

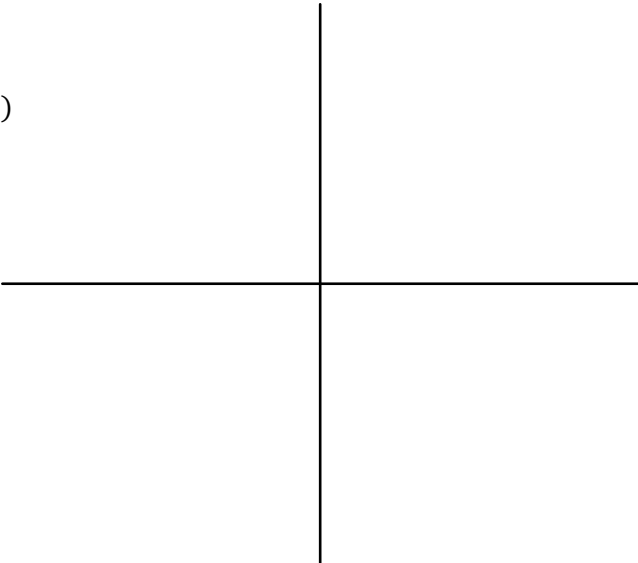
Chapter 4 Sec 41-45

Now that we have talked about functions and derivatives it is about time for us to talk about integration. While we will develop a fundamental theorem of calculus for complex functions first we need to talk about Contour Integrals.

A contour integral is an integral over piece-wise smooth arc in the complex plane. So let's define that!

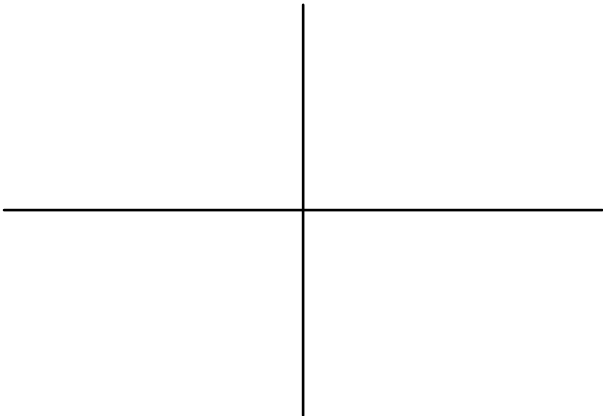
A set of points $z = (x, y)$ is said to be an arc if $x = x(t), y = y(t)$ for $a \leq t \leq b$ Where $x(t), y(t)$ are continuous functions of the real paramater t .

Example of an arc: Suppose $z(t) = (x(t), y(t)) = x(t) + iy(t)$
where $x(t) = 2t$ and $y(t) = t^2$



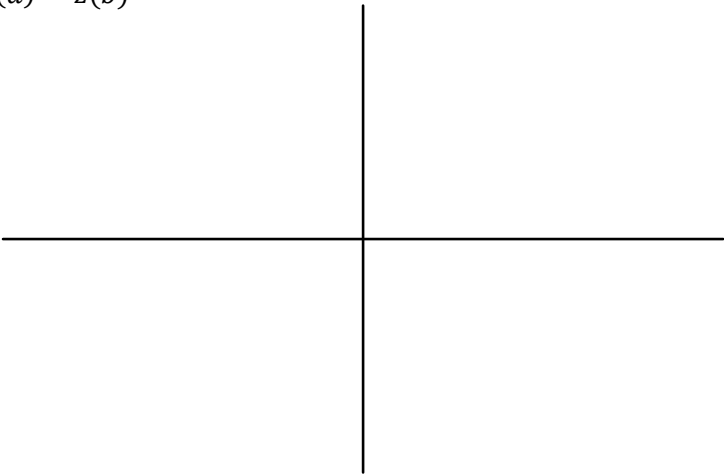
We of course have some classifications of arcs...

A **Simple Arc** or **Jordan arc** is an arc that does not cross itself.
That is if $t_1 \neq t_2$ then $z(t_1) \neq z(t_2)$



A **simple closed curve** or **Jordan curve** is a simple arc which only crosses itself at the endpoints of its time domain $a \leq t \leq b$ that is $z(a) = z(b)$

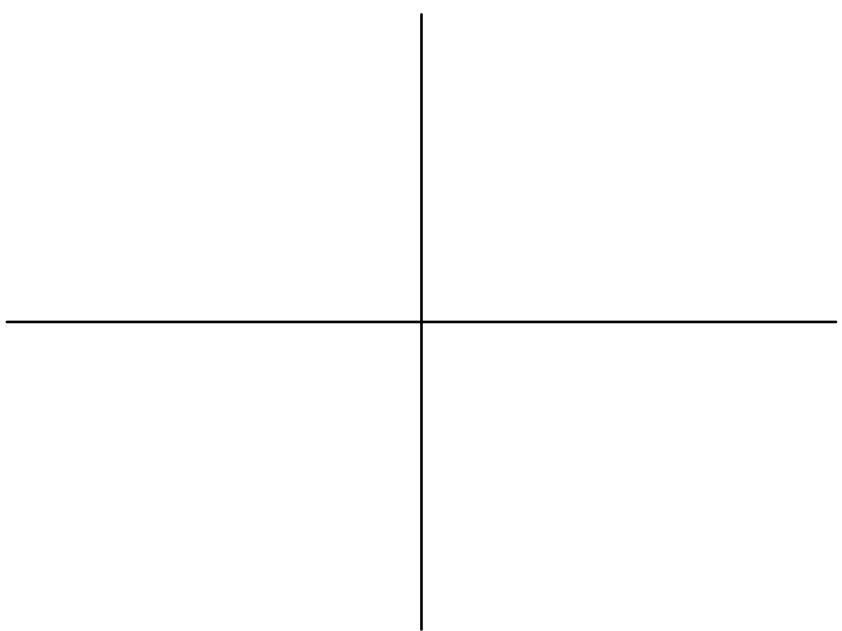
Further we say the simple closed curve is positively oriented
If when we go from $t = a$ to $t = b$ we traverse the curve
Counterclockwise.



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We said a contour was a piecewise smooth arc. By Smooth we mean that $z'(t)$ is continuous on the closed interval $a \leq t \leq b$

A **Contour** is a piece wise smooth arc . By piecewise we mean that we have joined together a finite number of smooth arcs so that $z'(t)$ is peicewise continuous Incidentally if $z'(t)$ is peicewise continuous $z(t)$ is continuous.



Homework Example

Find parameterized representations $z(t)$ of the following contours in the plane including t -ranges.

- (a) A straight line from point $(1 + 2i)$ to point $(i + 2)$
- (b) A line from $(0, 0)$ to point $(1 + \sqrt{3}i)$
- (c) A half-ellipse from point 2 to -2 passing through i centered at the origin.

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We mentioned that we require $z'(t)$ to be piecewise continuous for a contour, we should talk about what $z'(t)$ means here.

Let $z(t) = x(t) + iy(t)$ where $x(t)$ and $y(t)$ are real valued functions of t then

$$\frac{dz}{dt} = z'(t) = x'(t) + iy'(t)$$

where the prime denotes the full derivative with respect to t .

Most of the rules for differentiation of real functions of variables apply here but care should be taken. The things we do get for free are: (Provided everything is analytic on our smooth arc)

Constant multiple rule

$$\frac{d}{dt}(z_0 z(t)) = z_0 \frac{d}{dt} z(t)$$

The sum rule:

$$\frac{d}{dt}(z(t) + w(t)) = \frac{d}{dt} z(t) + \frac{d}{dt} w(t)$$

The Product rule:

$$\frac{d}{dt}(z(t)w(t)) = z(t) \frac{d}{dt} w(t) + w(t) \frac{d}{dt} z(t)$$

The Quotient Rule

$$\frac{d}{dt} \left(\frac{z(t)}{w(t)} \right) = \frac{z'(t)w(t) - w'(t)z(t)}{(w'(t))^2}$$

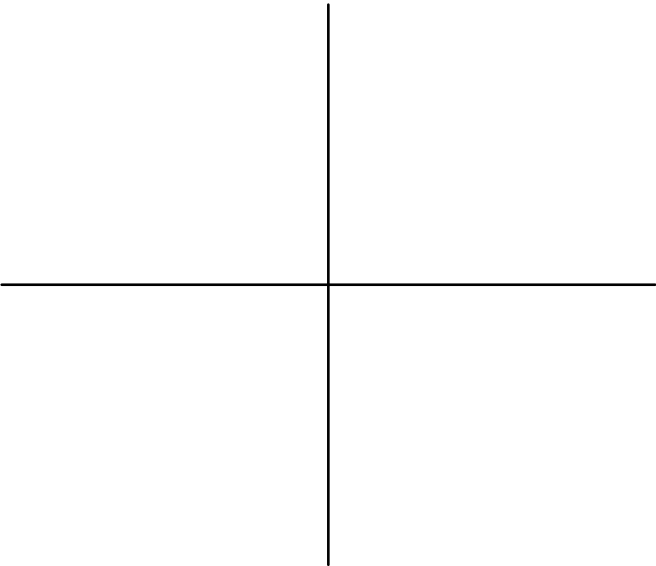
The Chain Rule

$$\frac{d}{dt} f(z(t)) = f'(z(t))z'(t)$$

Exponential Differentiation

$$\frac{d}{dt} e^{z_0 t} = z_0 e^{z_0 t}$$

Example: Suppose $f(t) = e^{it}$ find $f'(t)$ and interpret this in the sense of vectors.



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Definite integrals of functions $w(t) = u(t) + iv(t)$

If $w(t) = u(t) + iv(t)$ for $a \leq t \leq b$ and $u(t), v(t)$ are integrable on (a, b) then:

$$\int_a^b w(t) dt = \int_a^b u(t) dt + i \int_a^b v(t) dt$$

Homework Example:

Evaluate the following integrals and sketch the set that is generated from the integrand $w(t)$

(a) $\int_1^2 \left(\frac{1}{t} - i\right)^2 dt$

(b) $\int_0^{\pi/6} e^{i2t} dt$

As long as $u(t), v(t)$ are piecewise continuous so is $w(t)$ and we can also say

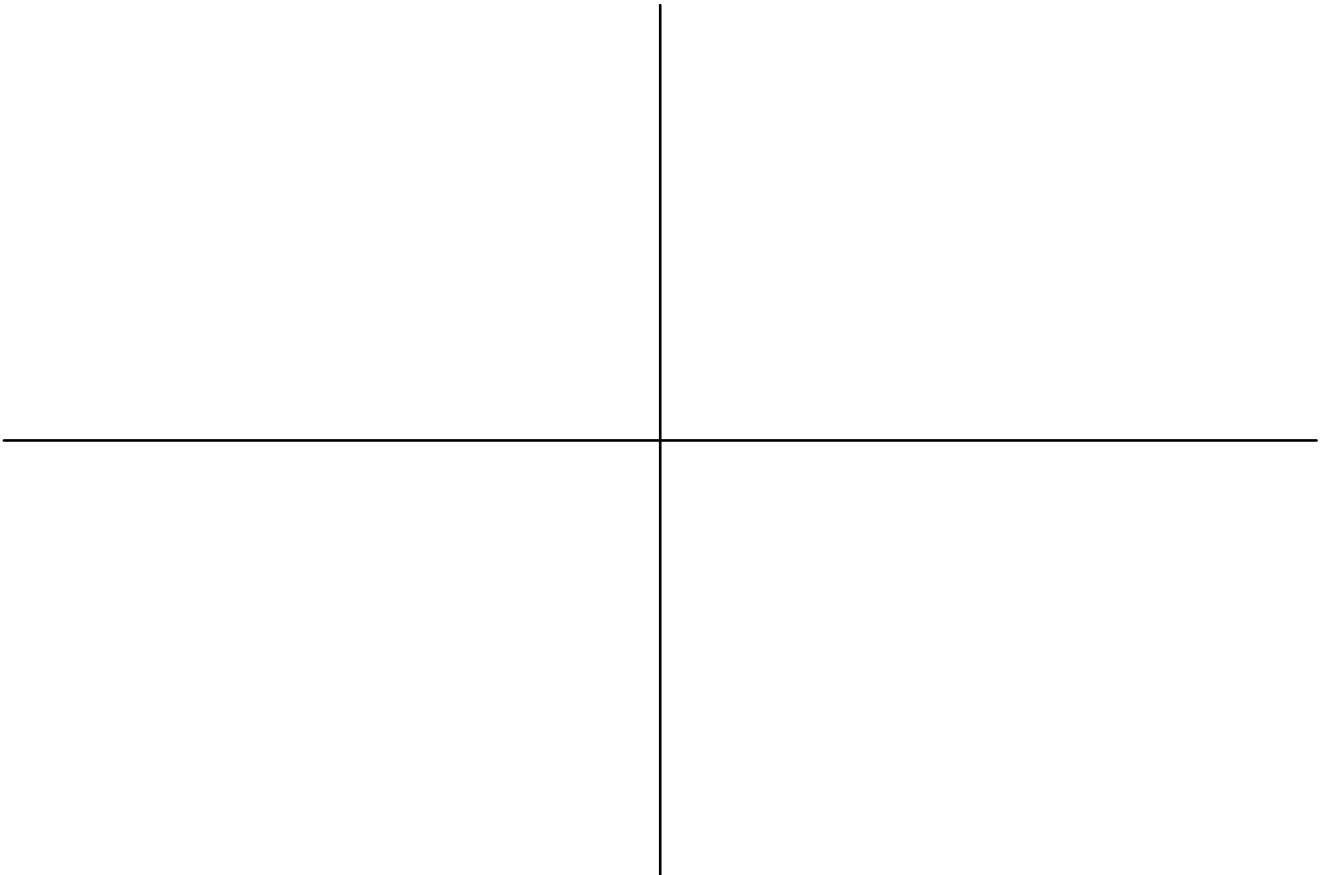
$$\int_a^b w(t) dt = \int_a^c w(t) dt + \int_c^b w(t) dt \quad \text{where } a \leq c \leq b$$

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The Length of a Contour

We calculate the length of any given contour in the same way as you might in the real numbers with a parametric function.

$$L = \int_a^b |z'(t)| dt = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2} dt$$



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Contour Integrals

A contour integral is really just a line integral in the complex plane, however we will see later some useful theorems can help us evaluate them simply. For now we will just focus on the actual integration along a contour.

Definition: For a given contour C we define the contour integral of f along C to be:

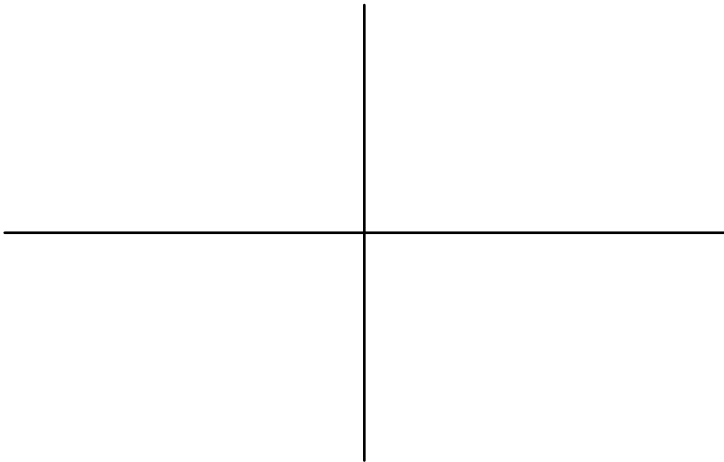
$$\int_C f(z)dz = \int_a^b f(z(t))z'(t)dt \quad \text{where } C \text{ is paramaterized by } z(t) \text{ for } a \leq t \leq b$$

Some important rules which go along with contour integrals are the following

$$\int_C z_0 f(z)dz = z_0 \int_C f(z)dz$$

$$\int_C f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz \quad \text{where } C = C_1 + C_2$$

$$\int_{-C} f(z)dz = - \int_C f(z)dz$$



If the value of the contour integral between $z_1 = f(a)$ and $z_2 = f(b)$ is the same regardless of the contour we often write:

$$\int_C f(z)dz = \int_{z_1}^{z_2} f(z)dz$$

Later we will use functions like this to define our fundamental theorem for integrals. It will turn out that as long as $f(z)$ is continuous and has an antiderivative everywhere in some domain D then the contour integrals will be path independent.

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Homework Example

Compute $\int_C f(z)dz$ for $f = z - 1$ for the contours

- (a) A semicircle $z = 2e^{i\theta}$ for $\theta \in [\pi, 2\pi]$.
- (b) A line on the real axis from 2 to -2 .

