

Chapter 4 Sec 55-57

We now have the amazing ...

Cauchy Integral Formula.

Let $f(z)$ be analytic everywhere inside and on a simple closed contour C which is positively oriented. If z_0 is any point in the interior of C then

$$f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_0} dz$$

$$\int_C \frac{f(z)}{z - z_0} dz = 2\pi i f(z_0)$$

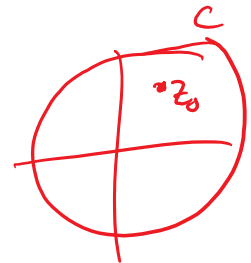


Now we will study an extension of this formula which relates the derivatives of $f(z)$ to the value of a contour integral. As well as some consequences of our nifty formula.

The extended Cauchy Integral Formula.

Let $f(z)$ be analytic inside and on a simple closed contour C (positively oriented), taken in the positive sense. If z_0 is any point interior to C then

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{n+1}} dz \quad n = 0, 1, 2, \dots$$



Or equivalently

$$\int_C \frac{f(z)}{(z - z_0)^{n+1}} dz = \frac{2\pi i}{n!} f^{(n)}(z_0) = \quad n = 0, 1, 2, \dots$$

Here $f^{(n)}$ is the n^{th} derivative of f

This formula is useful as now we have any power of $(z - z_0)$ in the denominator instead of just the first power. This means that if we have a tough contour integral to deal with we can try to work the integrand into a form for which this theorem applies, if we can do that then evaluating the integral is as simple as plugging in the point into the derivative.

For example: Let C be the contour $|z| = 1$ that has the origin in its interior let's evaluate: $\int_C \frac{\sin(z)}{(z^4 + 2z^2)^2} dz$

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$$\int_C \frac{\sin(z)}{(z^4+2z^2)^2} dz \quad \int_C \frac{f(z)}{(z-z_0)^{n+1}} dz = \frac{2\pi i}{n!} f^{(n)}(z_0)$$

$$= \int_C \frac{\sin(z)}{(z^2(z^2+2))^2} dz$$

$$= \int_C \frac{\sin(z)}{(z^2)^2 (z^2+2)^2} dz$$

$$= \int_C \frac{\sin(z)}{(z^4)(z^2+2)^2}$$

$$= \int_C \frac{\sin(z)}{(z^2+2)^2} \frac{1}{z^4} dz$$

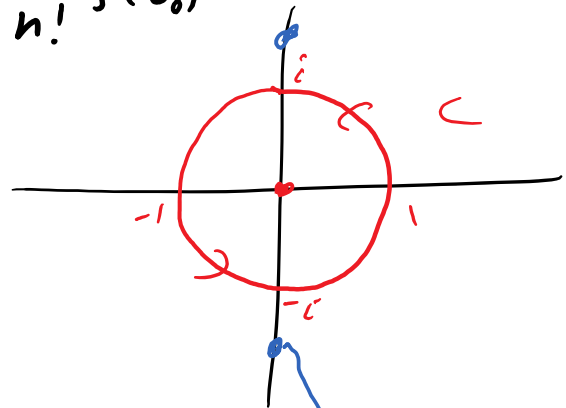
$$= \int_C \frac{\sin(z)}{(z^2+2)^2} \frac{1}{(z-0)^4}$$

$\uparrow$
 z_0

$$\int_C \frac{f(z)}{(z-z_0)^{n+1}} dz = \frac{2\pi i}{n!} f^{(n)}(z)$$

$$f(z) = \frac{\sin(z)}{(z^2+2)^2}$$

$\pm \sqrt{2}i$ not analytic



$$\int_C \frac{\sin(z)}{(z^2+2)^2} \frac{1}{(z-0)^4} dz = \frac{2\pi i}{3!} \frac{d^3}{dz^3} \frac{\sin(z)}{(z^2+2)^2} \Big|_{z=0}$$

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The extended Cauchy Integral Formula.

Let $f(z)$ be analytic inside and on a simple closed contour C (positively oriented), taken in the positive sense. If z_0 is any point interior to C then

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{n+1}} dz \quad n = 0, 1, 2, \dots$$

Or equivalently

$$\int_C \frac{f(z)}{(z - z_0)^{n+1}} dz = \frac{2\pi i}{n!} f^{(n)}(z_0) = \quad n = 0, 1, 2, \dots$$

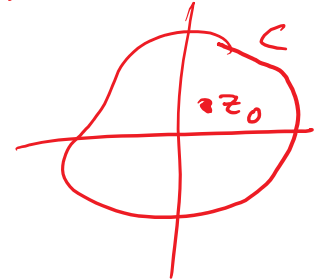
$$f'(z_0) = \frac{1!}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{1+1}} dz$$

The actual proof of this formula is done by induction and is a bit messy, we will show it for the first derivative though to give you a sense as to why this is true.

That is we want to show that :

$$f'(z_0) = \frac{1!}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^2} dz$$

$f(z)$ analytic

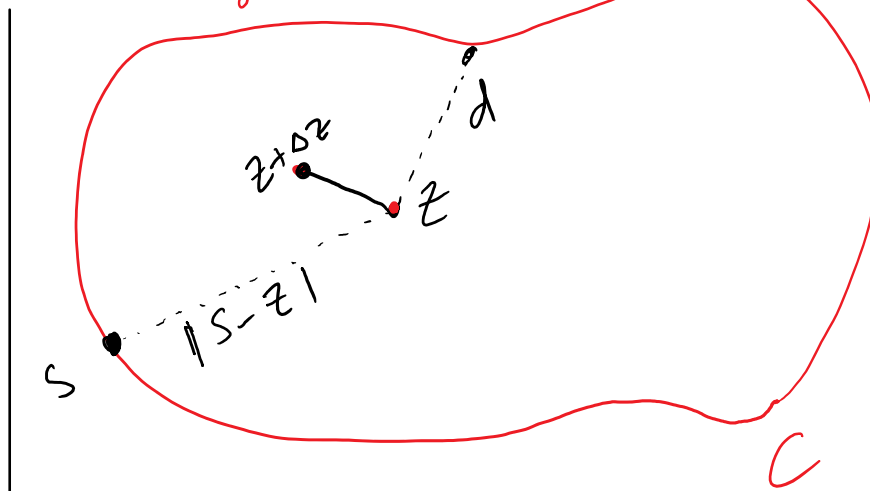


First we turn to the definition of the derivative:

$$f'(z) = \lim_{\Delta z \rightarrow 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$$

From the original Cauchy formula we know that if C is a contour that contains both z and $z + \Delta z$ Then we can say:

$$f(z + \Delta z) = \frac{1}{2\pi i} \int_C \frac{f(s)}{s - (z + \Delta z)} ds \quad \text{and} \quad f(z) = \frac{1}{2\pi i} \int_C \frac{f(s)}{s - z} ds$$



We let d be the smallest distance between z and the edge of the contour, then we enforce $|\Delta z| < d$ that way we know $z + \Delta z$ is in the region enclosed by C .

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So now we have $f(z + \Delta z) = \frac{1}{2\pi i} \int_C \frac{f(s)}{s - (z + \Delta z)} ds$ and $f(z) = \frac{1}{2\pi i} \int_C \frac{f(s)}{s - z} ds$

Let's use these to rewrite our definition of a derivative in terms of these integrals.

$$\begin{aligned} \lim_{\Delta z \rightarrow 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} &= \lim_{\Delta z \rightarrow 0} \frac{1}{\Delta z} \left(\frac{1}{2\pi i} \int_C \frac{f(s)}{s - (z + \Delta z)} ds - \frac{1}{2\pi i} \int_C \frac{f(s)}{s - z} ds \right) \\ &= \lim_{\Delta z \rightarrow 0} \frac{1}{\Delta z} \left(\frac{1}{2\pi i} \int_C \left(\frac{f(s)}{s - (z + \Delta z)} - \frac{f(s)}{s - z} \right) ds \right) \\ &= \lim_{\Delta z \rightarrow 0} \left(\frac{1}{2\pi i} \int_C \left(\frac{1}{s - z - \Delta z} - \frac{1}{s - z} \right) \frac{f(s)}{\Delta z} ds \right) \end{aligned}$$

So we can say then that

$$f'(z) = \lim_{\Delta z \rightarrow 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} = \lim_{\Delta z \rightarrow 0} \frac{1}{2\pi i} \int_C \left(\frac{1}{s - z - \Delta z} - \frac{1}{s - z} \right) \frac{f(s)}{\Delta z} ds = \lim_{\Delta z \rightarrow 0} \frac{1}{2\pi i} \int_C \frac{f(s)}{(s - z - \Delta z)(s - z)} ds$$

Finding a common denominator and adding the fractions in the integrand gives us.

$$f'(z) = \lim_{\Delta z \rightarrow 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} = \lim_{\Delta z \rightarrow 0} \frac{1}{2\pi i} \int_C \left(\frac{f(s)}{(s - z - \Delta z)(s - z)} \right) ds$$

Let's show that :)

$$\begin{aligned} &\frac{(s - z)}{(s - z)(s - z - \Delta z)} - \frac{1}{s - z} \frac{(s - z - \Delta z)}{(s - z - \Delta z)} \\ &= \frac{(s - z) - (s - z - \Delta z)}{(s - z - \Delta z)(s - z)} = \frac{s - z - s + z + \Delta z}{(s - z - \Delta z)(s - z)} \\ &= \frac{\Delta z}{(s - z - \Delta z)(s - z)} \end{aligned}$$

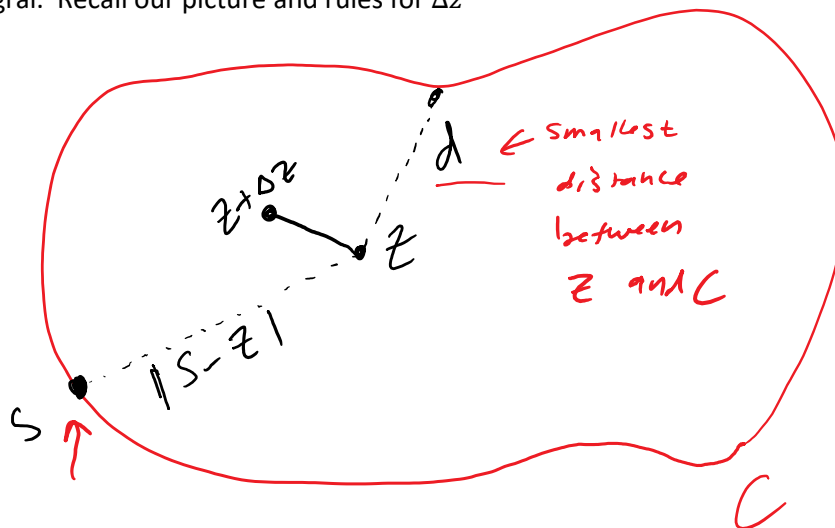
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$$\frac{f(s)}{(s-z)(s-z)} = \frac{f(s)}{(s-z)^2}$$

So we now have:

$$f'(z) = \lim_{\Delta z \rightarrow 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} = \lim_{\Delta z \rightarrow 0} \frac{1}{2\pi i} \int_C \left(\frac{f(s)}{(s-z-\Delta z)(s-z)} \right) ds$$

At this point we might be tempted to just pass our limit under the integral sign, while this might be ok for some functions it is not for all.. So to be general we will find another way to show our formula by estimating the integral. Recall our picture and rules for Δz



We let d be the smallest distance between z and the edge of the contour, then we enforce $|\Delta z| < d$ that way we know $z + \Delta z$ is in the region enclosed by C .

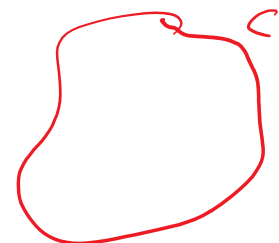
With this in mind we can say that $|\Delta z| < d$ $|s - z| \geq d$ and through the use of the reverse triangle inequality $|z_1 - z_2| \geq ||z_1| - |z_2||$

$$|s - z - \Delta z| = |(s - z) - \Delta z| \geq ||s - z| - |\Delta z|| \geq d - |\Delta z|$$

We will also use an old theorem which says that if $f(z)$ is analytic on a bounded domain D then there exists some M such that $|f(z)| \leq M$ on the whole domain D . Armed with these inequalities we again turn to showing that the difference of the two things we want to be equal is zero to prove our equality, that is we will now show

$$\lim_{\Delta z \rightarrow 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} - \frac{1}{2\pi i} \int_C \left(\frac{f(s)}{(s-z-\Delta z)(s-z)} \right) ds = 0$$

$f'(z)$



In order to do this we will need to cleverly rewrite $\frac{f(s)}{(s-z-\Delta z)(s-z)}$

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It turns out that

$$\frac{f(s)}{(s-z-\Delta z)(s-z)} = \frac{f(s)}{(s-z)^2} + \frac{f(s)\Delta z}{(s-z-\Delta z)(s-z)^2}$$

Let's verify it....

$$\begin{aligned} & \frac{f(s)}{(s-z)^2} + \frac{f(s)\Delta z}{(s-z-\Delta z)(s-z)^2} \\ &= \frac{f(s)(s-z-\Delta z) + f(s)\Delta z}{(s-z-\Delta z)(s-z)^2} = \\ &= \frac{f(s)s - f(s)z - f(s)\Delta z + f(s)\Delta z}{(s-z-\Delta z)(s-z)^2} \end{aligned}$$

$\rightarrow \frac{f(s)(s-z)}{(s-z-\Delta z)(s-z)^2}$
 $\rightarrow \frac{f(s)}{(s-z-\Delta z)(s-z)}$

With the identity above we can write:

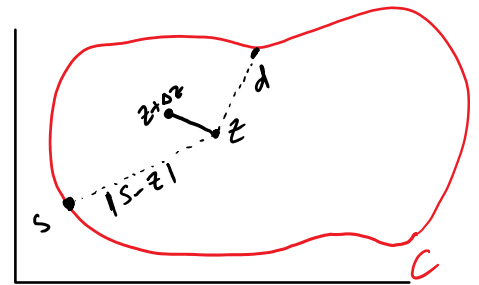
$$\begin{aligned} & \lim_{\Delta z \rightarrow 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} - \frac{1}{2\pi i} \int_C \left(\frac{f(s)}{(s-z)^2} \right) ds = 0 \\ &= \lim_{\Delta z \rightarrow 0} \frac{1}{2\pi i} \int_C \left(\frac{f(s)}{(s-z-\Delta z)(s-z)} \right) ds - \frac{1}{2\pi i} \int_C \left(\frac{f(s)}{(s-z)^2} \right) ds \\ &= \lim_{\Delta z \rightarrow 0} \frac{1}{2\pi i} \int_C \left(\frac{f(s)}{(s-z)^2} + \frac{f(s)\Delta z}{(s-z-\Delta z)(s-z)^2} \right) ds - \frac{1}{2\pi i} \int_C \left(\frac{f(s)}{(s-z)^2} \right) ds \\ &= \lim_{\Delta z \rightarrow 0} \frac{1}{2\pi i} \int_C \frac{f(s)\Delta z}{(s-z-\Delta z)(s-z)^2} ds \end{aligned}$$

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Now with our estimates.. $|f(s)| \leq M$ $|\Delta z| < d$ $|s - z| \geq d$ and $|s - z - \Delta z| \geq d - |\Delta z|$

We estimate our integral

$$\lim_{\Delta z \rightarrow 0} \frac{1}{2\pi i} \int_C \frac{f(s)\Delta z}{(s - z - \Delta z)(s - z)^2} ds$$



$$\begin{aligned} \lim_{\Delta z \rightarrow 0} \frac{1}{2\pi i} \int_C \frac{f(s)\Delta z}{(s - z - \Delta z)(s - z)^2} ds &\leq \lim_{\Delta z \rightarrow 0} \left| \frac{1}{2\pi i} \int_C \frac{f(s)\Delta z}{(s - z - \Delta z)(s - z)^2} ds \right| \\ &\leq \lim_{\Delta z \rightarrow 0} \frac{1}{2\pi i} \int_C \frac{|f(s)| |\Delta z|}{|s - z - \Delta z| |s - z|^2} ds \\ &\leq \lim_{\Delta z \rightarrow 0} \frac{1}{2\pi i} \int_C \frac{M |\Delta z|}{(d - |\Delta z|) d^2} ds \\ &= \lim_{\Delta z \rightarrow 0} \frac{1}{2\pi i} \frac{M |\Delta z|}{(d - |\Delta z|) d^2} \int_C ds \\ &= \lim_{\Delta z \rightarrow 0} \frac{1}{2\pi i} \frac{M |\Delta z| L}{(d - |\Delta z|) d^2} = 0 \end{aligned}$$

$|f(s)| \leq M$
 $|s - z|^2 = |s - z|^2$
 $|s - z|^2 \geq d^2$

$\int_C ds = \text{Length of } C = L$

Thus

$$\lim_{\Delta z \rightarrow 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} - \frac{1}{2\pi i} \int_C \left(\frac{f(s)}{(s - z)^2} \right) ds = 0$$

$\downarrow$
 $f'(z)$

And we finally have that

$$f'(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^2} dz$$

If we wanted to prove the general formula $f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{n+1}} dz$ $n = 0, 1, 2, \dots$ we would need to use induction, we would assume it is true for n then show because of that it is true for $n + 1$. Once we have done that we can argue that since it is true for $n = 1$ it must be true for $n + 1 = 2$ and if true for $n = 2$ then it must be for $n = 3$ etc...

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Ok so we have an extended version of the Cauchy Integral Formula which in itself is great but it turns out that we get even more great theorems out of it!

Theorem: If a function f is analytic at a given point, then it's derivatives of all orders are analytic there too.

To see why first we recall the definition of analytic, a function f is analytic in an open set S if it has a derivative everywhere in that set. It is analytic at a point z_0 if it is analytic in some neighborhood of z_0 .

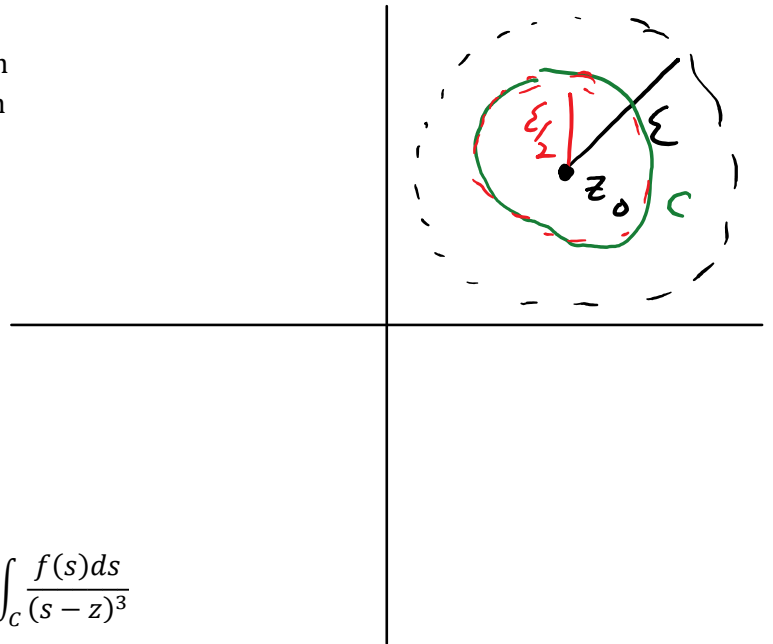
This means that if f is analytic at a point z_0 then it must also be analytic in some ϵ neighborhood of z_0 . That is it is analytic for some z values which satisfy $|z - z_0| < \epsilon$

If it is analytic in an ϵ neighborhood it certainly is in an $\frac{\epsilon}{2}$ neighborhood. If this is the case then we can take a contour C to be the boundary of the $\frac{\epsilon}{2}$ neighborhood. In which case we can apply our extended Cauchy integral formula to the second derivative.

$f'(z)$ is analytic

$$f''(z) = \frac{2!}{2i\pi} \int_C \frac{f(s)}{(s-z)^3} ds$$

$$f''(z) = \frac{2!}{2i\pi} \int_C \frac{f(s)ds}{(s-z)^3}$$



Further we can do that at each point z in the interior of C . This means that the derivative of $f'(z)$ exists in some open neighborhood of z_0 which means that it is analytic at that point. To show the rest of the derivatives are analytic we simply play the same game with f''' and so on.

From this theorem we get a nice corollary.

If a function $f(z) = u(x, y) + iv(x, y)$ is analytic at a point $z = x + iy$, then the component functions u, v have continuous partial derivatives of all orders at that point!

Why? Well recall that for analytic functions $f'(x) = u_x + iv_x = v_y - iu_y$ and that if a function is analytic it is also differentiable and if it is differentiable it is also continuous. Since f' is analytic because f was this means it is differentiable, and if it is differentiable it is continuous. Thus if $f'(x) = u_x + iv_x = v_y - iu_y$, each of the components of the sum which makes up f' must be continuous and we have our result.

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Yet another theorem we get from our Cauchy Formula is the following!

Theorem: Let f be continuous on a domain D . If

$$\int_C f(z) dz = 0$$

for every closed contour C in D , then f is analytic throughout D



Proof:

We start with our old antiderivative theorem..

Theorem: Suppose that a function $f(z)$ is continuous in a domain D . If any one of the following statements are true, then so are all of the others

- 1. $f(z)$ has an antiderivative $F(z)$ Throughout D
- 2. the integrals of $f(z)$ along contours lying entirely in D and extending from any z_1 to any fixed point z_2 all have the same value, that is

$$\int_{z_1}^{z_2} f(z) dz = F(z_2) - F(z_1)$$

- 3. the integrals of $f(z)$ around closed contours lying entirely in D are all 0.

Note that 3 in this theorem is exactly the statement of our newest theorem and if 3 is true so are 1 and 2. In particular 1. ensures we have an antiderivative $F(z)$ in our domain and that antiderivative clearly is analytic since its derivative is $f(z)$ which means that $f(z) = F'(z)$ is just the derivative of an analytic function which by our previous result means it is also analytic. Woot!

Last but not lest we also get the next Theorem.

Theorem: Suppose that a function f is analytic inside and on a positively oriented circle C_R , centered at z_0 and with radius R . If M_R is the maximum value of $|f(z)|$ of C_R then.

$$|f^{(n)}(z_0)| \leq \frac{n! M_R}{R^n}$$

To see why we just estimate the Cauchy integral...

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Theorem: Suppose that a function f is analytic inside and on a positively oriented circle C_R , centered at z_0 and with radius R . If M_R is the maximum value of $|f(z)|$ of C_R then.

$$\underline{|f(z)| \leq M_R}$$

$$|f^{(n)}(z_0)| \leq \frac{n! M_R}{R^n}$$

To see why we just estimate the Cauchy integral on our region ...

$$\sum \frac{f^{(n)}(a)}{n!} (x-a)^n$$

Proof:

$$\left| f^{(n)}(z_0) \right| = \left| \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^{n+1}} dz \right| \leq$$

$$\underline{\left| \int f dz \right| \leq \int |f| dz}$$

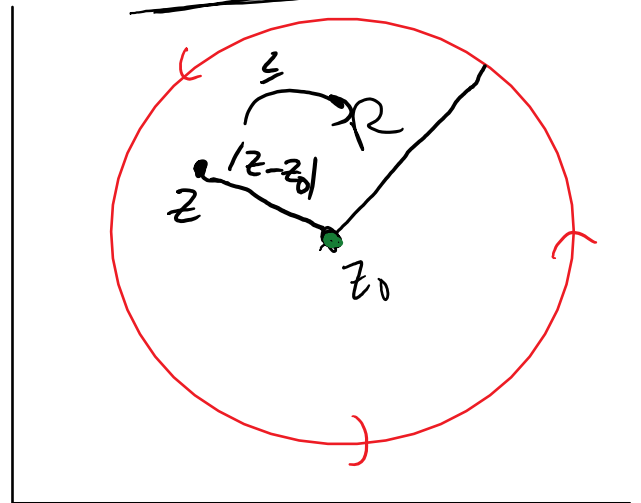
$$|f^{(n)}(z_0)| = \left| \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^{n+1}} dz \right|$$

$$= \left| \frac{n!}{2\pi i} \right| \left| \int_C \frac{f(z)}{(z-z_0)^{n+1}} dz \right| \leq \frac{n!}{2\pi} \int_C \left| \frac{f(z)}{(z-z_0)^{n+1}} \right| dz$$

$$\leq \frac{n!}{2\pi} \int_C \frac{|f(z)|}{|z-z_0|^{n+1}} dz \leq \frac{n!}{2\pi} \int_C \frac{M_R}{R^{n+1}} dz$$

$$= \frac{n!}{2\pi} \frac{M_R}{R^{n+1}} \int_C dz = \frac{n!}{2\pi} \frac{M_R}{R^{n+1}} 2\pi R$$

$$f^{(n)}(z_0) \leq \frac{n! M_R}{R^n}$$



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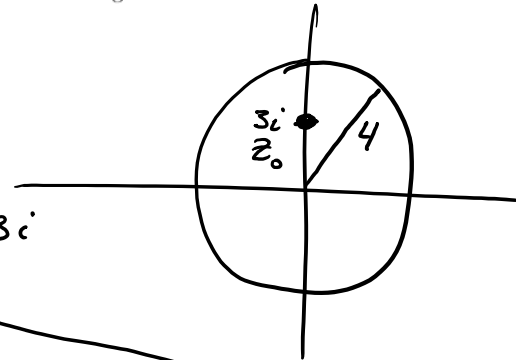
5. Using the Cauchy integral extension, compute

$$\frac{2\pi i}{n!} f^{(n)}(z_0) = \int_C \frac{f(z)}{(z-z_0)^{n+1}} dz$$

$n=3$

on the contour C being a positive-oriented circle with radius 4 centered at the origin.

$$\int_C \frac{e^{2iz}}{(z-3i)^4} dz = \frac{2\pi i}{3!} \left. \frac{d^3}{dz^3} (e^{2iz}) \right|_{z=3i}$$



$$\frac{d}{dz} e^{2iz} = 2ie^{2iz}$$

$$\frac{d^2}{dz^2} (e^{2iz}) = 2i(2i)e^{2iz} = -4e^{2iz}$$

$$\frac{d^3}{dz^3} e^{2iz} = -4(2i)e^{2iz} = -8ie^{2iz}$$

$$\left. \frac{d^3}{dz^3} e^{2iz} \right|_{z=3i} = -8ie^{2i(3i)} = (-8ie^{-6})$$

$$\int_C \frac{e^{2iz}}{(z-3i)^4} dz = \frac{2\pi i}{6} (-8ie^{-6}) = \boxed{\frac{8\pi}{3} e^{-6}}$$

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4. Suppose we wish to integrate $\int_C f(z) dz$ for $f(z) = \frac{1}{(z-i)(z-1)}$ over a contour C being a positive-oriented circle at the origin with radius 2.

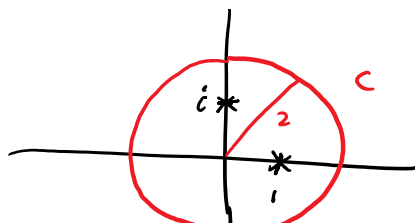
(a) Draw the non-analytic points of the function and the contour C

(b) Draw an appropriate multiply connected domain D that has C as its outer boundary, within which f is analytic.

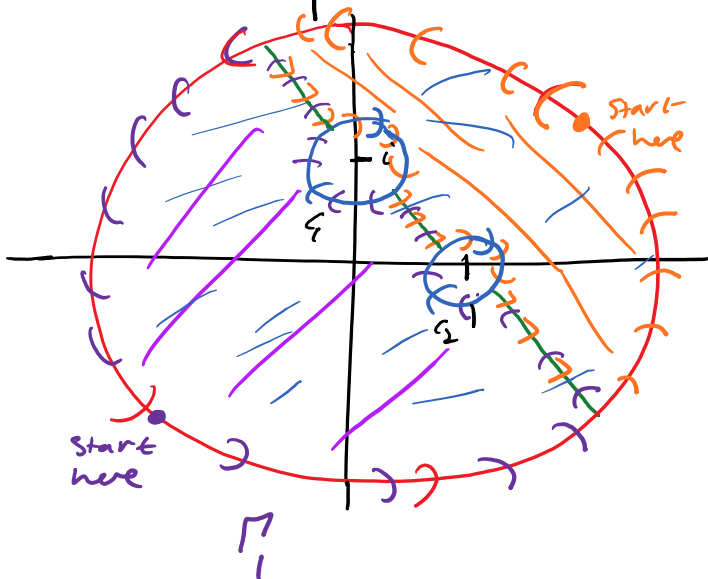
(c) Use the Cauchy integral formula on the boundary of the domain D you have specified to compute the integral.

$f(z)$ is not analytic
at $z=i$ or $z=1$

a)



b)



$$\int_C f(z) dz = 0$$

$$\int_C f(z) + \int_{C_1} f(z) = \int_{C_1} f(z) + \int_{C_2} f(z)$$

c)

$$\int_C \frac{f(z)}{(z-z_0)} dz = 2\pi i f(z_0)$$

$$\int_C \frac{1}{(z-i)(z-1)} dz$$

$$\frac{1}{(z-i)(z-1)} = \frac{A}{(z-i)} + \frac{B}{(z-1)}$$

$$z=i$$

$$1 = A(i-1) + B(i-i)$$

$$1 = A(z-1) + B(z-i) \quad z=1$$

$$1 = A(1-1) + B(1-i)$$

$$1 = B(1-i)$$

$$B = \frac{1}{1-i}$$

$$A = \frac{1}{i-1}$$

cont

$$\frac{1}{(z-i)(z-1)} = \frac{A}{(z-i)} + \frac{B}{(z-1)}$$

$$A = \frac{1}{i-1}$$

$$B = \frac{1}{1-i}$$

$$\frac{1}{(z-i)(z-1)} = \frac{1}{(i-1)(z-i)} + \frac{1}{(1-i)(z-1)}$$

$$\int_C \frac{1}{(z-i)(z-1)} dz = \int_C \frac{1}{(i-1)(z-i)} dz + \int_C \frac{1}{(1-i)(z-1)} dz$$

$$= \frac{1}{i-1} \int_C \frac{\overset{f}{-1}}{(z-i)} dz + \frac{1}{1-i} \int_C \frac{\overset{f}{1}}{(z-1)} dz$$

$\uparrow$ z_0 $\uparrow$ z_0
 $\parallel$ $\parallel$

$$\int_C \frac{f(z)}{z-z_0} dz = 2\pi i f(z_0)$$

$$\frac{1}{i-1} 2\pi i (1) + \frac{1}{1-i} 2\pi i (1)$$

$$\frac{2\pi i}{-(1-i)} + \frac{2\pi i}{1-i} = \boxed{0}$$