

Laurent Series

We know that if a function is analytic everywhere on an open disk that we have a Taylor series on that disk. It is a shame that even just one point of non-analyticity prevent us from having series expansion everywhere else. It turns out that there is a type of series which allows us to still expand our function in powers of $(z - z_0)$ even if we have point where the function is not analytic.

Laurent's Theorem: Suppose that a function f is analytic throughout an annular domain $R_1 < |z - z_0| < R_2$, centered at a point z_0 , and let C be any positively oriented simple closed contour around z_0 lying in the annular region. Then at each point in the annular domain $f(z)$ has the series representation.

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}$$

$$R_1 < |z - z_0| < R_2$$

$$\frac{1}{(z - z_0)^n} = (z - z_0)^{-n}$$

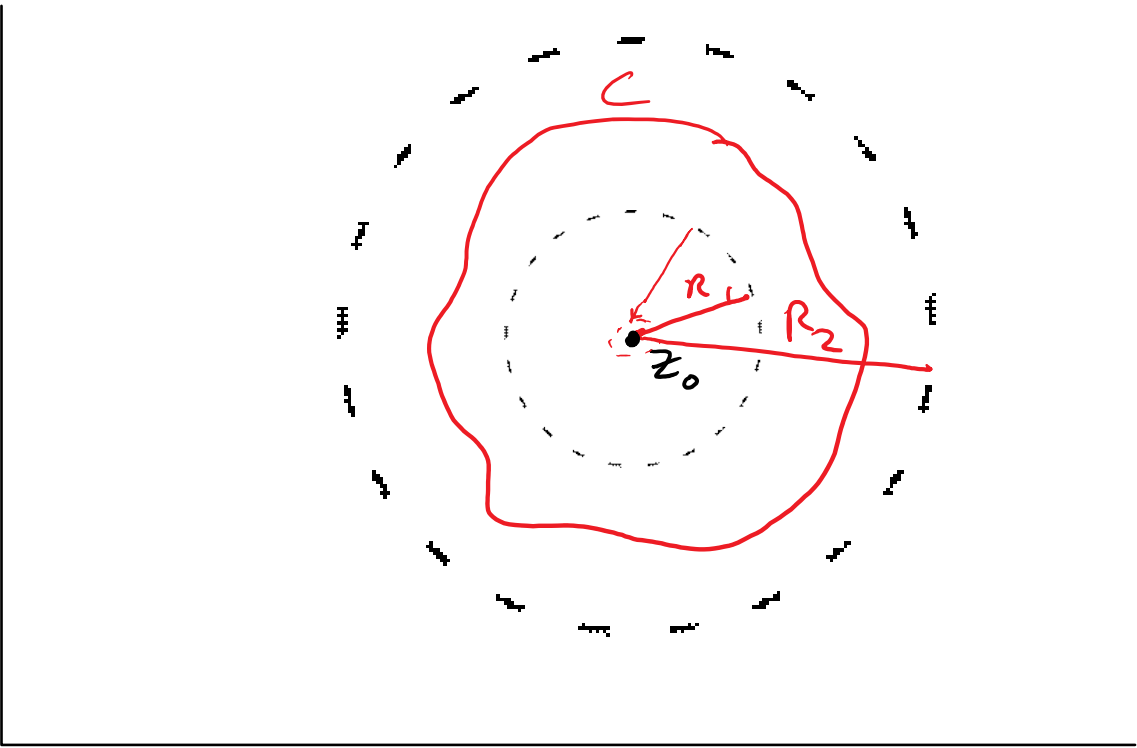
Where

$$a_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{n+1}} dz$$

$$n = 0, 1, 2, \dots$$

$$b_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{-n+1}} dz$$

$$n = 1, 2, \dots$$



Laurent Series

Another way Laurent series are represented is to allow negative values for n in which case our theorem becomes:

Laurent's Theorem: Suppose that a function f is analytic throughout an annular domain $R_1 < |z - z_0| < R_2$, centered at a point z_0 , and let C be any positively oriented simple closed contour around z_0 lying in the annular region. Then at each point in the annular domain $f(z)$ has the series representation.

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n \qquad R_1 < |z - z_0| < R_2$$

$$c_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{n+1}} dz \qquad n = 0, \pm 1, \pm 2, \pm 3$$

$\sum_{n=-\infty}^{\infty} c_n (z - z_0)^n = \dots + c_{-N} (z - z_0)^{-N} + c_{-N+1} (z - z_0)^{-N+1} + \dots + c_{-1} (z - z_0)^{-1} + c_0 + c_1 (z - z_0) + c_2 (z - z_0)^2 + \dots$

We can show this by interchanging $-n$ with n in our expression for b_n $b_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{-n+1}} dz \quad n = 1, 2, \dots$

Then $b_{-n} = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{-(-n)+1}} = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{n+1}}$

And we can then write:

$$\sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n} = \sum_{n=-\infty}^{-1} \frac{b_{-n} (z - z_0)^n}{(z - z_0)^n}$$

Then
$$f(z) = \sum_{n=-\infty}^{-1} b_{-n} (z - z_0)^n + \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

Or just that

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n \qquad R_1 < |z - z_0| < R_2$$

$$c_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{n+1}} dz \qquad n = 0, \pm 1, \pm 2, \pm 3$$

Chapter 5 Sec 65-68

Proof!

Laurent's Theorem: Suppose that a function f is analytic throughout an annular domain $R_1 < |z - z_0| < R_2$, centered at a point z_0 , and let C be any positively oriented simple closed contour around z_0 lying in the annular region. Then at each point in the annular domain $f(z)$ has the series representation.

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}$$

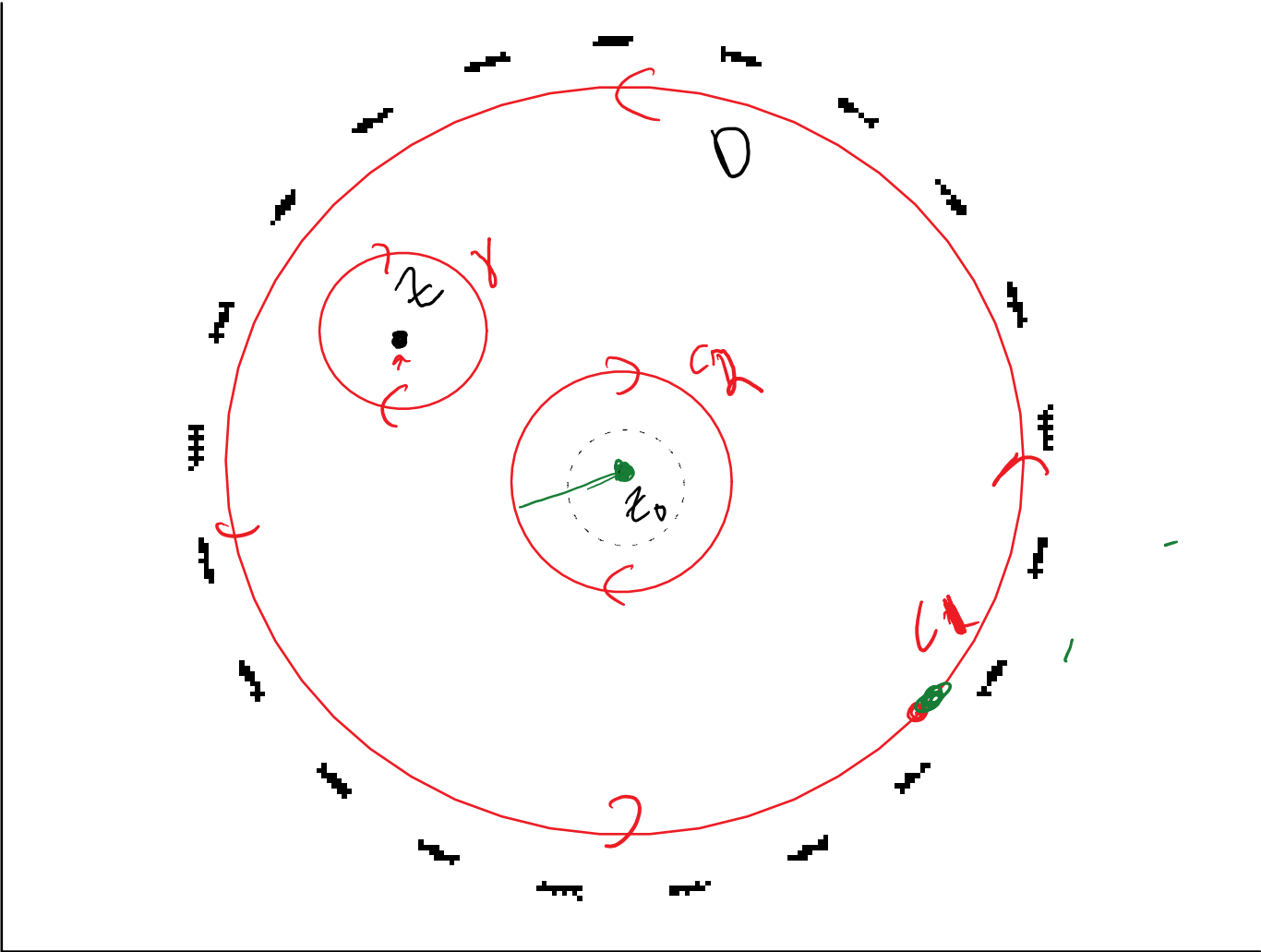
$$R_1 < |z - z_0| < R_2$$

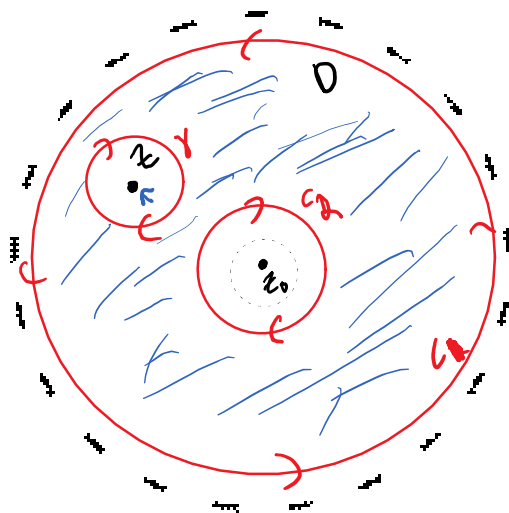
Where

$$a_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{n+1}} dz \quad n = 0, 1, 2, \dots$$

$$b_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{-n+1}} dz \quad n = 1, 2, \dots$$

To prove this theorem we will first show it for $z_0 = 0$. Choose an arbitrary z in the annulus where f is analytic and cleverly re-write $f(z)$ in terms of some contour integrals which will give us our Laurent series.





Notice that this is a multiply connected domain, so if we were to orient C_2 positively and γ, C_1 negatively we have the following theorem that will apply:

- Suppose that C is a simple closed contour positively oriented
1. C is a simple closed contour which is positively oriented
 2. C_k ($k = 1, 2, \dots, n$) are simple closed contours interior to C , all negatively oriented (clockwise) and disjoint with no common interior points.

Then if $f(z)$ is analytic on all of these contours and throughout the multiply connected domain consisting of the points inside C and exterior to each C_k then

$$\int_C f(z) dz + \sum_{k=1}^n \int_{C_k} f(z) dz = 0$$

not analytic at z_0

What we will do is define $g(z) = \frac{f(s)}{s-z}$ which is analytic on the multiply connected domain and further more $f(z)$ is analytic on and inside the region defined by γ

not analytic at z

We can then apply our theorem $\int_{C_1} g(z) ds + \int_{C_2} g(z) ds + \int_{\gamma} g(z) ds = 0$ or in terms of $f(s)$

$$\int_{C_1} \frac{f(s)}{s-z} ds + \int_{C_2} \frac{f(s)}{s-z} ds + \int_{\gamma} \frac{f(s)}{s-z} ds = 0$$

Note that if instead we orient C_2 and γ positively we can just change the signs on their integrals that is we can say:

$$\int_{C_1} \frac{f(s)}{s-z} ds - \int_{C_2} \frac{f(s)}{s-z} ds - \int_{\gamma} \frac{f(s)}{s-z} ds = 0$$

Where C_2 and γ are now positively oriented, that is we just used the fact that $\int_C f(z) dz = - \int_{-C} f(z) dz$

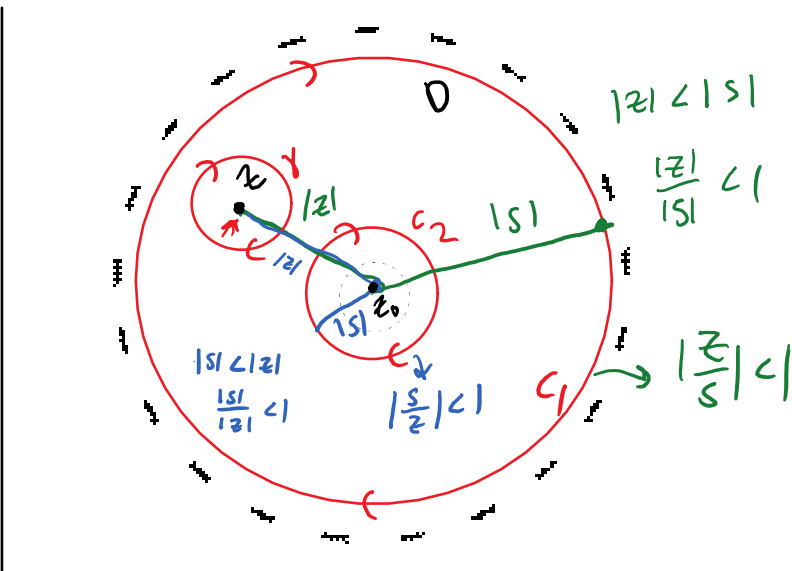
Chapter 5 Sec 65-68

$\int_{\gamma} \frac{f(z)}{z-z_0} dz = 2\pi i f(z_0)$

Ok so we have the statement that :

$\int_{C_1} \frac{f(s)}{s-z} ds - \int_{C_2} \frac{f(s)}{s-z} ds - \int_{\gamma} \frac{f(s)}{s-z} ds = 0$

Now the only point where $f(s)$ would not be analytic is at z_0 this means it is analytic on and in the region defined by γ which means that the cauchy integral formula applies for that contour.



$\int_{\gamma} \frac{f(s)}{s-z} ds = 2\pi i f(z)$

So we may say

$\int_{C_1} \frac{f(s)}{s-z} ds - \int_{C_2} \frac{f(s)}{s-z} ds - 2\pi i f(z) = 0$
 $+ 2\pi i f(z)$

Or

$f(z) = \frac{1}{2\pi i} \int_{C_1} \frac{f(s)}{s-z} ds - \frac{1}{2\pi i} \int_{C_2} \frac{f(s)}{s-z} ds$

Now we will just multiply that minus sign through the second integral to get

$f(z) = \frac{1}{2\pi i} \int_{C_1} \frac{f(s)}{s-z} ds + \frac{1}{2\pi i} \int_{C_2} \frac{f(s)}{z-s} ds$

If we then factor out a $\frac{1}{s}$ from the integral around C_1 and a $\frac{1}{z}$ from the integral around C_2 we can say

$f(z) = \frac{1}{2\pi i} \int_{C_1} \frac{1}{s} \frac{f(s)}{1-\frac{z}{s}} ds + \frac{1}{2\pi i} \int_{C_2} \frac{1}{z} \frac{f(s)}{1-\frac{s}{z}} ds$

Any ideas on another way we can represent $\frac{1}{1-\frac{z}{s}}$ and $\frac{1}{1-\frac{s}{z}}$?

$\frac{1}{1-\frac{z}{s}} = \sum_{n=0}^{\infty} \left(\frac{z}{s}\right)^n$

$\sum_{h=0}^{\infty} z^h = \frac{1}{1-z}$

$\frac{1}{1-\frac{s}{z}} = \sum_{h=0}^{\infty} \left(\frac{s}{z}\right)^h$

$$1 - \frac{1}{s} \quad \checkmark \quad n=0 \quad 1$$

$$C_2 \quad \downarrow \quad \left| \frac{z}{s} \right| < 1$$

$$\frac{1}{1 - \frac{s}{z}} = \sum_{n=0}^{\infty} \left(\frac{s}{z} \right)^n \quad \checkmark \quad (z)$$

$$C_1 \quad \left| \frac{s}{z} \right| < 1$$

Chapter 5 Sec 65-68

So we have:

$$f(z) = \frac{1}{2\pi i} \int_{C_1} \frac{f(s)}{s} \frac{1}{1 - \frac{z}{s}} ds + \frac{1}{2\pi i} \int_{C_2} \frac{f(s)}{z} \frac{1}{1 - \frac{s}{z}} ds$$

$$= \frac{1}{2\pi i} \int_{C_1} \frac{f(s)}{s} \sum_{n=0}^{\infty} \left(\frac{z}{s}\right)^n ds + \frac{1}{2\pi i} \int_{C_2} \frac{f(s)}{z} \sum_{n=0}^{\infty} \left(\frac{s}{z}\right)^n ds$$

Our goal will be to interchange the summation and the integration, which we can certainly do for finite sums however these are infinite sums so we have to be a bit more careful, the trick will be to re-write our geometric series

$$\sum_{n=0}^{\infty} z^n = \frac{1}{1-z} - \frac{z^N}{1-z} + \frac{z^N}{1-z} = \frac{1-z^N}{1-z} + \frac{z^N}{1-z} = \sum_{n=0}^{N-1} z^n + \frac{z^N}{1-z}$$

If we put this into our integral equation for $f(z)$ we get:

$$f(z) = \frac{1}{2\pi i} \int_{C_1} \frac{f(s)}{s} \sum_{n=0}^{N-1} \left(\frac{z}{s}\right)^n + \frac{1}{2\pi i} \frac{\left(\frac{z}{s}\right)^N}{1 - \frac{z}{s}} ds + \frac{1}{2\pi i} \int_{C_2} \frac{f(s)}{z} \sum_{n=0}^{N-1} \left(\frac{s}{z}\right)^n + \frac{1}{2\pi i} \frac{\left(\frac{s}{z}\right)^N}{1 - \frac{s}{z}} ds$$

Now that we have a finite sum we can interchange the summation with the integration:

$$f(z) = \frac{1}{2\pi i} \int_{C_1} \frac{f(s)}{s} \sum_{n=0}^{N-1} \left(\frac{z}{s}\right)^n + \frac{1}{2\pi i} \int_{C_1} \frac{\left(\frac{z}{s}\right)^N}{1 - \frac{z}{s}} ds + \frac{1}{2\pi i} \int_{C_2} \frac{f(s)}{z} \sum_{n=0}^{N-1} \left(\frac{s}{z}\right)^n + \frac{1}{2\pi i} \int_{C_2} \frac{\left(\frac{s}{z}\right)^N}{1 - \frac{s}{z}} ds$$

$$= \sum_{n=0}^{N-1} \frac{1}{2\pi i} \int_{C_1} \frac{f(s)}{s} \frac{z^n}{s^n} ds + \frac{1}{2\pi i} \int_{C_1} \frac{\left(\frac{z}{s}\right)^N}{1 - \frac{z}{s}} ds + \sum_{n=0}^{N-1} \frac{1}{2\pi i} \int_{C_2} \frac{f(s)}{z} \left(\frac{s^n}{z^n}\right) + \frac{1}{2\pi i} \int_{C_2} \frac{\left(\frac{s}{z}\right)^N}{1 - \frac{s}{z}} ds$$

$$= \sum_{n=0}^{N-1} \frac{1}{2\pi i} \int_{C_1} \frac{f(s)}{s^{n+1}} ds z^n + \frac{1}{2\pi i} \int_{C_1} \frac{\left(\frac{z}{s}\right)^N}{1 - \frac{z}{s}} ds + \sum_{n=0}^{N-1} \frac{1}{2\pi i} \int_{C_2} f(s) s^n ds \frac{1}{z^{n+1}} + \int_{C_2} \frac{\left(\frac{s}{z}\right)^N}{1 - \frac{s}{z}} ds$$

So we have:

$$f(z) = \sum_{n=0}^{N-1} \frac{1}{2\pi i} \int_{C_1} \frac{f(s)}{s^{n+1}} ds z^n + \frac{1}{2\pi i} \int_{C_1} \frac{\left(\frac{z}{s}\right)^N}{1 - \frac{z}{s}} ds + \sum_{n=0}^{N-1} \frac{1}{2\pi i} \int_{C_2} f(s) s^n ds \frac{1}{z^{n+1}} + \frac{1}{2\pi i} \int_{C_2} \frac{\left(\frac{s}{z}\right)^N}{1 - \frac{s}{z}} ds$$

If we re-index $\sum_{n=0}^{N-1} \frac{1}{2\pi i} \int_{C_2} f(s) s^n ds \frac{1}{z^{n+1}}$ by starting at $n = 1$ and sue negative exponents with s we can write

$$\sum_{n=0}^{N-1} \frac{1}{2\pi i} \int_{C_2} f(s) s^n ds \frac{1}{z^{n+1}} = \sum_{n=1}^N \frac{1}{2\pi i} \int_{C_2} \frac{f(s)}{s^{-n+1}} ds \frac{1}{z^n}$$

$$\sum_{n=1}^N \frac{1}{2\pi i} \int_{C_2} f(s) s^{n-1} ds \frac{1}{z^n}$$

$$\sum_{n=1}^N \frac{1}{2\pi i} \int_{C_2} f(s) \frac{1}{s^{-n+1}} ds \frac{1}{z^n}$$

$$\frac{1}{s^{-n+1}} = (s^{-n+1})^{-1} = s^{-(-n+1)} = s^{n-1}$$

So we now have that

$$f(z) = \sum_{n=0}^{N-1} \frac{1}{2\pi i} \int_{C_1} \frac{f(s)}{s^{n+1}} ds z^n + \frac{1}{2\pi i} \int_{C_1} \frac{\left(\frac{z}{s}\right)^N}{1 - \frac{z}{s}} ds + \sum_{n=1}^N \frac{1}{2\pi i} \int_{C_2} \frac{f(s)}{s^{-n+1}} ds \frac{1}{z^n} + \frac{1}{2\pi i} \int_{C_2} \frac{\left(\frac{s}{z}\right)^N}{1 - \frac{s}{z}} ds$$

Chapter 5 Sec 65-68

Now recall that our point z_0 is the origin, so another way I can write this is

$\int_{C_1} \left(\frac{z}{s}\right)^N \frac{1}{1-\frac{z}{s}}$

$N \rightarrow \infty$

$$f(z) = \sum_{n=0}^{N-1} \frac{1}{2\pi i} \int_{C_1} \frac{f(s)}{(s-0)^{n+1}} ds (z-0)^n + \frac{1}{2\pi i} \int_{C_1} \frac{\left(\frac{z}{s}\right)^N}{1-\frac{z}{s}} ds + \sum_{n=1}^N \frac{1}{2\pi i} \int_{C_2} \frac{f(s)}{(s-0)^{-n+1}} ds \frac{1}{(z-0)^n} + \frac{1}{2\pi i} \int_{C_2} \frac{\left(\frac{s}{z}\right)^N}{1-\frac{s}{z}} ds$$

Now we take the limit as $N \rightarrow \infty$ it turns out that both

$$\lim_{N \rightarrow \infty} \left(\frac{1}{2\pi i} \int_{C_1} \frac{\left(\frac{z}{s}\right)^N}{1-\frac{z}{s}} ds \right) = 0$$

And

$$\lim_{N \rightarrow \infty} \frac{1}{2\pi i} \int_{C_2} \frac{\left(\frac{s}{z}\right)^N}{1-\frac{s}{z}} ds = 0$$

$$\geq \frac{1}{2\pi} \int_{C_1} \frac{|f(s)|^N}{|1-\frac{z}{s}|}$$

This means in our limit we obtain

$$f(z) = \sum_{n=0}^{\infty} \frac{1}{2\pi i} \int_{C_1} \frac{f(s)}{(s-0)^{n+1}} ds (z-0)^n + \sum_{n=1}^{\infty} \frac{1}{2\pi i} \int_{C_2} \frac{f(s)}{(s-0)^{-n+1}} ds \frac{1}{(z-0)^n}$$

Compare this series to the Laurent series coefficients if $z_0 = 0$

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n} \quad R_1 < |z-z_0| < R_2$$

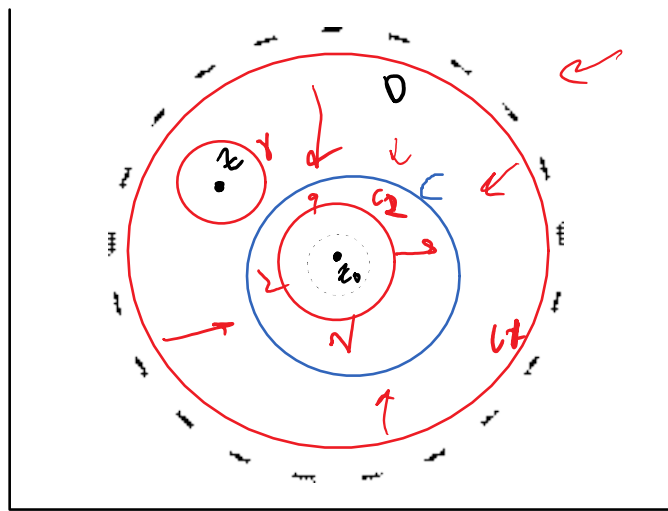
Where

$$a_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^{n+1}} dz$$

$$n = 0, 1, 2, \dots$$

$$b_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^{-n+1}} dz \quad n = 1, 2, \dots$$

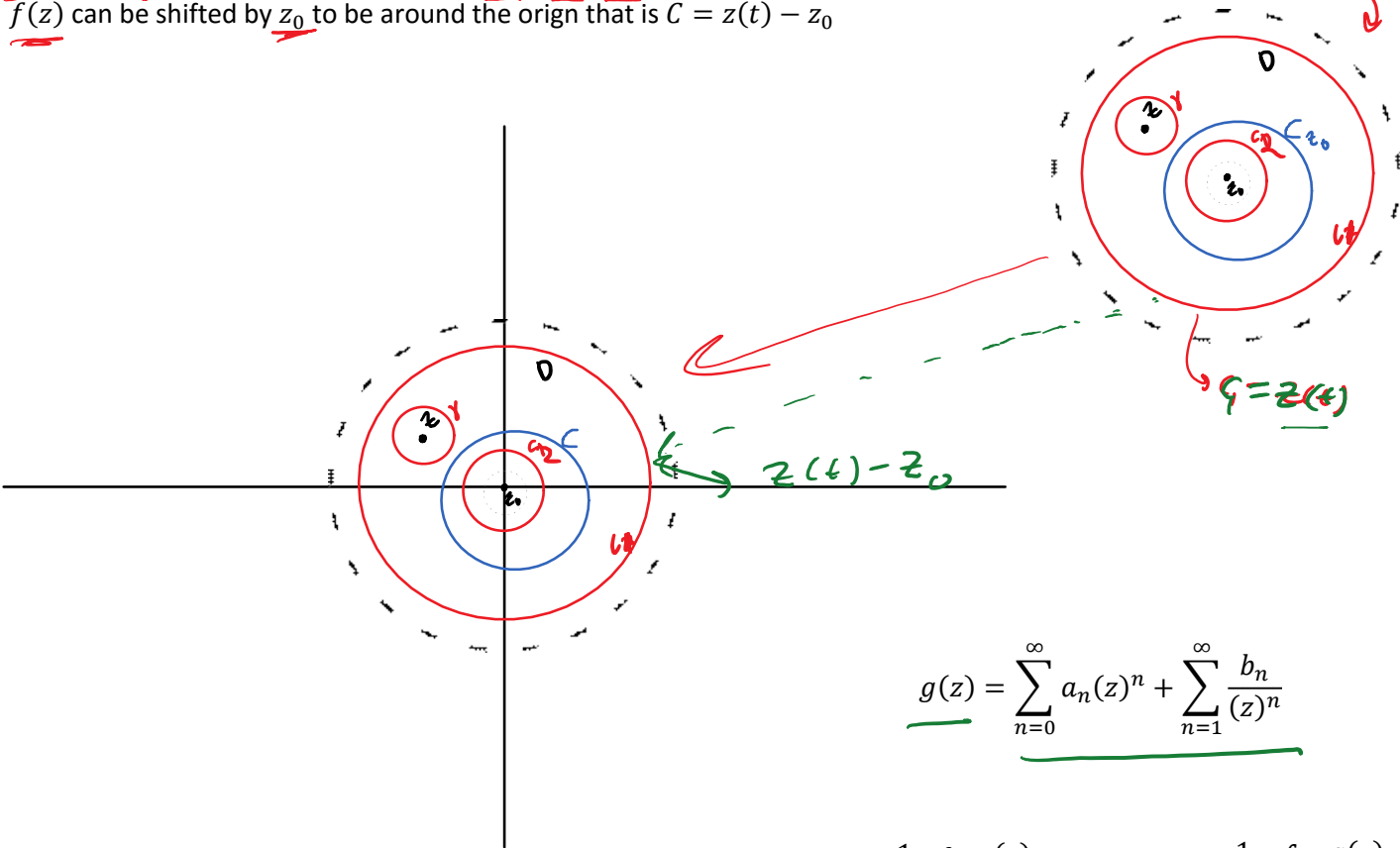
We need to keep in mind our deformation theorems, let C be some contour in between C_1, C_2



Chapter 5 Sec 65-68

So we found our Laurent series when $f(z)$ is not analytic at the origin!

To extend it to the case for any z_0 we simply define $g(z) = f(z + z_0)$ if $f(z)$ is analytic on $R_1 < |z - z_0| < R_2$ then $g(z)$ is on $R_1 < |z + z_0 - z_0| < R_2 \rightarrow R_1 < |z| < R_2$ which is centered at the origin now. Also the contours for $f(z)$ can be shifted by z_0 to be around the origin that is $C = z(t) - z_0$



$$g(z) = \sum_{n=0}^{\infty} a_n(z)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z)^n}$$

$$a_n = \frac{1}{2\pi i} \int_C \frac{g(z)}{(z)^{n+1}} dz \quad b_n = \frac{1}{2\pi i} \int_C \frac{g(z)}{(z)^{-n+1}} dz$$

This means what we just proves applies to $g(z)$ that is:

But $g(z) = f(z + z_0) = \sum_{n=0}^{\infty} a_n(z)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z)^n}$ $f(z) = f(z + z_0 - z_0) = \sum_{n=0}^{\infty} a_n(z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}$

and

$$a_n = \frac{1}{2\pi i} \int_C \frac{g(z)}{(z)^{n+1}} dz = \int_a^b \frac{g(z(t) - z_0)}{(z(t) - z_0)^{n+1}} z'(t) dt = \int_a^b \frac{f(z(t) - z_0 + z_0)}{(z(t) - z_0)^{n+1}} z'(t) dt = \int_a^b \frac{f(z(t))}{(z(t) - z_0)^{n+1}} z'(t) dt = \int_{C_{z_0}} \frac{f(z)}{(z - z_0)^{n+1}} dz$$
$$b_n = \frac{1}{2\pi i} \int_C \frac{g(z)}{(z)^{-n+1}} dz = \int_a^b \frac{g(z(t) - z_0)}{(z(t) - z_0)^{-n+1}} z'(t) dt = \int_a^b \frac{f(z(t) - z_0 + z_0)}{(z(t) - z_0)^{-n+1}} z'(t) dt = \int_a^b \frac{f(z(t))}{(z(t) - z_0)^{-n+1}} z'(t) dt = \int_{C_{z_0}} \frac{f(z)}{(z - z_0)^{-n+1}} dz$$

And there we go!

Chapter 5 Sec 65-68

Now we can very easily prove Taylor's theorem

Recall that if $g(z)$ is analytic on and in the region
Defined by a simple closed contour

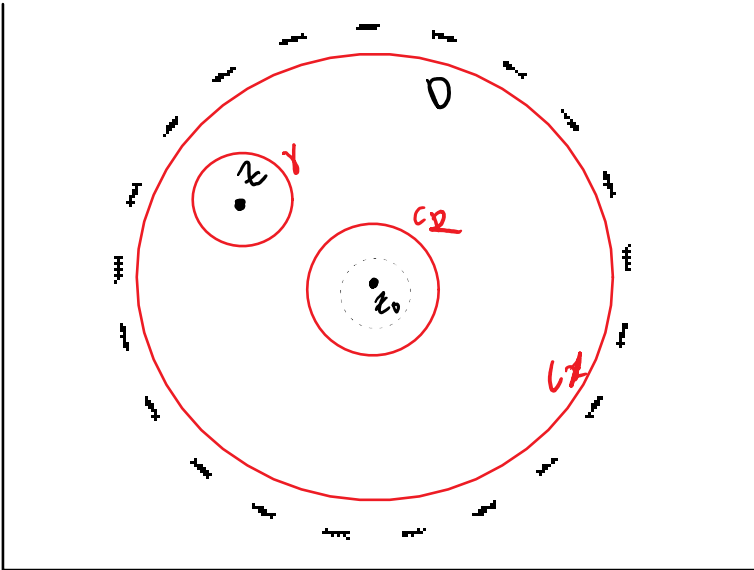
$$\int_C g(z) dz = 0$$

Well suppose our $f(z)$ is analytic at z_0 and consider

$\leftarrow$ analytic at z_0

$$b_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{-n+1}} dz \quad n = 1, 2, \dots$$

$\uparrow$



$\leftarrow$ analytic at z_0

$$b_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{-n+1}} dz = \frac{1}{2\pi i} \int_C \underbrace{f(z)}_{\text{analytic at } z_0} (z - z_0)^{n-1} dz = 0$$

$\leftarrow$ analytic at z_0

Note then that $g(z) = f(z)(z - z_0)^{n-1}$ is analytic on the contour around z_0 and inside since z_0 is not a problem here as the denominator comes up on top. This means that $b_n = 0$

If that is the case then we only retain the part of our Laurent series with the a'_n 's

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0) + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}$$

$\nearrow 0$

$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$

$a_n = \frac{f^{(n)}(z_0)}{n!}$

$$a_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{n+1}} dz \quad n = 0, 1, 2, \dots$$

$\nearrow 0$

$$b_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{-n+1}} dz \quad n = 1, 2, \dots$$

Then since $f(z)$ is analytic at z_0 we can apply the extended cauchy integral formula

$$a_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{n+1}} dz = \frac{1}{n!} f^{(n)}(z_0) = \quad n = 0, 1, 2, \dots$$

Thus

$$a_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{n+1}} dz = \frac{f^n(z_0)}{n!}$$

Our Taylor Series Coefficients!!!!

Chapter 5 Sec 65-68

Homework Example

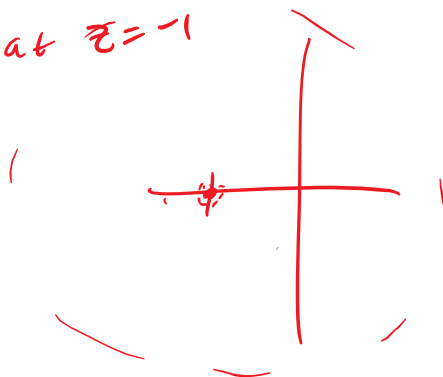
Derive the Laurent series representation of

$$f(z) = \frac{e^z}{(z+1)^2}$$

not analytic at $z = -1$

about the point $z_0 = -1$.

$$0 < |z - (-1)| < \infty$$



$$\frac{e^z}{(z+1)^2} = \frac{1}{(z+1)^2} e^z = \frac{1}{(z+1)^2} \sum_{n=0}^{\infty} \frac{(z+1)^n}{n!} = \frac{1}{(z+1)^2} \sum_{n=0}^{\infty} \frac{(z+1)^n}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{(z+1)^n}{(z+1)^2 n!} = \sum_{n=0}^{\infty} \frac{(z+1)^{n-2}}{n!} = \frac{(z+1)^{-2}}{0!} + \frac{(z+1)^{-1}}{1!} + \frac{(z+1)^0}{2!} + \frac{(z+1)^1}{3!} + \frac{(z+1)^2}{4!} + \dots$$

$$= \frac{1}{(z+1)^2} + \frac{1}{(z+1)} + \frac{1}{2} + \frac{1}{3!}(z+1) + \frac{1}{4!}(z+1)^2 + \frac{1}{5!}(z+1)^3 + \dots$$

$$b_2 = 1, \quad b_1 = 1, \quad \sum_{n=1}^{\infty} b_n \frac{1}{(z-z_0)^n} + \sum_{n=0}^{\infty} a_n (z-z_0)^n$$

$a_0 = \frac{1}{2!}, \quad a_1 = \frac{1}{3!}, \quad a_2 = \frac{1}{4!}$

$$\boxed{\frac{1}{(z+1)^2} + \frac{1}{(z+1)} + \sum_{n=0}^{\infty} \frac{1}{(n+2)!} (z+1)^n = f(z)}$$

Let's also just verify that $b_n = 0$ for $n > 2$ with the integral definition of the coefficient $b_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^{-n+1}} dz$

$$b_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^{-n+1}} dz = \frac{1}{2\pi i} \int_C \frac{e^z}{(z+1)^2} \frac{1}{(z+1)^{-n+1}} dz$$

$$= \frac{1}{2\pi i} \int_C \frac{e^z}{(z+1)^{n-3}} dz \quad n=3, \int_C e^z dz = 0$$

$$\frac{1}{2\pi i} \int_C \overline{(z+1)}^0 \frac{1}{z} dz = 0$$

$$n=4 \quad \frac{1}{2\pi i} \int_C \frac{e^z}{(z+1)^{-4+3}}$$

$$= \frac{1}{2\pi i} \int_C \frac{e^z}{(z+1)^{-1}} = \frac{1}{2\pi i} \int_C e^z (z+1)' dz = 0$$

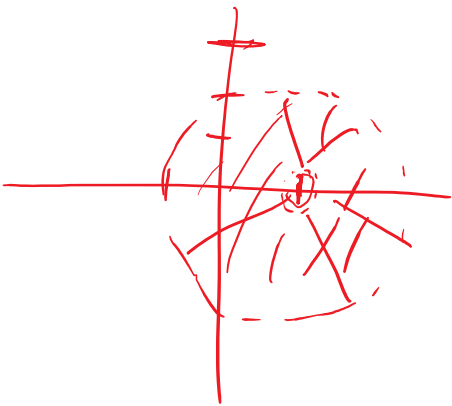
Chapter 5 Sec 65-68

Homework Example

On the domain $0 < |z-1| < 2$, find the Laurent series about $z_0 = 1$ of

$$f(z) = \frac{z}{(z-1)(z-3)}$$

circle
center
at 1
 $re^{i\theta} + (-1)$



$$\frac{1}{1-(z-z_0)} = \sum_{n=0}^{\infty} (z-z_0)^n$$

$$\frac{z}{(z-1)(z-3)} = \frac{A}{(z-1)} + \frac{B}{(z-3)}$$

$$z = A(z-3) + B(z-1)$$

$z=1$

$$1 = A(1-3) + B(1-1)$$

$$1 = A(-2)$$

$$A = -1/2$$

$z=3$

$$3 = A(3-3) + B(3-1)$$

$$3 = B(2) \quad B = 3/2$$

$$\frac{z}{(z-1)(z-3)} = \frac{-1}{2(z-1)} + \frac{3}{2(z-3)}$$

$$\frac{3}{2} \left(\frac{1}{z-3+1-1} \right) = \frac{3}{2} \left(\frac{1}{-2+(z-1)} \right)$$

$$\frac{3}{2} \left(\frac{-1}{2} \left(\frac{1}{1-(z-1)} \right) \right)$$

Chapter 5 Sec65-68 Page 17

Chapter 5 Sec 65-68

Now just a couple of statements on term by term differentiation and contour integration of power series.

Suppose we have a power series:

$$S(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$$

Theorem: Let C denote any contour interior to the circle of convergence of a powerseries, and let $g(z)$ be any function that in continuous on C . The series formed by multiplying the power series by $g(z)$ can be integrated term by term over C

$$\int_C g(s)S(z)dz = \sum_{n=0}^{\infty} a_n \int_C g(z) (z - z_0)^n dz$$

Note that the function $g(z) = 1$ is continuous on any contour.

Theorem: The power series can be differentiated term by term, that is at each point z interior to the circle of convergence of that series

$$S'(z) = \sum_{n=0}^{\infty} na_n(z - z_o)^{n-1}$$

A statement on Uniqueness

Theorem: If a series

$$\sum_{n=0}^{\infty} a_n(z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}$$

Converges to $f(z)$ at all points in some annular domain about z_0 , Then it is the Laurant series expansion for f in powers of $z - z_0$ in that domain, if $f(z)$ is analytic in the whole disk $|z - z_0| < R$ then $b_n = 0$ and the weries is the taylor series.

Chapter 5 Sec 65-68

Homework Example

Find the Taylor Series for $f(z) = \frac{1}{(1-z)^2}$ for the disk $|z| < 1$