

Chapter 5 Sec 60-64

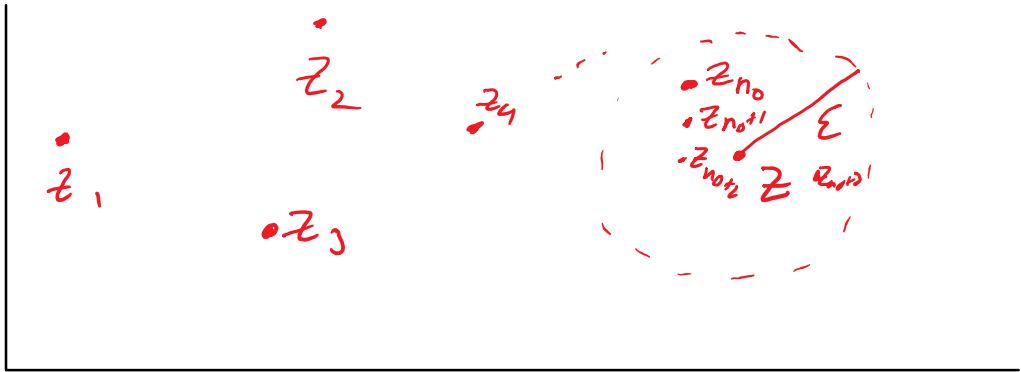
We will now move on to study of series and in particular series representations of analytic functions. In general a Taylor series is a physicists and engineer's best friend. Many complicated problems can be solved or greatly simplified using series representations.

First we need to run through a few definitions and Theorems we will use along the way.

Definition: An infinite sequence $z_1, z_2, \dots, z_n, \dots$ of complex numbers has a limit z if for each $\epsilon > 0$ there exists a positive integer n_0 such that whenever $n > n_0$

$|z - z_n| < \epsilon$ whenever $n > n_0$ $\lim_{n \rightarrow \infty} z_n = z$
↑
limit

If the limit z exists the sequence is said to converge, if the sequence has not limit is it said to diverge.



Theorem: Suppose that $z_n = x_n + iy_n$ $n \in \mathbb{N}$ and the limit of the sequence is $z = x + iy$, Then

$\lim_{n \rightarrow \infty} z_n = z$ if and only if $\lim_{n \rightarrow \infty} x_n = x$ and $\lim_{n \rightarrow \infty} y_n = y$

Proof:

1) $\lim_{n \rightarrow \infty} z_n = z \Rightarrow \lim_{n \rightarrow \infty} x_n = x$ and $\lim_{n \rightarrow \infty} y_n = y$ $z = x + iy$

Assume $\lim_{n \rightarrow \infty} z_n = z$ Then by definition there is an n_0 such that $|z_n - z| < \epsilon$ for $n \geq n_0$ what we want to show is that this means there exist n_1 and n_2 such that $|x_n - x| < \epsilon$ when $n > n_1$ and $|y_n - y| < \epsilon$ whenever $n > n_2$

From our assumption we can say:

$|z_n - z| = |x_n + iy_n - (x + iy)| = |(x_n - x) + i(y_n - y)| < \epsilon$ whenever $n > n_0$

I claim that

and $|x - x_n| \leq |z - z_n| < \epsilon$ whenever $n > n_0$

$|y - y_n| \leq |z - z_n| < \epsilon$

Let's verify that

Chapter 5 Sec 60-64

From the definition of the modulus it is easy to check that

$$|x - x_n| \leq |(x_n - x) + i(y_n - y)|$$

$|z - z_n|$
↓

$$\sqrt{(x - x_n)^2} \leq \sqrt{(x_n - x)^2 + (y_n - y)^2}$$

And for the same reasons

$$|y - y_n| \leq |(x_n - x) + i(y_n - y)|$$
$$\sqrt{(y - y_n)^2} \leq \sqrt{(x_n - x)^2 + (y_n - y)^2}$$

$$\text{And } |(x_n - x) + i(y_n - y)| = |x_n + iy_n - (x + iy)| = |z - z_n|$$

So we have that

$$\text{and } \underbrace{|x - x_n|} \leq \underbrace{|z - z_n|} < \underbrace{\epsilon}$$

whenever $n > n_0$

$$\underbrace{|y - y_n|} \leq \underbrace{|z - z_n|} < \underbrace{\epsilon}$$

Thus let $\underline{n_1 = n_0}$ and $\underline{n_2 = n_0}$ and we have the result that:

$$\underline{\lim_{n \rightarrow \infty} z_n = z} \Rightarrow \underline{\lim_{n \rightarrow \infty} x_n = x} \text{ and } \underline{\lim_{n \rightarrow \infty} y_n = y}$$

Chapter 5 Sec 60-64

Now on to the other half of the if and only if

$$2) \lim_{n \rightarrow \infty} x_n = x \text{ and } \lim_{n \rightarrow \infty} y_n = y \Rightarrow \lim_{n \rightarrow \infty} z_n = z$$

$n_1 = 10$

$$|x_n - x| < \epsilon/2 \quad n_1 = 15$$

Here we are assuming that there exist n_1 and n_2 such that $|x_n - x| < \epsilon$ whenever $n > n_1$ and $|y_n - y| < \epsilon$ whenever $n > n_2$

What we want to show is that this implies that there is an n_0 such that $|z - z_n| \leq \epsilon$ whenever $n > n_0$

So Let $\epsilon > 0$ we know that since the limit for the x_n and y_n sequences exist that we can find an n_1, n_2 such that

$$|x_n - x| < \frac{\epsilon}{2} \text{ whenever } n > n_1$$

$$|y_n - y| < \frac{\epsilon}{2} \text{ whenever } n > n_2$$

So then :

$$|z_n - z| = |x_n + iy_n - (x + iy)| = |(x_n - x) + i(y_n - y)|$$

Now we just need apply the triangle inequality and choose n_0 to be the larger of $n_1, n_2, \dots$

$$|z_n - z| = |(x_n - x) + i(y_n - y)|$$

$$\leq |x_n - x| + |i(y_n - y)| = |x_n - x| + |y_n - y|$$

$$= |x_n - x| + |y_n - y| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

$$|z_n - z| < \epsilon$$

$$n_0 = \max(n_1, n_2)$$

Chapter 5 Sec 60-64

Quick Example

Show that the $\lim_{n \rightarrow \infty} \left(\pi + i \frac{(-1)^n}{n^2} \right) = \pi$

$$\left| \pi + i \frac{(-1)^n}{n^2} - \pi \right| < \varepsilon \quad \text{for } n > n_0$$

$$\left| \cancel{\pi} + \frac{i(-1)^n}{n^2} - \cancel{\pi} \right| = \left| \frac{i(-1)^n}{n^2} \right| = \frac{|i(-1)^n|}{|n^2|} = \frac{i i (-1)^n}{|n^2|} = \frac{1 \cdot 1}{n^2}$$
$$= \frac{1}{n^2} < \varepsilon$$

$$\Rightarrow n^2 > \frac{1}{\varepsilon}$$

$$n_0 \geq \frac{1}{\sqrt{\varepsilon}}$$

Chapter 5 Sec 60-64

Convergence of a Series

An infinite series converges to the sum S if the sequence of partial sums

$$S_N = \sum_{n=1}^N z_n = z_1 + z_2 + \cdots + z_N$$

Converges to S in the limit as $N \rightarrow \infty$

If this is so we say:

$$\lim_{N \rightarrow \infty} \sum_{n=1}^N z_n = \sum_{n=1}^{\infty} z_n = S$$

Example: The partial sum

$$z_n = z^n$$

$$\sum_{n=0}^N z^n = 1 + z + z^2 + \cdots + z^N$$

Converges to $\frac{1}{1-z}$ as $N \rightarrow \infty$

First we will find a closed form for $1 + z + z^2 + \cdots + z^N = S_N$

$$z z^N = z^{N+1}$$

$$S_N - z S_N = (1 + z + z^2 + \cdots + z^N) - z(1 + z + z^2 + \cdots + z^N)$$

$$= 1 + \cancel{z} + \cancel{z^2} + \cdots + \cancel{z^N} - \cancel{z} - \cancel{z^2} - \cancel{z^3} - \cdots - \cancel{z^N} - z^{N+1}$$

$$S_N - z S_N = 1 - z^{N+1}$$

$$S_N - z S_N = 1 - z^{N+1}$$

$$S_N(1 - z) = 1 - z^{N+1}$$

$$S_N = \frac{1 - z^{N+1}}{1 - z}$$

Chapter 5 Sec 60-64

Now we want to show that $\lim_{n \rightarrow \infty} \frac{1+z^{N+1}}{1-z} = \frac{1}{1-z}$

Checking the modulus of the difference of the sequence of partial sums and the thing we believe to be its limit

$$\left| \frac{1+z^{N+1}}{1-z} - \frac{1}{1-z} \right| = \left| \frac{z^{N+1}}{1-z} \right| = \frac{|z|^{N+1}}{|1-z|}$$

Handwritten note: $\left| \frac{1}{1-z} - \frac{1+z^{N+1}}{1-z} \right| = \left| \frac{-z^{N+1}}{1-z} \right| = \frac{|z|^{N+1}}{|1-z|}$

With a bit of thought we can see that as long as $|z| < 1$ we could find an n_0 large enough so that for any $\epsilon > 0$

$$\left| \frac{1+z^{N+1}}{1-z} - \frac{1}{1-z} \right| = \left| \frac{z^{N+1}}{1-z} \right| = \frac{|z|^{N+1}}{|1-z|} < \epsilon \text{ whenever } N > n_0$$

Handwritten note: $\left(\frac{1}{2}\right)^{N+1} = \left(\frac{1}{2}\right)^2 = \frac{1}{4}$

So we can now say that

$$\sum_{n=0}^{\infty} z^n = \frac{1}{1-z} \text{ as long as } |z| < 1$$

This is our old favorite from Calculus, the geometric series! It turns out to take the same form in the complex plane!

We will use this series quite often to find other series, for example Let's find the series for $\frac{1}{1+z^2}$

$\frac{1}{1+z^2} = ? \sum$

$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$

$|z| < 1$
 $\Rightarrow |z^2| < 1$

Substitute

$\frac{1}{1-(z^2)} = \frac{1}{1+z^2} = \sum_{n=0}^{\infty} (-z^2)^n = \sum_{n=0}^{\infty} (-1)^n z^{2n}$

Chapter 5 Sec 60-64

We also have some important theorems which go along with our infinite series.

Theorem: Suppose that $z_n = x_n + iy_n$ and $S = X + iY$ then

If and only if
$$\sum_{n=1}^{\infty} z_n = S$$

$$\sum_{n=1}^{\infty} x_n = X$$

$$\sum_{n=1}^{\infty} y_n = Y$$

This also allows us to be able to write

$$\sum_{n=1}^{\infty} z_n = \sum_{n=1}^{\infty} x_n + i \sum_{n=1}^{\infty} y_n$$

Corollary: If a series of complex numbers converges, the nth term *of the sum* converges to zero as n tends to infinity.

That is to say if $\sum_{n=1}^{\infty} z_n = S$ converges it must be that $\lim_{n \rightarrow \infty} z_n = 0$

This is true since it is true for the real numbers and we can split our sum as above, $\sum_{n=1}^{\infty} z_n = \sum_{n=1}^{\infty} x_n + i \sum_{n=1}^{\infty} y_n$

$\sum_{n=1}^{\infty} x_n$ converges $\lim_{n \rightarrow \infty} x_n = 0$ $\sum_{n=1}^{\infty} y_n$ converges $\lim_{n \rightarrow \infty} y_n = 0$
 $\lim_{n \rightarrow \infty} x_n + i y_n = \lim_{n \rightarrow \infty} z_n = 0$

We do need to be careful however, the statement does not work in reverse that is just because

$\lim_{n \rightarrow \infty} z_n = 0$	Does <u>not</u> mean	$\sum_{n=1}^{\infty} z_n$ converges
	$\sum \frac{1}{n}$ where $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$	
However		
$\lim_{n \rightarrow \infty} z_n \neq 0$	Does <u>indeed</u> mean	$\sum_{n=1}^{\infty} z_n$ diverges

Chapter 5 Sec 60-64

We can also extend the notion of absolute convergence to the complex plane.

Corollary 2 The absolute convergence of a series of complex numbers implies the convergence of that series.

Here a series $\sum_{n=1}^{\infty} z_n$ is said to converge absolutely if $\sum_{n=1}^{\infty} |z_n|$ converges

$\sum -\frac{1}{n^2}$ converges because $\sum \frac{1}{n^2}$ converges

The Remainder of a series

Suppose the sum $\sum_{n=1}^{\infty} z_n = S$ We define the remainder of a finite partial sum and the infinite sum as

$\rho_N = S - S_N$

Where $S_N = \sum_{n=1}^N z_n$

As it turns out the series $\sum_{n=1}^{\infty} z_n$ converges if and only if the remainders tend to zero as $N \rightarrow \infty$

Just a few other useful Identities before Taylor's Theorem

If $\sum_{n=1}^{\infty} z_n = S = x + iy$ then $\sum_{n=1}^{\infty} \overline{z_n} = \overline{S} = x - iy$

If $\sum_{n=1}^{\infty} z_n = S$ then $\sum_{n=1}^{\infty} cz_n = cS$

If $\sum_{n=1}^{\infty} z_n = S$ $\sum_{n=1}^{\infty} w_n = T$ then $\sum_{n=1}^{\infty} (z_n + w_n) = S + T$

Chapter 5 Sec 60-64

Taylor's Theorem!!!

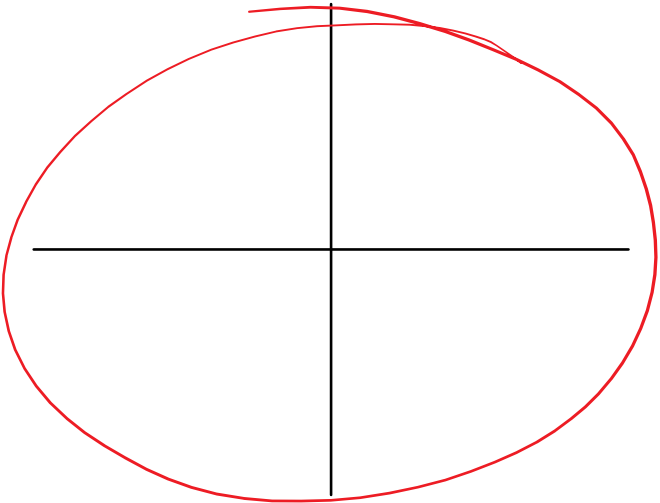
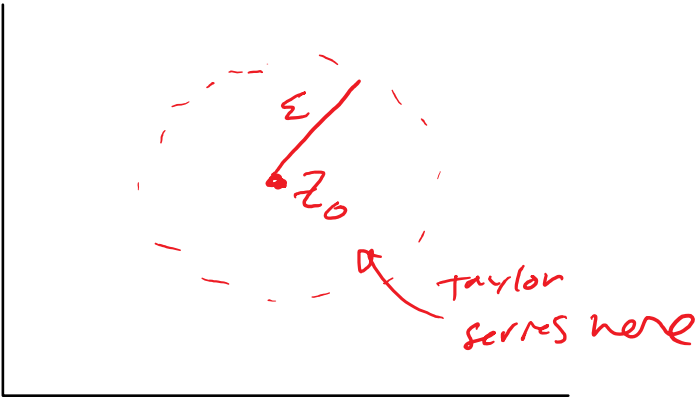
Theorem: Suppose that a function f is analytic throughout a disk $|z - z_0| < R_0$ centered at z_0 and with radius R_0 . Then $f(z)$ has the power series representation

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n \quad \text{When } |z - z_0| < R_0$$



Note that the requirement for the existence of the Taylor series was that it was analytic in the open disk $|z - z_0| < R_0$

You might recall that if $f(z)$ is analytic at a point this means that it is analytic in some neighborhood of that point, this means that if $f(z)$ is analytic at a point it must have a Taylor series in some neighborhood around that point. Also if a function is entire then its Taylor series will converge in the whole complex plane.



So ϵ becomes the radius of convergence if $f(z)$ is analytic at a point

If $f(z)$ is entire the Taylor series converges everywhere!

Chapter 5 Sec 60-64

Maclaurin series

If we take $z_0 = 0$ then just like in calculus we call that Taylor series the Maclaurin series.

$$f(z) = \sum_{n=0}^{\infty} \frac{f^n(0)}{n!} (z-0)^n$$

When $|z| < R_0$

Here is a list of a whole bunch of useful ones!

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n = 1 + z + z^2 + \dots \quad |z| < 1$$

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots \quad |z| < \infty$$

$$\sin(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots \quad |z| < \infty$$

$$\cos(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!} = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \dots \quad |z| < \infty$$

$$\sinh(z) = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!} = z + \frac{z^3}{3!} + \frac{z^5}{5!} + \dots \quad |z| < \infty$$

$$\cosh(z) = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!} = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \dots \quad |z| < \infty$$

Having a list is nice as we can use known series to find others.

Chapter 5 Sec 60-64

Let's use the definition

$$f(z) = \sum_{n=0}^{\infty} \frac{f^n(z_0)}{n!} (z - z_0)^n$$

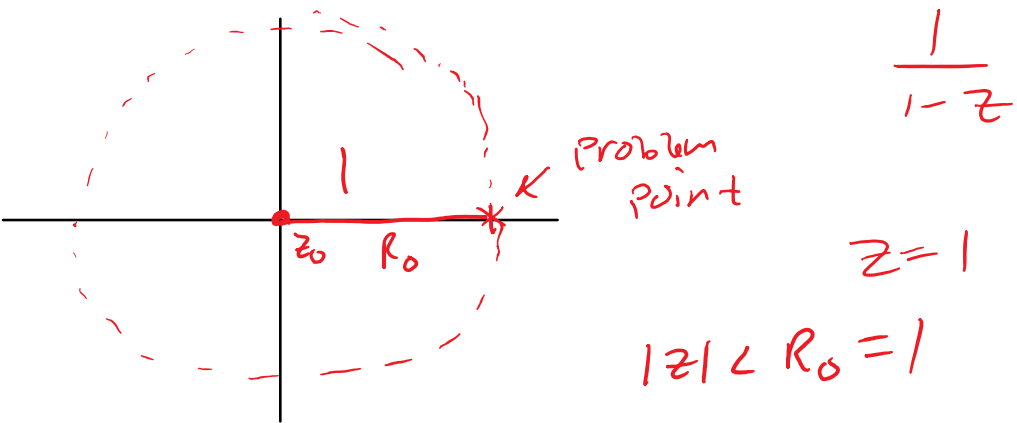
When $|z - z_0| < R_0$

To verify that

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n = 1 + z + z^2 + \dots$$

$|z| < 1$

Note that this is the series for $z_0 = 0$ Let's think about the largest neighborhood around zero for which $\frac{1}{1-z}$ is analytic



Ok so then we can only go out to $z = 1$ Let's apply the theorem then to the open disk $|z| < 1$

$$f(z) = \sum_{n=0}^{\infty} \frac{f^n(z_0)}{n!} (z - z_0)^n$$

$z_0 = 0$

$$f(z) = \frac{1}{1-z} = (1-z)^{-1}$$

$(1-z)^{-2} = -2(1-z)^{-3}$

$$f'(z) = - (1-z)^{-2} \cdot (-1) = (1-z)^{-2} = \frac{1}{(1-z)^2}$$

$$2(1-z)^3 = 6(1-z)^4$$

$$f''(z) = \frac{2}{(1-z)^3}$$

$$f'''(z) = \frac{2 \cdot 3}{(1-z)^4}$$

$4 \cdot 3 \cdot 2 \cdot 1 = 4!$

$$f^{(4)}(z) = \frac{2 \cdot 3 \cdot 4}{(1-z)^5}$$

$5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5!$

$$f^{(n)}(z) = \frac{n!}{(1-z)^{n+1}}$$

$$f^{(n)}(z) = \frac{n!}{(1-z)^{n+1}}$$

$$f(z) = \frac{1}{1-z} = \sum_{n=0}^{\infty} \frac{n!}{n!} (z-0)^n$$

$$= \sum_{n=0}^{\infty} z^n$$

$$= \left[\sum_{n=0}^{\infty} z^n \right] \checkmark$$

$$f^{(4)}(z) = \frac{2 \cdot 3 \cdot 4}{(1-z)^5}$$

$$f^{(n)}(0) = \frac{n!}{(1-0)^{n+1}} = n!$$

$$= \boxed{\sum_{n=0}^{\infty} z^n} \checkmark$$

Chapter 5 Sec 60-64

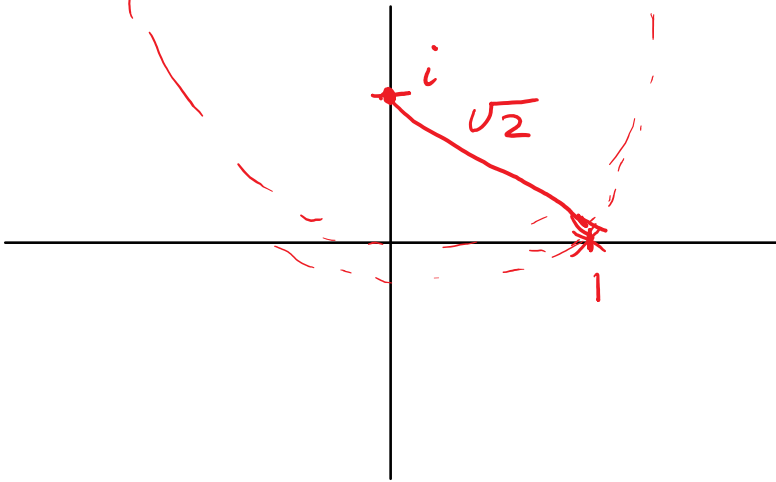
We can also just use the given series to find expansions around other points.

Find the Taylor series for $f(z) = \frac{1}{1-z}$ expanded around $z_0 = i$

We will use $\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n = 1 + z + z^2 + \dots$

If we cleverly add and subtract i in the denominator, that is we write $\frac{1}{1-z} = \frac{1}{(1-i) - (z-i)} = \frac{1}{1-i} \left(\frac{1}{1 - \frac{(z-i)}{(1-i)}} \right)$

First though let's think about the neighborhood around i where our function is actually analytic



$\frac{1}{1-z}$

$z=1$

So our series will be good for $|z-i| < \sqrt{2}$

Now let's modify away...

$$\frac{1}{1-i} \left(\frac{1}{1 - \frac{(z-i)}{(1-i)}} \right) = \frac{1}{1-i} \sum_{n=0}^{\infty} \left(\frac{(z-i)}{(1-i)} \right)^n$$
$$= \sum_{n=0}^{\infty} \frac{(z-i)^n}{(1-i)^{n+1}}$$

Homework Example

4. Find the Taylor series expansion of

$f(z) = \frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$

at the following points, if it exists.

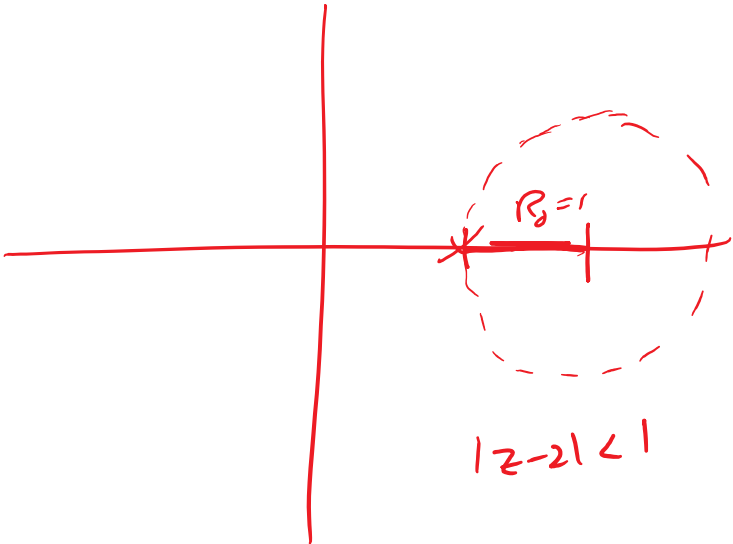
- (a) $z = 0$
- (b) $z = 1$
- (c) $z = 2$

$\sum \frac{f^{(n)}(2)}{n!} (z-2)^n$

$\frac{1}{1-z} = \frac{1}{1-2-z+2} = \frac{1}{-1-(z-2)}$
 $= \frac{-1}{1-(-(z-2))}$

$= - \sum_{n=0}^{\infty} (-(z-2))^n$
 $= - \sum_{n=0}^{\infty} (-1)^n (z-2)^n$
 $= \sum_{n=0}^{\infty} (-1)^{n+1} (z-2)^n$

$\frac{1}{1-z}$



Chapter 5 Sec 60-64

We can also find new series from old ones

Homework example

6. Obtain the Maclaurin series representation

$$\underline{z \cosh(z^2)} = \sum_{n=0}^{\infty} \underline{\frac{z^{4n+1}}{(2n)!}}$$

when $|z| < \infty$

$$\cosh(z) = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!}$$

$$\cosh(z^2) = \sum_{n=0}^{\infty} \frac{(z^2)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{z^{4n}}{(2n)!}$$

$$z \cosh(z^2) = z \sum_{n=0}^{\infty} \frac{z^{4n}}{(2n)!}$$

$$= \sum_{n=0}^{\infty} \frac{z z^{4n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{z^{4n+1}}{(2n)!}$$

$$\frac{1}{(z-2)(z+1)(z-3)(z-ji)} \quad z=1$$

