

Chapter 6 Sec 74-78

Residues and Poles

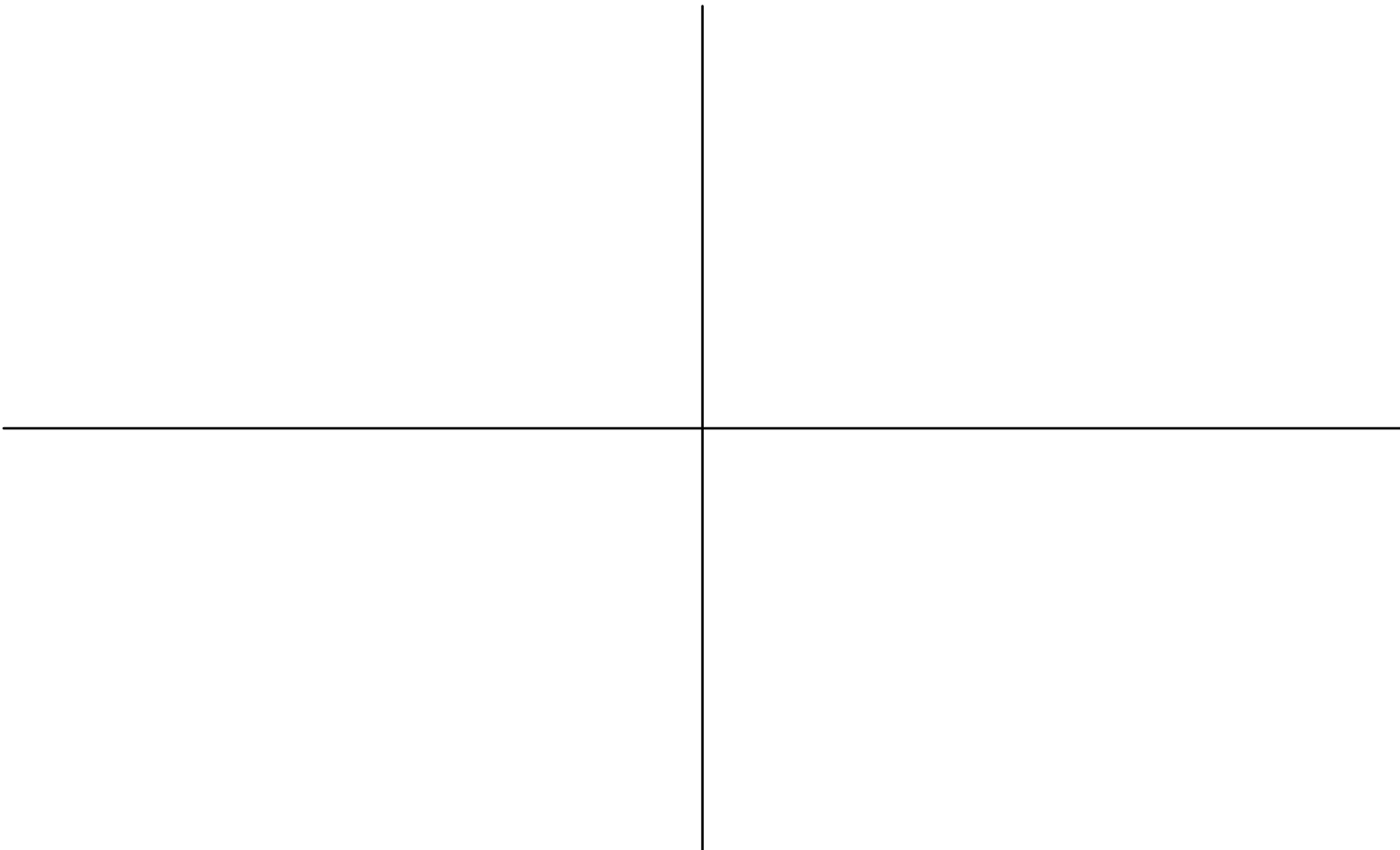
Before we talk about Residues we need to introduce the concept of an isolated singular point.

A singular point of a function $f(z)$ is a point where $f(z)$ is not analytic

An *isolated singular point* is a point at which $f(z)$ is analytic and there is a deleted neighborhood around that point where $f(z)$ is analytic.

Example $f(z) = \frac{z}{(z+1)(z^2+4)}$

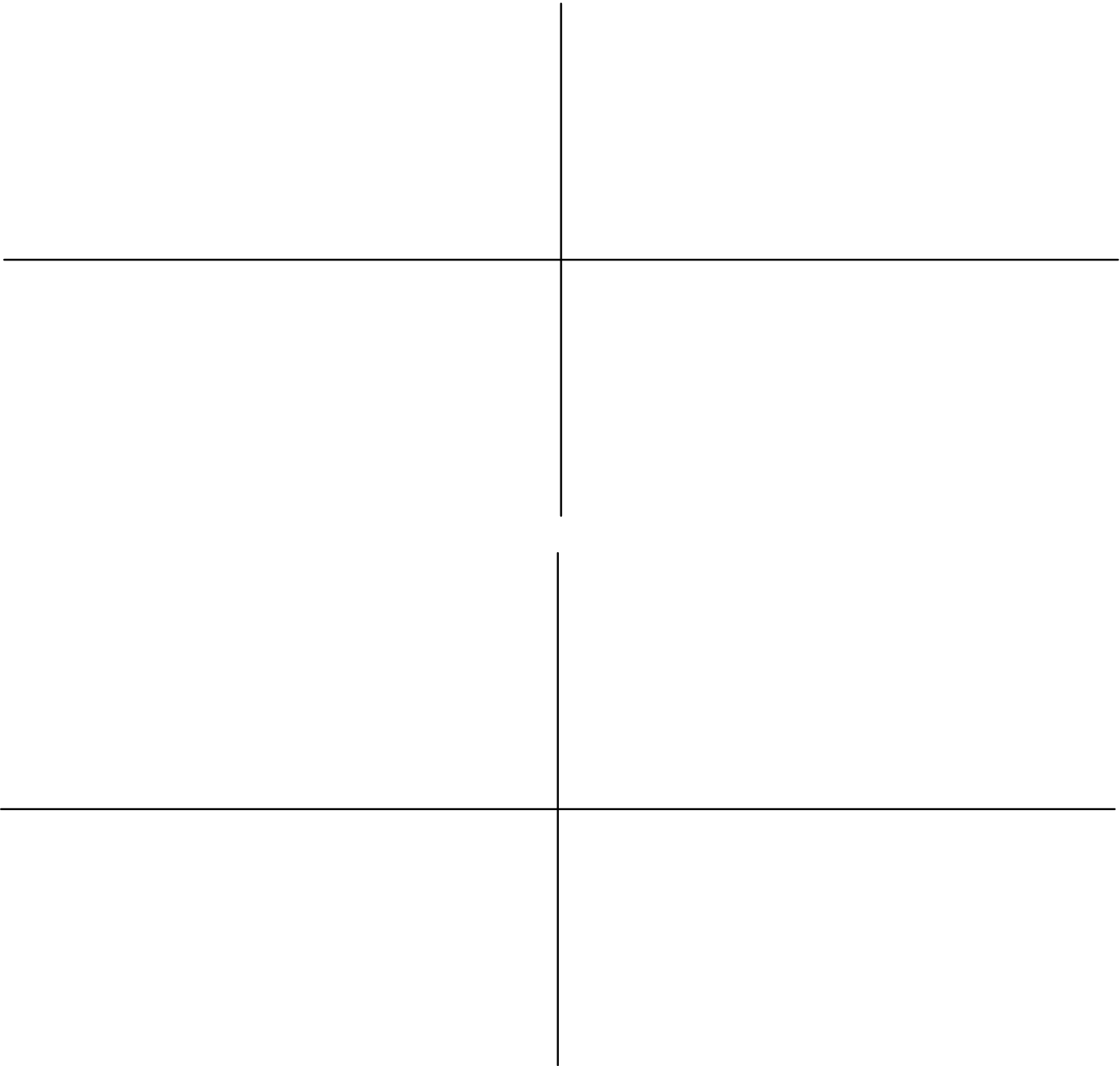
Where is $f(z)$ not analytic? Can we find some deleted neighborhood around those points where $f(z)$ is analytic?



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Another Example $f(z) = \frac{1}{\sin\left(\frac{\pi}{z}\right)}$

Where is $f(z)$ not analytic? Is there a deleted neighborhood around each for which $f(z)$ is analytic throughout?



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Residues

When z_0 is an isolated singular point of a function $f(z)$ we know that there is some deleted neighborhood $0 < |z - z_0| < R_2$ for which $f(z)$ has a Laurent series representation.

That is:

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n} = \sum_{n=0}^{\infty} a_n(z - z_0)^n + \frac{b_1}{z - z_0} + \frac{b_2}{(z - z_0)^2} + \dots + \frac{b_n}{(z - z_0)^n} + \dots$$

Where

$$b_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{-n+1}} dz$$

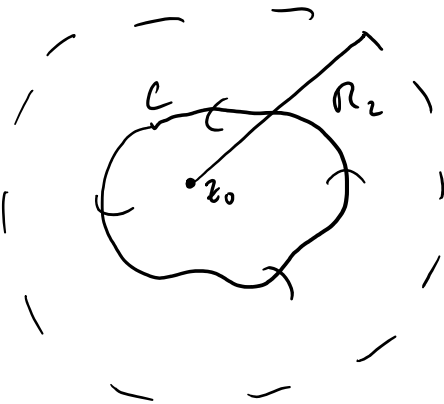
Note that if $n = 1 \dots$

So we now have a formula for the contour integral of $f(z)$ around one of it's isolated singular points!

$$\int_C f(z) dz = 2\pi i b_1$$

We often call $b_1 = \text{Res}_{z=z_0} f(z)$

$$\int_C f(z) dz = 2\pi i \text{Res}_{z=z_0} f(z)$$



So the residue of a function $f(z)$ at one of it's isolated singular points is the first coefficient of the Laurent series expansion around that isolated singular point. Further this gives us a way to evaluate a contour integral that otherwise would be difficult to calculate.

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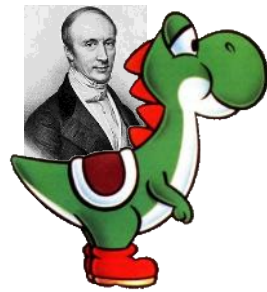
Homework Example

4. Find the residue of $\frac{z+1}{z-1}$ at

(a) $z_0 = 0$

(b) $z_0 = 1$

Incidentally we can now calculate $\int_C \frac{z+1}{z-1} dz$ easily!

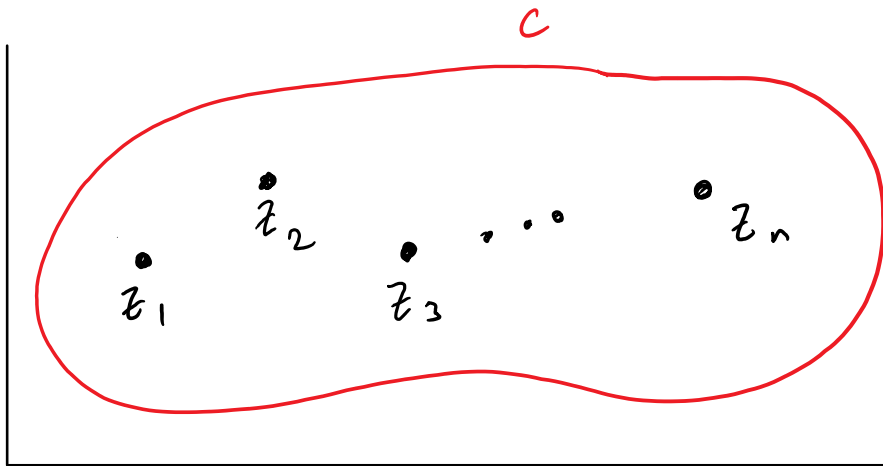


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Cauchy's Residue Theorem

Let C be a simple closed contour, positively oriented. If a function $f(z)$ is analytic on and inside C except at finitely many isolated singular points z_k for $k = 1 \dots n$ which lie within the contour C then:

$$\int_C f(z) dz = 2\pi i \sum_{k=1}^n \text{Res}_{z=z_k} f(z)$$



Proof! Draw positively oriented contours around each point z_k that lie within C . Now we have a multiply connected domain and the following theorem applies

Suppose that C is a simple closed contour positively oriented

1. C is a simple closed contour which is positively oriented
2. C_k ($k = 1, 2, \dots, n$) are simple closed contours interior to C , all negatively oriented (clockwise) and disjoint with no common interior points.

Then if $f(z)$ is analytic on all of these contours and throughout the multiply connected domain consisting of the points inside C and exterior to each C_k then

$$\int_C f(z) dz + \sum_{k=1}^n \int_{C_k} f(z) dz = 0$$

For us since we positively oriented our inside contours we have

$$\int_C f(z) dz - \sum_{k=1}^n \int_{C_k} f(z) dz = 0$$

Or

$$\int_C f(z) dz = \sum_{k=1}^n \int_{C_k} f(z) dz \quad \text{And} \quad \int_{C_k} f(z) dz = 2\pi i \text{Res}_{z=z_k} f(z)$$

So

$$\int_C f(z) dz = 2\pi i \sum_{k=1}^n \text{Res}_{z=z_k} f(z)$$

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In practice using this theorem can be tedious as for each isolated singular point we would need to find a Laurent Series for $f(z)$ centered around that point and look for the corresponding coefficient of $\frac{1}{z-z_k}$. We can actually rephrase this with one isolated singular point at infinity but before we do let's use this theorem to illustrate it.

5. Use the residue theorem to compute all possible values for the integral

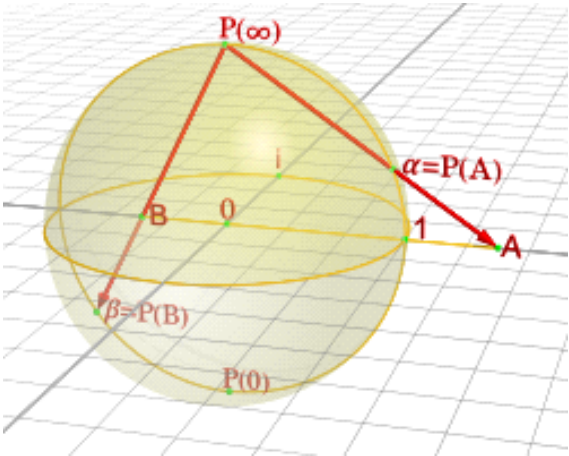
$$\oint_C \frac{1}{(z-1)(z+i)} dz,$$

where C is any positively oriented simple closed path in the plane that does not include the points $z = 1, -i$.

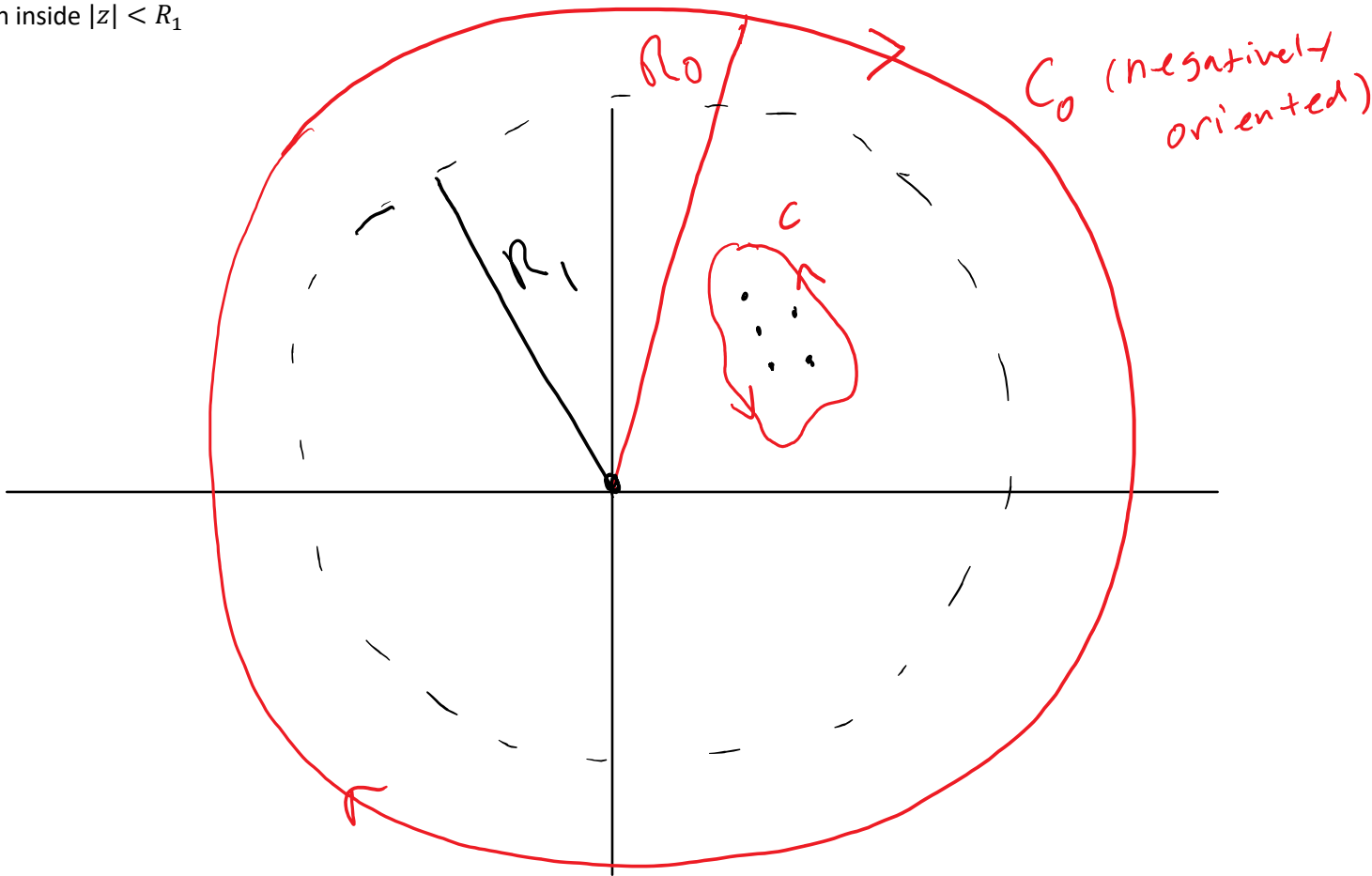
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Residue at Infinity

Here we will suppose that our function $f(z)$ is analytic throughout the finite plane except for a finite number of singular points which can be enclosed in a simple closed contour C .



To define the residue at infinity we will need to enclose the contour C with a circle of radius R_1 centered at the origin outside of which $f(z)$ is certainly analytic as all of its singular points live in the contour C . Then we will define a contour C_0 that encloses the region inside $|z| < R_1$



The residue at infinity is then defined by the integral equation

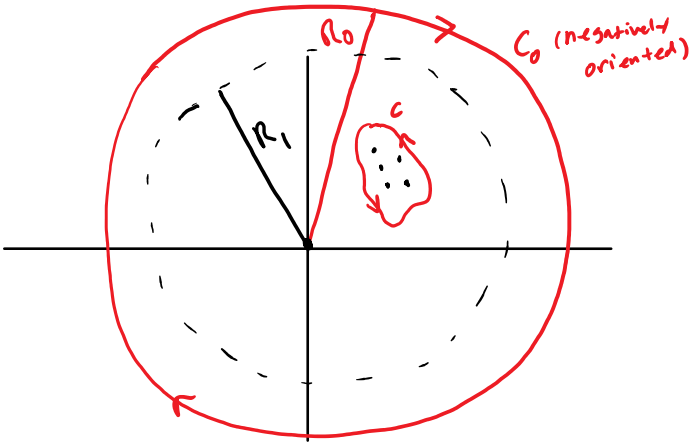
$$\int_{C_0} f(z) dz = 2\pi i \operatorname{Res}_{z=\infty} f(z)$$

Just a note on why c_0 is negatively oriented, Imagine yourself walking along the contour, infinity is to your left if you

move clockwise, that is this is positively oriented when we think about the point at infinity.

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Now note that $f(z)$ is analytic in the region between C_0 and C which means that if we were to integrate along either in the positive directions we would get the same answer, this is the principle of deformation of path at play again. However C_0 is negatively oriented so we will just get the negative of the result from C_0 that we would get from C



That is.

$$\int_C f(z) dz = \int_{-C_0} f(z) dz = - \int_{C_0} f(z) dz$$

We defined the residue at infinity with $\int_{C_0} f(z) dz = 2\pi i \operatorname{Res}_{z=\infty} f(z)$ Throwing on our minus sign gives us

$$\int_C f(z) dz = -2\pi i \operatorname{Res}_{z=\infty} f(z)$$

Ok, so now we theoretically have a way to calculate the integral around a contour which encloses more than one singular point with just one residue instead of a residue for each point, question is how do we calculate the residue at infinity?

Here will be the trick, first we know that there is a Laurent series for $f(z)$ in the region outside of the circle R_1 expanded around $z = 0$. That is

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n \quad R_1 < |z - z_0| < R_2 \quad c_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{n+1}} dz \quad n = 0, \pm 1, \pm 2, \pm 3$$

Gives us:

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z)^n \quad 0 < |z| < R_1 \quad c_n = \frac{1}{2\pi i} \int_{C=-C_0} \frac{f(z)}{(z)^{n+1}} dz \quad n = 0, \pm 1, \pm 2, \pm 3$$

Ok, so what we want then is $c_{-1} = \frac{1}{2\pi i} \int_{-C_0} f(z) dz$ since then $\int_{-C_0} f(z) dz = 2\pi i c_{-1} = -2\pi i \operatorname{Res}_{z=\infty} f(z)$

What we will do then is find c_{-1} using a contour around just one point of singularity at $z = 0$ by manipulating the Laurant series for $f(z)$

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Here is the trick... bear with me

Now we will look at $f\left(\frac{1}{z}\right)$ so that $f(z) = \sum_{n=-\infty}^{\infty} c_n(z)^n$ Becomes:

So
$$f\left(\frac{1}{z}\right) = \sum_{n=-\infty}^{\infty} c_n\left(\frac{1}{z^n}\right) \qquad 0 < |z| < \frac{1}{R_1}$$

Writing out a few terms of this gives us:

$$f\left(\frac{1}{z}\right) = \cdots + c_{-n}z^n + \cdots + c_{-2}z^2 + c_{-1}z + c_0 + c_1\left(\frac{1}{z}\right) + c_2\left(\frac{1}{z^2}\right) + \cdots$$

We want to pick of c_{-1} so of I can make it the residue of something at zero that is the coefficent of $\frac{1}{z}$ then we can use the residue theorem for the one point at zero. The trick is to knock the power down by two on c_{-1} above, to do this we can just multiply by $\frac{1}{z^2}$

$$f\left(\frac{1}{z}\right) = \cdots + c_{-n}z^n + \cdots + c_{-2}z^2 + c_{-1}z + c_0 + c_1\left(\frac{1}{z}\right) + c_2\left(\frac{1}{z^2}\right) + \cdots$$

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So

$$\frac{1}{z^2} f\left(\frac{1}{z}\right) = \dots + c_{-n} z^{n-2} + \dots + c_{-2} + c_{-1} \frac{1}{z} + c_0 \frac{1}{z^2} + c_1 \frac{1}{z^3} + c_2 \frac{1}{z^4} + \dots$$

Ok so now

$$c_{-1} = \text{Res}_{z=0} \frac{1}{z^2} f\left(\frac{1}{z}\right)$$

Also we had that

$$c_{-1} = -\text{Res}_{z=\infty} f(z)$$

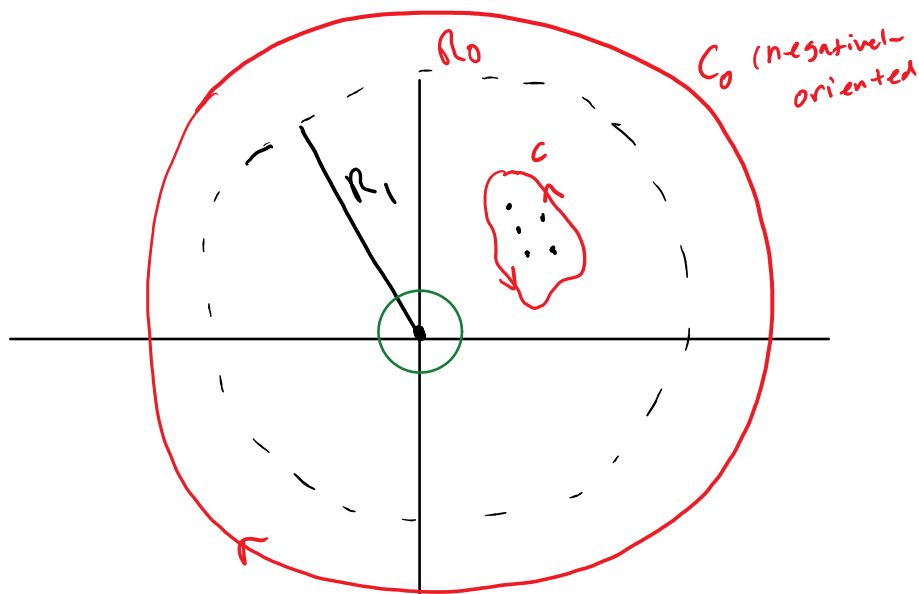
So we can say

$$\text{Res}_{z=\infty} f(z) = -\text{Res}_{z=0} \frac{1}{z^2} f\left(\frac{1}{z}\right)$$

Quite often the Laurent series for $\frac{1}{z^2} f\left(\frac{1}{z}\right)$ expanded around zero will be easier to find than the original series which will allow us to find our residue much more easily.

That is we can calculate now instead

$$\int_C f(z) dz = 2\pi i \text{Res}_{z=0} \left(\frac{1}{z^2} \right) f\left(\frac{1}{z}\right)$$



7. Find the residue at infinity of

(a) $f(z) = \frac{1}{(z+1)(z-i)}$

(b) $f(z) = \frac{z}{z^2+3z+2}$