^2(y) \qquad = 1 - \sin^2(x)$$

$$|\sin(z)|^2 = \sin^2(x) (1 + \sinh^2(y)) + (1 - \sin^2(x)) \sinh^2(y)$$

$$= \sin^2(x) + \cancel{\sin^2(x) \sinh^2(y)} + \sinh^2(y) - \cancel{\sin^2(x) \sinh^2(y)}$$

$$= \sin^2(x) + \sinh^2(y)$$

and

$$|\cos(z)|^2 = \cos^2(x) \cosh^2(y) + \sin^2(x) \sinh^2(y)$$

$$\text{①} \downarrow \qquad \qquad \text{②} \downarrow$$

$$= 1 + \sinh^2(y) \qquad = 1 - \cos^2(x)$$

$$= \cos^2(x) (1 + \sinh^2(y)) + (1 - \cos^2(x)) \sinh^2(y)$$

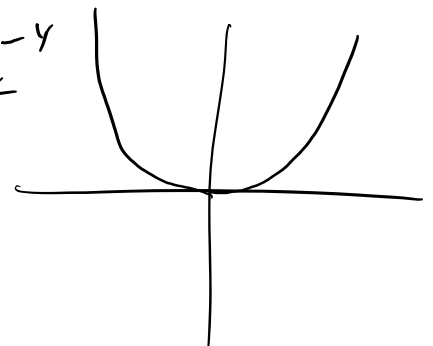
$$= \cos^2(x) + \cancel{\cos^2(x) \sinh^2(y)} + \sinh^2(y) - \cancel{\cos^2(x) \sinh^2(y)}$$

$$= \cos^2(x) + \sinh^2(y)$$

$$|\sin(z)|^2 = \sin^2(x) + \sinh^2(y) \geq 1$$

$$|\cos(z)|^2 = \cos^2(x) + \sinh^2(y)$$

$\frac{e^y - e^{-y}}{2}$



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$$x^2 + 1 = 0$$

$$x^2 = -1$$

$$x = \pm i$$

As it turns out unlike the case of some functions, like $x^2 + 1$, extending the sine and cosine functions into the complex plane does not change their zeros! In fact we have the following theorem.

Theorem: The zeros of $\sin(z)$ and $\cos(z)$ are the same in the complex plane as the zeros of $\sin(x)$ and $\cos(x)$ on the real line. That is:

$\sin(z) = 0$ if and only if $z = n\pi$ where n is an integer

And

$$x + iy = n\pi + i0$$

$\cos(z) = 0$ if and only if $z = \frac{\pi}{2} + n\pi$ where n is an integer

It is easy to show this, let's do it for $\sin(z)$

Recall $|\sin(z)|^2 = \sin^2(x) + \sinh^2(y)$

Let's suppose $\sin(z) = 0$ and see what that tells us.

$$\sin(z) = 0$$

$$|\sin(z)|^2 = 0 = \sin^2(x) + \sinh^2(y)$$

$$\Rightarrow y = 0$$

$z = x + iy$ that solves $\sin(z) = 0$

must have zero Im part.

$$\sin^2(x) = 0$$

$$\sinh(n\pi) \neq 0$$

$$\frac{e^y - e^{-y}}{2} \text{ only } 0 \text{ when } y=0$$

$$\frac{e^0 - e^{-0}}{2} = \frac{1-1}{2} = 0$$

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What about the other Trig functions? It turns out that they are define in the usual way .

$$\begin{aligned} \frac{e^{iz} - e^{-iz}}{2i} & \quad \leftarrow \tan(z) = \frac{\sin(z)}{\cos(z)} \\ \frac{e^{iz} + e^{-iz}}{2} & \quad \leftarrow \sec(z) = \frac{1}{\cos(z)} \\ \frac{e^{iz} - e^{-iz}}{i(e^{iz} + e^{-iz})} & \quad \leftarrow \cot(z) = \frac{\cos(z)}{\sin(z)} \\ & \quad \leftarrow \csc(z) = \frac{1}{\sin(z)} \end{aligned}$$

One can also show that they have the usual derivatives

$$\frac{d}{dz} \tan(z) = \sec^2(z)$$

$$\frac{d}{dz} \sec(z) = \sec(z) \tan(z)$$

$$\frac{d}{dz} \cot(z) = -\csc^2(z)$$

$$\frac{d}{dz} \csc(z) = -\csc(z) \cot(z)$$

Homework Example

$\sin$

4. Show that

$$(a) \sinh(z + i\pi) = -\sinh(z)$$

$$\sinh(z) = \frac{e^z - e^{-z}}{2} \quad \sinh(z + i\pi) = \frac{e^{z+i\pi} - e^{-(z+i\pi)}}{2}$$

$$= \frac{e^z e^{i\pi} - e^{-z} e^{-i\pi}}{2} = \frac{e^z (-1) - e^{-z} (-1)}{2}$$

$$= -\left(\frac{e^z - e^{-z}}{2}\right) = -\sinh(z)$$

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Hyperbolic Functions in the complex plane

The Hyperbolic trig functions retain their definition in the complex plane that is.

$$\sinh(z) = \frac{e^z - e^{-z}}{2}$$

$$\cosh(z) = \frac{e^z + e^{-z}}{2}$$

Their derivatives are also the same

$$\frac{d}{dz} \sinh(z) = \cosh(z)$$

$$\frac{d}{dz} \cosh(z) = \sinh(z)$$

In contrast to the regular trig functions the zeros of the hyperbolic trig functions lie on the imaginary axis instead of the real line.

Theorem: The zeros of $\sinh(z)$ and $\cosh(z)$ lie in the complex plane lie on the imaginary axis and,

$$\sinh(z) = 0 \text{ if and only if } z = n\pi i$$

And

$$\cosh(z) = 0 \text{ if and only if } z = \left(\frac{\pi}{2} + n\pi\right) i$$

Where n is an integer.

$$\sin(ix) = i \sinh(x)$$

$$\sinh(n\pi i) = \frac{e^{n\pi i} - e^{-n\pi i}}{2}$$

$$\sin(n\pi) = i \left(\frac{e^{n\pi i} - e^{-n\pi i}}{2} \right)$$

$$\sinh(x) = \frac{e^x - e^{-x}}{2} \quad \cosh(x) = \frac{e^x + e^{-x}}{2}$$

You might be realizing how related trig and hyperbolic trig functions are, after they are all definable with the exponential! Even their derivative structure is very similar.

$$\frac{d}{dz} \tanh(z) = \text{sech}^2(z)$$

$$\frac{d}{dz} (\coth(z)) = -\text{csch}^2(z)$$

$$\frac{d}{dz} \text{sech}(z) = -\text{sech}(z) \tanh(z)$$

$$\frac{d}{dz} \text{csch}(z) = -\text{csch}(z) \coth(z)$$

Where naturally,

$$\tanh(z) = \frac{\sinh(z)}{\cosh(z)}$$

$$\text{csch}(z) = \frac{1}{\sinh(z)}$$

$$\text{sech}(z) = \frac{1}{\cosh(z)}$$

$$\coth(z) = \frac{\cosh(z)}{\sinh(z)}$$

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$$\log(z) = \ln|z| + i\theta$$

Inverses of trig and hyperbolic trig functions

$$\log(e^{iw}) = \cancel{\ln(1)} + iw$$

We know how useful $\arcsin(x)$, $\arccos(x)$, and $\arctan(x)$ can be in calculus and we would like the same kind of function here.

Let's start with $\arcsin(z)$ if $w = \arcsin(z)$ then this must mean $z = \sin(w)$

That is $z = \frac{e^{iw} - e^{-iw}}{2i}$ Let's try and solve this for w ...

$$z = \frac{e^{iw} - e^{-iw}}{2i} \rightarrow (e^{iw}) 2iz = (e^{iw} - e^{-iw}) e^{iw}$$

$$2ize^{iw} = e^{2iw} - 1$$

$$e^{iw} = \frac{2iz + \sqrt{(2iz)^2 - 4(1)(-1)}}{2}$$

$$e^{iw} = iz + \frac{\sqrt{-4z^2 + 4}}{2}$$

$$(e^{iw})^2 - 2ize^{iw} + 1 = 0 \quad e^{iw} = iz + \sqrt{1-z^2} \quad \log(e^z) = z$$

$$a=1 \quad b=-2iz \quad c=-1$$

$$e^{iw} = iz + \sqrt{1-z^2}$$

$$\log(e^{iw}) = \log(iz + \sqrt{1-z^2})$$

$$iw = \log(iz + \sqrt{1-z^2}) \rightarrow w = -i \log(iz + \sqrt{1-z^2})$$

Then $\arcsin(z) = -i \log[iz + (1-z^2)^{\frac{1}{2}}]$. This function is very multivalued infact it has infinitely many values for a given z !

To illustrate, let's look at $\arcsin(-i) = -i \log(1 \pm \sqrt{2})$

$$-i \log(iz + (1-z^2)^{\frac{1}{2}}) \rightarrow -i \log(1 \pm \sqrt{2})$$

With the plus sign, $-i \log(1 + \sqrt{2}) = -i(\ln(1 + \sqrt{2}) + 2n\pi)$ and with the minus sign $-i(\ln(1 - \sqrt{2}) + \pi + 2n\pi)$

$$i(\log(1 + \sqrt{2})) = i \log(1 \pm 2)$$

We can play the same quadratic solution game as above to find that :

$$\boxed{\arccos(z) = -i \log[z + i(1-z^2)^{\frac{1}{2}}]} \quad \text{and} \quad \arctan(z) = \frac{i}{2} \log\left(\frac{i+z}{i-z}\right)$$

$$\sqrt{2} \rightarrow +\sqrt{2} \quad -\sqrt{2}$$

$$\sqrt{2} = 1.41\dots$$

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The derivatives follow those for the real counterparts of the inverse trig functions that is.

$$\frac{d}{dz} \arcsin(z) = \frac{1}{\sqrt{1-z^2}}$$

$$\frac{d}{dz} \arccos(z) = \frac{-1}{\sqrt{1-z^2}}$$

$$\frac{d}{dz} \arctan(z) = \frac{1}{1+z^2}$$

Our inverse hyperbolic functions are given by:

$$\operatorname{arcsinh}(z) = \log \left[z + (z^2 + 1)^{\frac{1}{2}} \right]$$

$$\operatorname{arccosh}(z) = \log \left[z + (z^2 - 1)^{\frac{1}{2}} \right]$$

$$\operatorname{arctanh}(z) = \frac{1}{2} \log \left(\frac{1+z}{1-z} \right)$$

Homework Example

5. Find all the values of

(a) $\tan^{-1}(2i)$

(b) $\tan^{-1}(1+i)$

(c) $\cosh^{-1}(-1)$

(d) $\tanh^{-1}(0)$

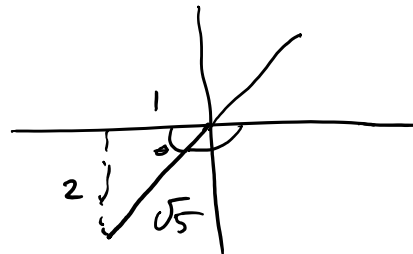
$$\tan^{-1}(z) = \frac{i}{2} \log \left(\frac{i+z}{i-z} \right)$$

b) $\tan^{-1}(1+i)$ $z = 1+i$

$$\frac{i}{2} \log \left(\frac{i+1+i}{i-(1+i)} \right) = \frac{i}{2} \log \left(\frac{1+2i}{-1} \right)$$

$$= \frac{i}{2} \log(-1-2i)$$

$$= \frac{i}{2} (\ln|z| + i \arg(z))$$



$$= \boxed{\frac{i}{2} (\ln(\sqrt{5}) + i [\arctan(2) - \pi] + 2n\pi)} \quad \text{①} = \arctan(2) - \pi$$

$$-\pi \leq \theta \leq \pi$$

$$|z| = \sqrt{x^2 + y^2} = \sqrt{1^2 + 2^2} = \sqrt{5}$$

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$$z_1 = (1-i) \quad z_2 = (-1-i)$$

$$(z_1 z_2)^i = (\sqrt{2} e^{-\pi/4 i} \sqrt{2} e^{-3\pi/4 i})^i$$

$$= (2 e^{-\cancel{4\pi/4} i})^i$$

$$= (-2)^i = e^{i \log(2e^{i\pi})}$$

$$= e^{i(\ln(2) + i\pi)}$$

$$= e^{i \ln(2) - \pi}$$

$$\boxed{e^{-\pi} e^{i \ln(2)}}$$

$$(1-i)^i (-1-i)^i = (\sqrt{2} e^{i(\pi/4)})^i (\sqrt{2} e^{i(-3\pi/4)})^i$$

$$= e^{i(\log(\sqrt{2} e^{i(\pi/4)}))} e^{i \log(\sqrt{2} e^{i(3\pi/4)})}$$

$$= e^{i(\ln(\sqrt{2}) - i\pi/4)} e^{i(\ln(\sqrt{2}) - i3\pi/4)}$$

$$= e^{i(\ln \sqrt{2} + \ln \sqrt{2} - i(\frac{\pi}{4} + \frac{3\pi}{4}))} = e^{i \ln(2)} e^{\pi} =$$

