

Chapter 4 Sec 58-59

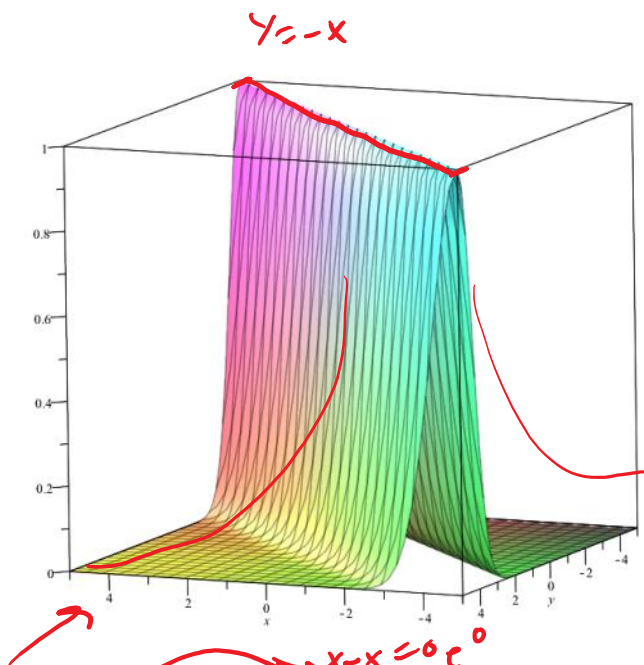
Liouville's Theorem and the Fundamental Theorem of Algebra

We still need to go through a few theorems that will be useful in our study of series in the complex plane. In order to understand them we should prove them....

Liouville's Theorem: If a function f is entire and bounded in the complex plane then, $f(z)$ is constant throughout the plane.

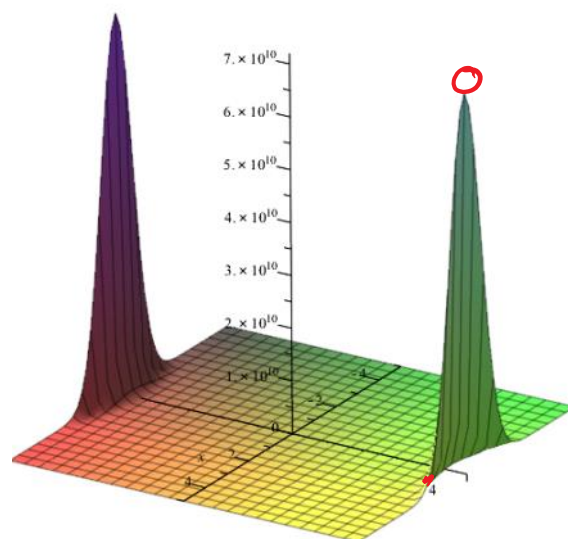
$$|f(z)| < M$$

This is actually kind of amazing, this theorem says that any entire function which is not constant is not bounded in the complex plane, this is a bit different than with functions in the real number system. For example the function $f(x, y) = e^{-(x+y)^2}$ is certainly not constant but it is bounded by 1. On the contrary the function $f(z) = e^{-z^2} = e^{-(x+iy)^2}$ is entire and not constant, thus it cannot be bounded.



$$f(x, y) = e^{-(x+y)^2}$$

$$y = -x$$



$$|f(z)| = |e^{-z^2}| = |e^{-(x+iy)^2}|$$

$$\frac{1}{e^{2x}}$$

You might think, but wait, we know e^z is entire yet $e^{i\theta}$ is bound by 1 for all θ no matter how large, how can that be given the theorem? Keep in mind $e^z = e^{x+iy}$ so when we consider $e^{i\theta}$ we are effectively looking at e^{iy} , that is we are restricting $z = 0 + iy$. So e^z is bounded on the set of complex numbers with zero real part, but not on the whole complex plane!

$$|e^{i\theta}| = 1$$

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Liouville's Theorem: If a function f is entire and bounded in the complex plane then, $f(z)$ is constant throughout the plane.

Proof:

First let's name some things, in the statement of the theorem we assumed that $f(z)$ was bounded in the entire complex plane. Let's call that maximum value of $|f(z)| = M$ that is M is the absolute maximum that f can take on in all of the complex plane. What we need to do to show that $f(z)$ is constant with this assumption is to simply show that $f'(z) = 0$

$$f'(z) = 0 \rightarrow f(z) = c$$

To prove this we will just estimate the derivative using the extended Cauchy integral formula.

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{n+1}} dz$$

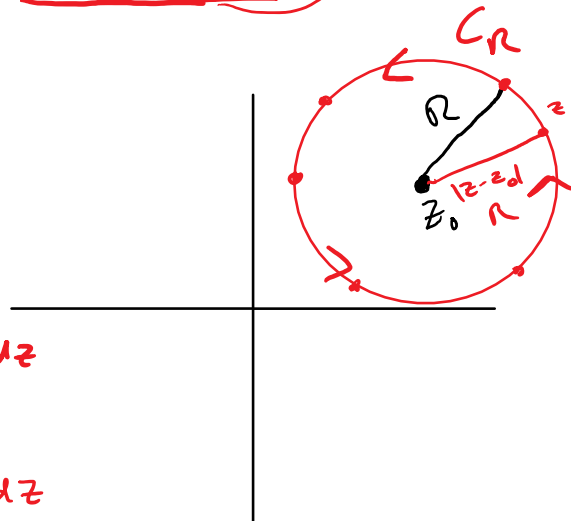
Choose any z_0 and take a positively oriented circle around that point with arbitrary radius R for our contour., then for the first derivative $n = 1$ so:

$$f'(z_0) = \frac{1!}{2\pi i} \int_{C_R} \frac{f(z)}{(z - z_0)^{1+1}} dz$$

$$f'(z_0) = \frac{1}{2\pi i} \int_{C_R} \frac{f(z)}{(z - z_0)^2} dz$$

$$|f'(z_0)| = \left| \frac{1}{2\pi i} \int_{C_R} \frac{f(z)}{(z - z_0)^2} dz \right| \leq \left| \frac{1}{2\pi i} \right| \int_{C_R} \left| \frac{f(z)}{(z - z_0)^2} \right| dz$$

$$\leq \frac{1}{2\pi} \int_{C_R} \frac{|f(z)|}{|z - z_0|^2} dz \leq \frac{1}{2\pi} \int_{C_R} \frac{M}{R^2} dz = \frac{M}{2\pi R^2} \int_{C_R} dz$$



Now we just note that since M was the absolute maximum of $f(z)$ anywhere and recall that $\int_C dz = 2\pi R$ when our contour is a circle. We then get,

$$|f'(z_0)| \leq \frac{M}{2\pi R^2} \int_{C_R} dz = \frac{M}{2\pi R^2} 2\pi R$$

$$\boxed{|f'(z_0)| \leq \frac{M}{R}}$$

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So we have that $|f'(z)| \leq \frac{M}{R}$. Now we said that R was arbitrary thus we can let R be as large as we want in which case $\frac{M}{R}$ gets closer and closer to zero. We also know that $|f'(z)| \leq \frac{M}{R}$ for all possible values of R .

Since $\lim_{R \rightarrow \infty} \frac{M}{R} = 0$ and $|f'(z)| \leq \frac{M}{R}$ for all R the only way this is possible is if $|f'(z)| = 0$ and this can only be the case if $f'(z) = 0$ which can only be the case if $f(z) = \text{constant}$.

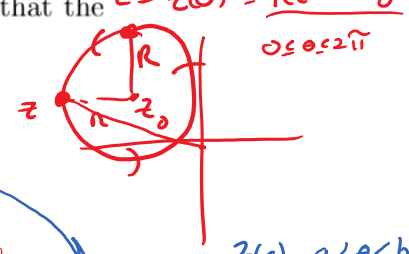
$$|f'(z)| = 0 \quad |f(z)| = 0$$

$$\frac{M}{R} = 0.000000000001$$

Homework Example

1. Let f be analytic entire such that $|f(z)| \leq A|z|$ for all z and A is a fixed positive number. Show that $f(z) = a_1 z$, where $a_1 \in \mathbb{C}$. Hint: use Cauchy's inequality to show that the second derivative f'' is zero.

$$f''(z) = 0 \quad f'(z) = a_1 \quad f(z) = a_1 z$$

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^{n+1}} dz \rightarrow f''(z) = \frac{2!}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^3} dz$$


$$|f''(z)| \leq \frac{1}{\pi} \int_C \frac{|f(z)|}{|z-z_0|^3} dz$$

$$\leq \frac{1}{\pi} \int_C \frac{A|z|}{R^3} dz = \frac{A}{\pi R^3} \int_C |z| dz$$

$$= \frac{A}{\pi R^3} \int_0^{2\pi} |Re^{i\theta} + z_0| iRe^{i\theta} d\theta \leq \frac{A}{\pi R^3} \int_0^{2\pi} (R + |z_0|) iRe^{i\theta} d\theta$$

$$= \frac{A}{\pi R^3} \int_0^{2\pi} (R + |z_0|) iRe^{i\theta} d\theta = \frac{A}{\pi R^3} \left[\int_0^{2\pi} iR^2 e^{i\theta} d\theta + \int_0^{2\pi} iR|z_0| e^{i\theta} d\theta \right]$$

$$= \frac{A}{\pi R^3} \left[iR^2 \int_0^{2\pi} e^{i\theta} d\theta + iR|z_0| \int_0^{2\pi} e^{i\theta} d\theta \right] = \frac{A}{\pi R^3} \left[iR^2 \left[\frac{1}{i} e^{i\theta} \right]_0^{2\pi} + iR|z_0| \left[\frac{1}{i} e^{i\theta} \right]_0^{2\pi} \right]$$

$$= \frac{A}{\pi R^3} \left[R^2 [e^{i2\pi} - e^0] + |z_0|R [e^{i2\pi} - e^0] \right] = \frac{A}{\pi R^3} [R^2(0) + |z_0|R(0)]$$

$$= 0 \rightarrow |f''(z)| \leq 0 \quad |f''(z)| = 0$$

$$\Rightarrow f''(z) = 0 \Rightarrow f(z) = a, z$$

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The Fundamental Theorem of Algebra

Any Polynomial $p(z) = a_0 + a_1z + a_2z^2 + a_3z^3 + \dots + a_nz^n$ **has at least one zero. That is there exists at least one point z_0 such that $p(z_0) = 0$**

Note that this is different than in the real numbers, in that set we can have polynomials with no zeros, for example $f(x) = x^2 + 1$. However, when we extend our set of numbers to the complex number system we are guaranteed at least one root and at most n for an n^{th} degree polynomial.

$$x^2 + 1 = 0 \quad x^2 = -1 \quad x = \pm i$$

Proof!

The proof of this theorem is done by good ol' contradiction. That is we assume that what we want to show is not true and then find that our assumption leads to an inconsistency which means it could have not been true in the first place.

With that, assume that some polynomial $p(z)$ has no roots, then $f(z) = \frac{1}{p(z)}$ is entire. Furthermore it is bounded.. To see why we just estimate using the triangle inequality.

Let $p(z) = a_0 + a_1z + a_2z^2 + a_3z^3 + \dots + a_nz^n$ and note that $a_0 \neq 0$ or else $z = 0$ is a root and we assumed no roots.

$$\text{Then } \left| \frac{1}{p(z)} \right| \leq \frac{A}{a_n R^n} = M$$

$\left| \frac{1}{p(z)} \right| < M$

$\frac{1}{p(z)}$ is entire and bounded $\Rightarrow \frac{1}{p(z)}$ is constant

$\Rightarrow p(z)$ is constant

but $p(z)$ was not constant.

So then $\left| \frac{1}{p(z)} \right| \leq \frac{1}{|a_0|}$ and thus is bounded, we just showed that any entire function which is bounded must be constant, but $p(z)$ can't be constant as we assumed the opposite, thus our assumption that $p(z)$ having no roots is wrong.

Once we know that there is at least one root we can actually show that every polynomial in the complex plane can be factored into linear factors. That is

$$p(z) = a_0 + a_1z + a_2z^2 + a_3z^3 + \cdots + a_nz^n = c(z - z_1)(z - z_2) \cdots (z - z_n)$$

First let's verify an identity we need.

$$\begin{aligned} z^k - z_1^k &= (z - z_1)(z^{k-1} + z_1z^{k-2} + \cdots + zz_1^{k-2} + z_1^{k-1}) \\ z(z^{k-1} + z_1z^{k-2} + \cdots + zz_1^{k-2} + z_1^{k-1}) &- z_1(z^{k-1} + z_1z^{k-2} + \cdots + zz_1^{k-2} + z_1^{k-1}) \\ z^k + z_1z^{k-1} + \cdots + z_1^{k-2}z &- z_1z^{k-1} - z_1^2z^{k-2} - \cdots - z_1^{k-1}z \\ &= z^k - z_1^k \end{aligned}$$

Armed with this let's examine $p(z) - p(z_1)$ where z_1 is our garenteed root for $p(z)$. Note that because $p(z_1) = 0$, $p(z) - p(z_1) = p(z)$. Let's group like powers of z for each part of our difference and apply the identity above.

$$p(z) - p(z_1) = p(z)$$

$$p(z) - p(z_1) = a_nz^n + a_{n-1}z^{n-1} + \cdots + a_1z + a_0 - (a_nz_1^n + a_{n-1}z_1^{n-1} + \cdots + a_1z_1 + a_0)$$

$$= a_nz^n + a_{n-1}z^{n-1} + \cdots + a_1z + a_0 - a_nz_1^n - a_{n-1}z_1^{n-1} - \cdots - a_1z_1 - a_0$$

$$= a_n(z^n - z_1^n) + a_{n-1}(z^{n-1} - z_1^{n-1}) + \cdots + a_1(z - z_1)$$

$$\begin{aligned} &a_n(z - z_1)(z^{n-1} + z_1z^{n-2} + \cdots + z_1^{n-2}z + z_1^{n-1}) + a_{n-1}(z - z_1)(z^{n-2} + z_1z^{n-3} + \cdots + z_1^{n-3}z + z_1^{n-2}) + \cdots + a_1(z - z_1) \\ &= (z - z_1)(a_n(z^{n-1} + \cdots) + a_{n-1}(z^{n-2} + \cdots) + \cdots + a_1) = p(z) \end{aligned}$$

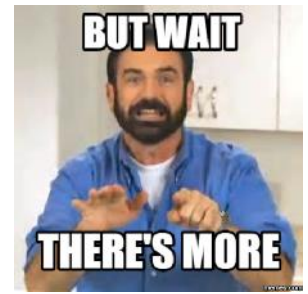
So this says that $p(z) = p(z) - p(z_1) = (z - z_1)Q(z)$ where $Q(z)$ is an $n-1$ degree polynomial. We could at this point do the same exact thing with $Q(z)$ as it is garenteed a root, call it z_2 , as well then we could keep going until we show that

$$p(z) = c(z - z_1)(z - z_2) \cdots (z - z_n)$$

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The Maximum Modulus Principle

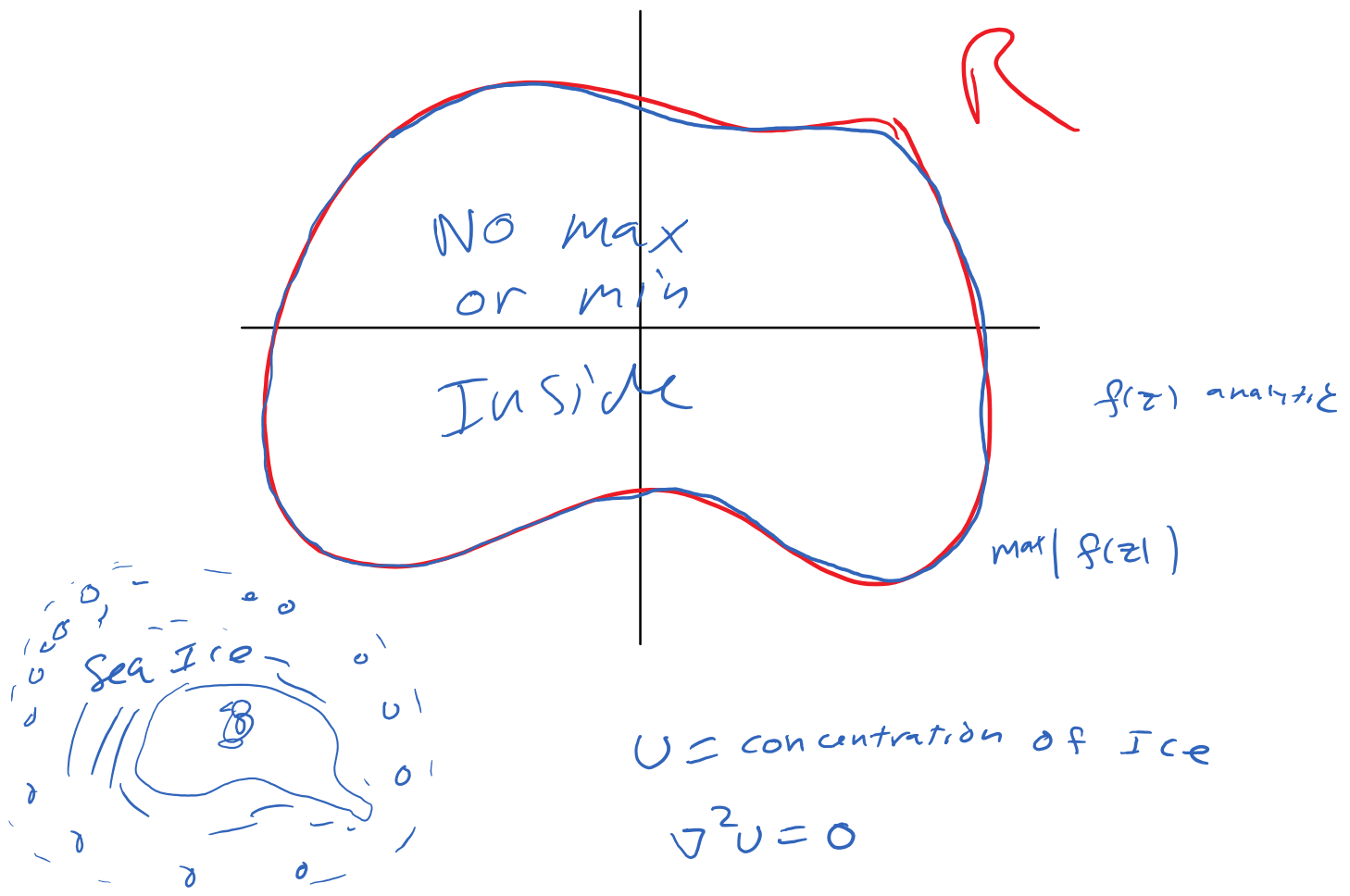
If a function f is analytic and not constant in a given domain D , then $|f(z)|$ has no maximum value in D . That is, there is no point z_0 in the domain such that $|f(z)| \leq |f(z_0)|$ for all points z in it.



We will prove this theorem in a moment but I'd first like to use it to explain a very important corollary.

Corollary: Suppose that a function $f(z)$ is continuous in on a closed bounded region R and that f is non constant and analytic in the interior of the Region. Then the maximum value of $|f(z)|$, which is always reached occurs on the boundary of R and never in the interior and the same goes for the minimum value of $|f(z)|$.

Why is this corollary so important? Well, recall that if a function $f(z) = u(x, y) + iv(x, y)$ is analytic then the component functions are harmonic. With corollary one can show that the same thing is true for $u(x, y)$. That is for a harmonic function on a closed bounded region in the x - y plane it can only attain its maximum or minimum values on the boundary of that region! Equations which are harmonic are equations which satisfy $\nabla^2 u = 0$ an equation that you will see often in your careers.

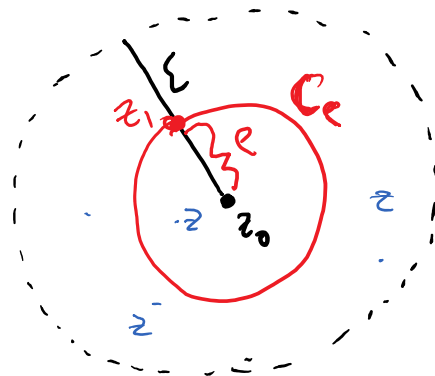


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To prove the Maximum Modulus principle we will first need the following lemma

Lemma: Suppose that $|f(z)| \leq |f(z_0)|$ at each point z in some neighborhood $|z - z_0| < \epsilon$ in which f is analytic. Then $f(z)$ has the constant value $f(z_0)$ throughout that neighborhood.

To prove this we take some point inside our neighborhood call it z_1 . Further we will say that the distance between z_1 and z_0 is ρ that is $|z_1 - z_0| = \rho$. We can then take as a contour around z_0 to be the circle of radius ρ centered at z_0 call it C_ρ .



Right away we can apply the Cauchy integral formula to the integral around C_ρ

$$f(z_0) = \frac{1}{2\pi i} \int_{C_\rho} \frac{f(z)}{(z - z_0)} dz$$

$$\int_C g(z) dz = \int_a^b g(z(\theta)) z'(\theta) d\theta$$

$$g(z) = \frac{f(z)}{z - z_0}$$

We could also write this with a parameterization of C_ρ $z(\theta) = z_0 + \rho e^{i\theta}$ $0 \leq \theta \leq 2\pi$

$$z'(\theta) = \rho i e^{i\theta}$$

$$f(z_0) = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(z_0 + \rho e^{i\theta})}{(z_0 + \rho e^{i\theta} - z_0)} \rho i e^{i\theta} d\theta$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \frac{f(z_0 + \rho e^{i\theta})}{\cancel{\rho e^{i\theta}}} \cancel{\rho e^{i\theta}} d\theta = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + \rho e^{i\theta}) d\theta$$

So we have that

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + \rho e^{i\theta}) d\theta$$

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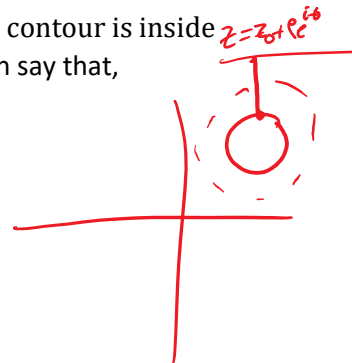
Now we estimate!

$$|f(z_0)| = \left| \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + \rho e^{i\theta}) d\theta \right| \leq \frac{1}{2\pi} \int_0^{2\pi} |f(z_0 + \rho e^{i\theta})| d\theta$$

$$|f(z_0)| \leq \frac{1}{2\pi} \int_0^{2\pi} |f(z_0 + \rho e^{i\theta})| d\theta$$

But we also assumed that $|f(z)| \leq |f(z_0)|$ for all z in our neighborhood, since our contour is inside that neighborhood certainly for the points parameterized by $z = z_0 + \rho e^{i\theta}$ we can say that,

$$|f(z_0 + \rho e^{i\theta})| \leq |f(z_0)|$$



Which If we integrate both sides of this inequality tells us that

$$\int_0^{2\pi} |f(z_0 + \rho e^{i\theta})| d\theta \leq \int_0^{2\pi} |f(z_0)| d\theta = 2\pi |f(z_0)|$$

$$\int_0^{2\pi} |f(z_0)| d\theta = |f(z_0)| \int_0^{2\pi} d\theta = |f(z_0)| \theta \Big|_0^{2\pi} = 2\pi |f(z_0)|$$

$$\frac{1}{2\pi} \int_0^{2\pi} |f(z_0 + \rho e^{i\theta})| d\theta \leq \frac{2\pi |f(z_0)|}{2\pi}$$

So we have that

$$|f(z_0)| \geq \frac{1}{2\pi} \int_{C_\rho} |f(z_0 + \rho e^{i\theta})| d\theta$$

So now we have the two statements,

$$|f(z_0)| \leq \frac{1}{2\pi} \int_0^{2\pi} |f(z_0 + \rho e^{i\theta})| d\theta$$

$$|f(z_0)| \geq \frac{1}{2\pi} \int_0^{2\pi} |f(z_0 + \rho e^{i\theta})| d\theta$$

The only way to satisfy both at once is with the equality part of the inequality that is

$$|f(z_0)| = \frac{1}{2\pi} \int_0^{2\pi} |f(z_0 + \rho e^{i\theta})| d\theta$$

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$$2\pi |f(z_0)| = \frac{1}{2\pi} \int_{C_\rho} |f(z_0 + \rho e^{i\theta})| d\theta$$

Multiplying both sides by 2π gives

$$2\pi |f(z_0)| = \int_0^{2\pi} |f(z_0 + \rho e^{i\theta})| d\theta$$

We just showed that $2\pi |f(z_0)| = \int_0^{2\pi} |f(z_0)| d\theta$ so we can say

$$\int_0^{2\pi} |f(z_0)| d\theta = \int_0^{2\pi} |f(z_0 + \rho e^{i\theta})| d\theta$$

$$\int_0^{2\pi} |f(z_0)| d\theta - \int_0^{2\pi} |f(z_0 + \rho e^{i\theta})| d\theta = 0$$

Or that:

$$\int_0^{2\pi} |f(z_0)| - |f(z_0 + \rho e^{i\theta})| d\theta = 0$$

The thing about this integral is that ρ was arbitrary, we could have chosen any circle inside our neighborhood. That means that the integral above is zero for any choice of ρ the only way that is possible is if the thing we are integrating is zero.

That is we can say:

$$|f(z_0)| - |f(z_0 + \rho e^{i\theta})| = 0$$

$$|f(z_0)| = |f(z_0 + \rho e^{i\theta})|$$

$$\text{Thus } f(z_0) = f(z_0 + \rho e^{i\theta})$$

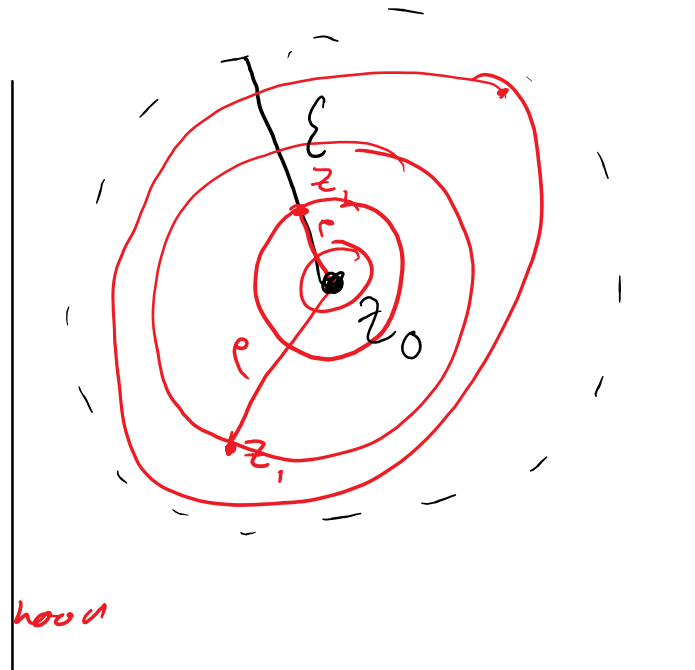
$$|f(z_0)| = |f(z)| \quad \forall z \text{ in neighborhood}$$

Further the arbitrariness of ρ means we could pick a circle through any point in the neighborhood. As a result we can finally say:

$$f(z_0) = f(z) \quad \forall z \in |z - z_0| \leq \epsilon$$

for all

$f(z) = \text{constant}$



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So we now have our Lemma

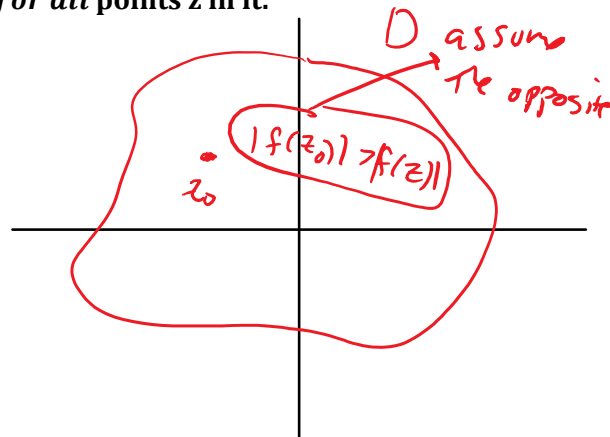
Lemma: Suppose that $|f(z)| \leq |f(z_0)|$ at each point z in some neighborhood $|z - z_0| \leq \epsilon$ in which f is analytic. Then $f(z)$ has the constant value $f(z_0)$ throughout that neighborhood.

Let's use this to prove the Maximum Modulus Principle.

The Maximum Modulus Principle

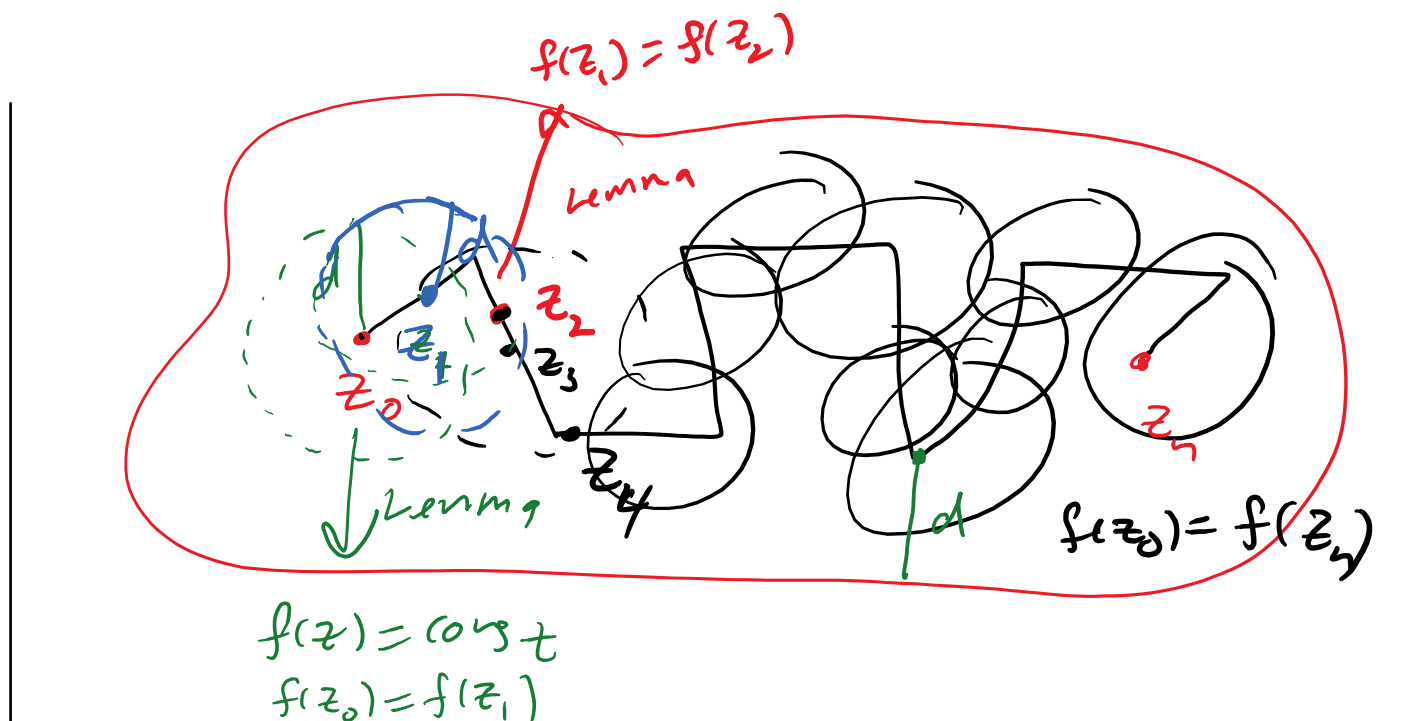
If a function f is analytic and not constant in a given domain D , then $|f(z)|$ has no maximum value in D . That is, there is no point z_0 in the domain such that $|f(z)| \leq |f(z_0)|$ for all points z in it.

We will again use the trick of proof by contradiction, we will assume that $f(z)$ indeed does have a maximum at some point z_0 inside some domain D and that $f(z)$ is not constant.

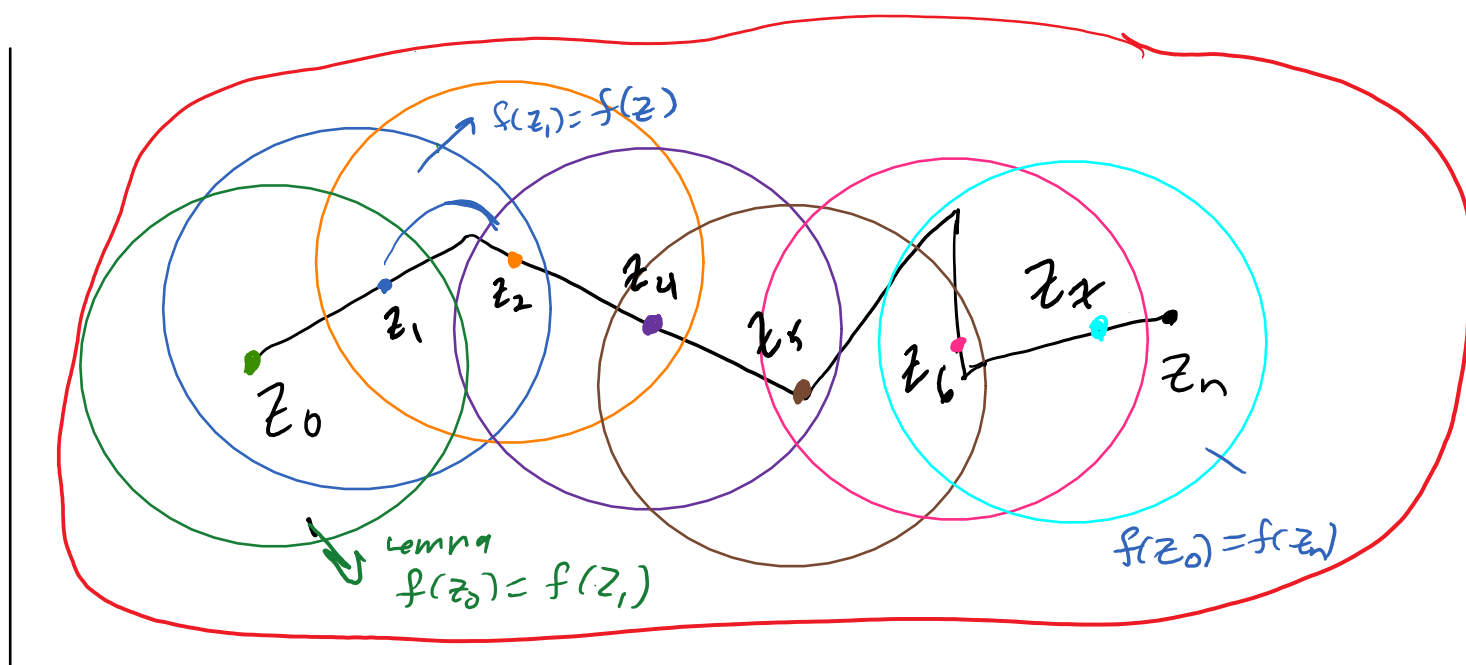


Let $|f(z)|$ have its maximum at z_0 in D , and make a random polygonal line connecting z_0 to any other point in D .

Further define d to be the shortest distance from any point on our line to the boundary of D . Now draw a neighborhood of size d around z_0 then choose a point in that neighborhood on our line and draw a neighborhood around that point, do this until you reach z_n .



D



So what we do is apply the Lemma

Lemma: Suppose that $|f(z)| \leq |f(z_0)|$ at each point z in some neighborhood $|z - z_0| \leq \epsilon$ in which f is analytic. Then $f(z)$ has the constant value $f(z_0)$ throughout that neighborhood.

First in the neighborhood that has z_1, z_0 in that green neighborhood $f(z_0) = f(z_1)$ since $f(z)$ has to be constant there. Next we look at the blue neighborhood that has z_1, z_2 in that neighborhood $f(z_1) = f(z_2)$ since f must be constant there, as a result $f(z_2) = f(z_0)$. We then continue our daisy chain all the way until the randomly chosen point z_n and can show that $f(z_n) = f(z_0)$

Since we could do this for any point $z = z_n$ in the whole of the domain D it must be that

$$\underline{f(z_0) = f(z) \quad \forall z \in D}$$

This means that f was constant which violates our assumptions and we have our contradiction. Instead it must be that:

If a function f is analytic and not constant in a given domain D , then $|f(z)|$ has no maximum value in D . That is, there is no point z_0 in the domain such that $|f(z)| \leq |f(z_0)|$ for all points z in it.

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Homework Example

2. Let f be the function $f(z) = e^z$ and R the rectangular region $0 \leq x \leq 1$, $0 \leq y \leq \pi$. Find the real part of $f(z)$, $u(x, y)$, and show that it is harmonic. Then find the place on the closed region R where it attains its maximum and minimum values.