

Chapter 6 Sec 78 , 80-83 More about Residues and Poles

The three types of isolated singular points.

Just a reminder

An isolated singular point is a point at which $f(z)$ is not analytic and there is a deleted neighborhood around that point where $f(z)$ is analytic.



The types..

1. Removable singular points: We know that around an isolated singular point we have a Laurent Series for $f(x)$ given by

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$$

$R_1 < |z-z_0| < R_2$

Where

$$a_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^{n+1}} dz \quad n = 0, 1, 2, \dots$$
$$b_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^{-n+1}} dz \quad n = 1, 2, \dots$$

If every b_n is zero so that

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n = \underline{a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \dots}$$

$\downarrow$
 $f(z_0)$

$b_n's = 0$

We call z_0 a removable singularity. What we really mean by this is that we were able to assign a value of a_0 to $f(z)$ at the point of singularity to make it analytic. Let's look at an example.

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Example of a removable singularity

$f(z) = \frac{1 - \cosh(z)}{z^2}$ → not analytic at $z=0$ $z \rightarrow 0$ ↓

To help us along recall that the Taylor series for $\cosh(z)$ is given by: $\cosh(z) = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!} = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \dots \quad |z| < \infty$

$f(z) = \frac{1}{z^2} (1 - \cosh(z))$

$f(z) = \frac{1}{z^2} \left[1 - \left(1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \frac{z^6}{6!} + \dots \right) \right]$

$= \frac{1}{z^2} \left[\cancel{1} - \cancel{1} - \frac{z^2}{2!} - \frac{z^4}{4!} - \frac{z^6}{6!} - \dots \right]$

$= \frac{1}{z^2} \left[-\frac{z^2}{2!} - \frac{z^4}{4!} - \frac{z^6}{6!} - \dots \right] = -\frac{\cancel{z^2}}{2! \cancel{z^2}} - \frac{\cancel{z^4}}{4! \cancel{z^2}} - \frac{\cancel{z^6}}{6! \cancel{z^2}} - \dots$

$= -\frac{1}{2} - \frac{z^2}{4!} - \frac{z^4}{6!} - \dots = f(z)$ ← must be Laurent series for $f(z)$

So since the b'_n s all disappear in the series we can redefine $f(z)$ as

$\rightarrow f(z) = \begin{cases} \frac{1 - \cosh(z)}{z^2} & \text{if } z \neq 0 \\ -\frac{1}{2} & \text{if } z = 0 \end{cases}$

$f(0) = -\frac{1}{2} - \frac{0}{4!} - \frac{0}{6!} - 0 - \dots = -\frac{1}{2}$

To make it analytic, thus removing the singularity!

$$\cancel{f(x)} = \frac{(x+2)}{x^2+5x+6} = \frac{\cancel{(x+2)}}{(x+3)\cancel{(x+2)}}$$

$$\frac{1}{x+3}$$

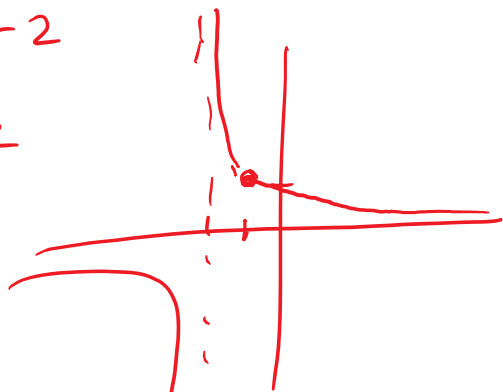
$$x = -3$$

$$\boxed{x = -2}$$

↓
remove

$$\frac{1}{-2+3} = +1$$

$$\rightarrow f(x) = \begin{cases} \frac{x+2}{x^2+5x+6} & x \neq -2 \\ +1 & x = -2 \end{cases}$$



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2. Essential Singularity: If $f(z)$ is not analytic at z_0 and we have an infinite number of NON-ZERO b_n 's in the Laurent series for $f(z)$ we call z_0 an *essential singularity*

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n} \quad R_1 < |z-z_0| < R_2$$

Where

$$a_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^{n+1}} dz \quad n = 0, 1, 2, \dots$$

$$b_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^{-n+1}} dz \quad n = 1, 2, \dots$$

So the singularity is essential if we have an infinite number of b_n 's in our Laurent series, Let's look at an example:

$f(z) = e^{\frac{1}{z}}$ $\rightarrow$ not analytic $z=0$
 $0 < |z-0| < \infty$

This has an essential singularity at $z = 0$ let's see why:

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

$$e^{1/z} = \sum_{n=0}^{\infty} \frac{(\frac{1}{z})^n}{n!} = \sum_{n=0}^{\infty} \frac{1}{z^n n!}$$

$$= \frac{1}{0!} + \frac{1}{z(1)} + \frac{1}{z^2(2)} + \frac{1}{z^3 3!} + \frac{1}{z^4 4!}$$

$$b_4 = \frac{1}{4!}$$

$b_1 = 1$

$$(z-0)^{-4}$$

$$a_0 = 1$$

$$\text{Res}_{z=0} e^{1/z} = 1$$

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3. Pole of order m: If at some number m all of the b_n's = 0 for n > m in the Laurent series for f(z)

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n} \qquad R_1 < |z-z_0| < R_2$$

Where

$$a_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^{n+1}} dz \qquad n = 0,1,2, \dots \qquad b_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^{-n+1}} dz \quad n = 1,2, \dots$$

That is b_m ≠ 0 and b_{m+1} = b_{m+2} = ... = 0 Then we say that f(z) has a pole of order m at z₀

That means out expansion might look like

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n} = \sum_{n=0}^{\infty} a_n(z-z_0)^n + \frac{b_1}{(z-z_0)} + \frac{b_2}{(z-z_0)^2} + \frac{b_3}{(z-z_0)^3} + \dots + \frac{b_m}{(z-z_0)^m} + \dots + 0$$

Example: f(z) = 1 / (z^2(1-z)) 0 < z < 1 Has a pole of order 2 at z = 0



$$\frac{1}{z^2} \left(\frac{1}{1-z} \right) = \frac{1}{z^2} \sum_{n=0}^{\infty} z^n = \sum_{n=0}^{\infty} \frac{z^n}{z^2} = \sum_{n=0}^{\infty} z^{n-2}$$

$$z^{-2} + z^{-1} + 1 + z + z^2 + \dots$$

$n=0 \quad n=1 \quad n=2 \quad n=3 \quad n=4$

$$\frac{1}{z^2} + \frac{1}{z} + 1 + z + z^2 + \dots$$

$\uparrow \qquad \uparrow$
 $b_2 \quad b_1$

$b_1 = 1 \quad b_2 = 1$
 $b_3 = 0 \quad b_4 = 0$
no $1/z^3$ no $1/z^4$

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Residues at Poles

Theorem: Let z_0 be an isolated singular point of a function $f(z)$ the following two statements are equivalent.

- 1. z_0 is a pole of order m
- 2. $f(z)$ can be written in the form

$$\frac{\phi(z)}{(z-z_0)^m} = f(z) \quad m = 1, 2, \dots$$

Where $\phi(z)$ is analytic and nonzero at z_0

More over if statements 1,2 are true

← simple pole

$$\text{Res}_{z=z_0} f(z) = \phi(z_0) \quad \text{When } m = 1$$

↓ derivative

$$\text{Res}_{z=z_0} f(z) = \frac{\phi^{(m-1)}(z_0)}{(m-1)!} \quad \text{When } m = 2, 3, \dots$$

Proof assume 1), Then we know $f(z)$ has a laurant series expansion with the b_n 's stopping at some b_m .

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \frac{b_1}{z-z_0} + \frac{b_2}{(z-z_0)^2} + \dots + \frac{b_m}{(z-z_0)^m}$$

$$\text{Choose } \phi(z) = \begin{cases} (z-z_0)^m f(z) & \text{when } z \neq z_0 \\ b_m & \text{when } z = z_0 \end{cases}$$

If we write out the series for $\phi(z)$ we get..

$$\begin{aligned} \phi(z) &= (z-z_0)^m f(z) = (z-z_0)^m \left(\sum_{n=0}^{\infty} a_n (z-z_0)^n + \frac{b_1}{(z-z_0)} + \dots + \frac{b_m}{(z-z_0)^m} \right) \\ &= \sum_{n=0}^{\infty} a_n (z-z_0)^{m+n} + \frac{b_1 (z-z_0)^{m-1}}{(z-z_0)} + \frac{b_2 (z-z_0)^{m-2}}{(z-z_0)^2} + \dots + \frac{b_m (z-z_0)^m}{(z-z_0)^m} \end{aligned}$$

$$= \sum_{n=0}^{\infty} a_n (z-z_0)^{m+n} + \frac{b_1 (z-z_0)^{m-1}}{(z-z_0)^2} + \frac{b_2 (z-z_0)^{m-2}}{(z-z_0)^2} + \dots + \frac{b_m (z-z_0)^m}{(z-z_0)^m}$$

$$\frac{b_{m-1} (z-z_0)^{m-1}}{(z-z_0)^{m-1}}$$

So

$$\phi(z) = b_m + b_{m-1} (z-z_0) + \dots + b_1 (z-z_0)^{m-1} + \sum_{n=0}^{\infty} a_n (z-z_0)^{m+n}$$



Which is indeed analytic on the entire disk $|z - z_0| < R$

If you divide this by $(z - z_0)^m$ we end up with the Laurent series for $f(z)$

$$f(z) = \frac{\phi(z)}{(z-z_0)^m} =$$

$$\frac{1}{(z-z_0)^m} \left[b_m + b_{m-1} (z-z_0) + \dots + b_1 (z-z_0)^{m-1} + \sum_{n=0}^{\infty} a_n (z-z_0)^{m+n} \right]$$

$$= \frac{b_m}{(z-z_0)^m} + \frac{b_{m-1} (z-z_0)^{m-1}}{(z-z_0)^m} + \dots + \frac{b_1 (z-z_0)^{m-1}}{(z-z_0)^m} + \sum_{n=0}^{\infty} \frac{a_n (z-z_0)^{m+n}}{(z-z_0)^m}$$

$$= \frac{b_m}{(z-z_0)^m} + \frac{b_{m-1}}{(z-z_0)^{m-1}} + \dots + \frac{b_1}{(z-z_0)} + \sum_{n=0}^{\infty} a_n (z-z_0)^n$$

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Assume 2 $f(z)$ can be written in the form $\Rightarrow f(z)$ has a pole of order m at $z=z_0$

$f(z) = \frac{\phi(z)}{(z-z_0)^m}$

Where $\phi(z)$ is analytic and nonzero at z_0

$\phi(z) = \sum_{n=0}^{\infty} \frac{\phi^{(n)}(z_0)}{n!} (z-z_0)^n$
When $|z-z_0| < R_0$

Since we know that $\phi(z)$ is analytic it has a Taylor series given by

Let's write out the first m terms of the sum and then divide by $(z-z_0)^m$ to look at the series for $f(z)$

$\phi(z_0) + \frac{\phi'(z_0)}{1!} (z-z_0)^1 + \frac{\phi''(z_0)}{2!} (z-z_0)^2 + \dots + \frac{\phi^{(m)}(z_0)}{m!} (z-z_0)^m + \sum_{n=m+1}^{\infty} \frac{\phi^{(n)}(z_0)}{n!} (z-z_0)^n$

$= \frac{\phi(z_0)}{(z-z_0)^m} + \frac{\phi'(z_0)}{(z-z_0)^{m-1}} + \frac{\phi''(z_0)}{2! (z-z_0)^{m-2}} + \dots + \frac{\phi^{(m)}(z_0)}{m!} + \sum_{n=m+1}^{\infty} \frac{\phi^{(n)}(z_0)}{n!} (z-z_0)^{n-m}$

$n > m$
 $n-m > 0$

$a_0 = \frac{\phi^{(m)}(z_0)}{m!}$ $b_m = \phi(z_0)$ $b_n = 0 \quad n > m$

So

$f(z) = \frac{\phi(z_0)}{(z-z_0)^m} + \frac{\phi'(z_0)}{1! (z-z_0)^{m-1}} + \dots + \frac{\phi^{(m-1)}(z_0)}{(m-1)! (z-z_0)} + \sum_{n=m}^{\infty} \frac{\phi^{(n)}(z_0)}{n!} (z-z_0)^{n-m}$

So we see that $f(z)$ has a pole of order m at z_0 !

$\text{Res}_{z=z_0} f(z) = \frac{\phi^{(m-1)}(z_0)}{(m-1)!}$

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Examples:

The function $f(z) = \frac{z+4}{z^2+1}$

Find $\phi(z)$

not analytic at $z=i$

$$\frac{i+4}{i^2+1} = \frac{i+4}{0}$$

$$i = z_0$$

Can be written as $f(z) = \frac{\phi(z)}{z-i}$

Where $\phi(z)$ is analytic at $z = i$

$$f(z) = \frac{z+4}{z^2+1} = \frac{z+4}{(z+i)(z-i)} = \frac{z+4}{z-i} \phi(z)$$

$$\text{Res}_{z=i} \phi(i) = \frac{i+4}{i+i} = \frac{i+4}{2i}$$

Use our theorem to find the residue of $f(z)$ below at $z = i$

The function $f(z) = \frac{z^3+2z}{(z-i)^3} = \frac{\phi(z)}{(z-i)^m}$

$$\text{Res}_{z=z_0} f(z) = \frac{\phi^{(m-1)}(z_0)}{(m-1)!}$$

$$f(z) = \frac{\phi(z)}{(z-z_0)^m}$$

$$\phi(z) = \frac{z^3+2z}{(z-i)^3} \quad m=3$$

$$f(z) = \frac{\phi(z)}{(z-i)^3}$$

$$\phi(z) = z^3+2z \quad \text{analytic at } i, \quad \phi(i) \neq 0$$

$f(z)$ has a pole of order 3 at $z=i$

$$\text{Res}_{z=i} \frac{\phi^{(m-1)}(z_0)}{(m-1)!} = \frac{\phi^{(3-1)}(z_0)}{(3-1)!} = \frac{\phi^{(2)}(z_0)}{2!} = \frac{6(i)}{2}$$

$$\phi(z) = z^3+2z$$

$$\phi'(z) = 3z^2+2$$

$$\phi''(z) = 6z$$

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Zeros of Order m

We know that if a function f is analytic at a point z_0 then all of its derivatives exist, Suppose that

$f(z_0) = f'(z_0) = f''(z_0) = \dots f^{(m-1)}(z_0) = 0$ and $f^{(m)}(z_0) \neq 0$

We then call z_0 a zero of order m

$f(z_0) = 0$ z_0 is a zero of f

Theorem: Let f be analytic at z_0 then the following statements are equivalent.

- 1. $f(z)$ has a zero of order m at z_0 .
- 2. there is a function g , which is analytic and nonzero at z_0 such that

$f(z) = (z - z_0)^m g(z)$

Proof: 1 implies 2 To show that 1 implies 2 we can just write out the Taylor series for f and realize that all terms a_n for $n < m$ are zero by the assumption of 1 and then just factor out $(z - z_0)^m$

$f(z) = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \dots + a_{m-1}(z-z_0)^{m-1} + \sum_{n=m}^{\infty} a_n(z-z_0)^n$

Annotations: $a_0 = f(z_0)$, $a_1 = f'(z_0)/1!$, $a_2 = f''(z_0)/2!$, $a_{m-1} = f^{(m-1)}(z_0)/(m-1)!$

$f(z) = \sum_{n=m}^{\infty} a_n(z-z_0)^n = (z-z_0)^m \sum_{n=m}^{\infty} a_n(z-z_0)^{n-m}$

The sum is labeled $g(z)$.

$g(z) = a_m + a_{m+1}(z-z_0) + \dots$

Annotations: $n \geq m$ analytic, $g(z_0) \neq 0$ (labeled "pwr in z_0 ").

$a_m = \frac{f^{(m)}(z_0)}{m!} \leftarrow f^{(m)}(z) \neq 0$

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Theorem: Let f be analytic at z_0 then the following statements are equivalent.

- 1. $f(z)$ has a zero of order m at z_0 .
- 2. there is a function g , which is analytic and nonzero at z_0 such that $f(z) = (z - z_0)^m g(z)$

2 implies 1

To prove 2 implies 1 we just need to write out the taylor series for $g(z)$ multiply by $(z - z_0)^m$ and use the fact that taylor series are unique.

$$a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \dots$$

$$g(z) = g(z_0) + \frac{g'(z_0)}{1!}(z-z_0) + \frac{g''(z_0)}{2!}(z-z_0)^2 + \dots + \frac{g^{(m-1)}(z_0)}{(m-1)!}(z-z_0)^{m-1} + \sum_{n=m}^{\infty} \frac{g^{(n)}(z_0)}{n!}(z-z_0)^n$$

$$f(z) = (z-z_0)^m g(z)$$

$$= (z-z_0)^m g(z_0) + g'(z_0)(z-z_0)^{m+1} + \frac{g''(z_0)}{2}(z-z_0)^{m+2} + \dots + \frac{g^{(m-1)}(z_0)}{(m-1)!}(z-z_0)^{2m-1}$$

$$+ \sum_{n=m}^{\infty} \frac{g^{(n)}(z_0)}{n!}(z-z_0)^{n+m}$$

$a_0 = 0$ $a_1 = 0$ why? + here is no $(z-z_0)^1$

$$a_2 = 0 \quad a_3 = 0 \quad \dots \quad a_{m-1} = 0 \quad a_m \neq 0$$

$$a_0 = f(z_0) \quad a_1 = \frac{f'(z_0)}{1} = 0 \quad a_2 = \frac{f''(z_0)}{2!} = 0 \quad a_n = \frac{f^{(n)}(z_0)}{n!}$$

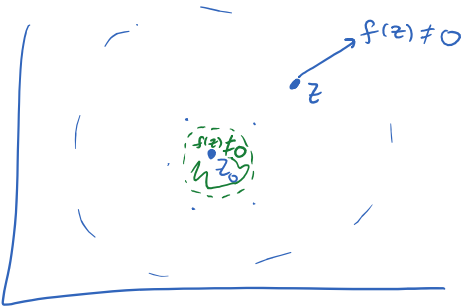
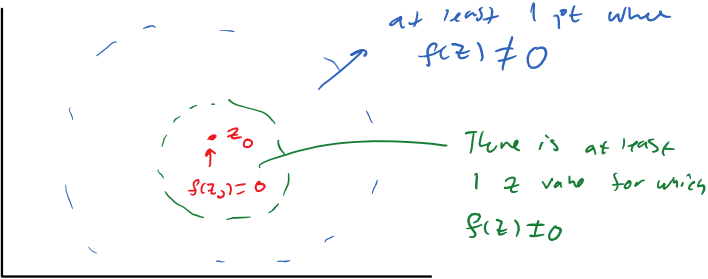
$$a_{m-1} = \frac{f^{(m-1)}(z_0)}{(m-1)!} = 0 \Rightarrow \text{zero of order } m$$

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Theorem: Given a function f and a point z_0 , suppose that

- a) f is analytic at z_0
- b) $f(z_0) = 0$ but $f(z)$ is not identically equal to zero in any neighborhood of z_0

Then $f(z) \neq 0$ throughout some deleted neighborhood $0 < |z - z_0| < \epsilon$ of z_0



Proof: $f(z_0) = 0$ but the fact that $f(z)$ is not identically zero means that not all of it's derivatives can be zero in some neighborhood of z_0 if so then the taylor series which is equal to f would be identically zero which would mean f was

$f(z) = f(z_0) + f'(z_0)(z - z_0) + \frac{f''(z_0)}{2!}(z - z_0)^2 + \frac{f^{(m)}(z_0)}{m!}(z - z_0)^m + \dots$

at least 1 derivative is not zero at $z = z_0$

So let m be the first non zero derivative then z_0 is a zero of order m this means that according to the theorem we just proved

$f(z) = (z - z_0)^m g(z)$

$f(z_0) \neq 0$ and analytic at z_0 then there is some $\epsilon > 0$ where $f(z) \neq 0$

For some $g(z)$ where $g(z) \neq 0$ is some ϵ neighborhood of z_0 , a theorem we had from long ago Consequently this means that if we are not at z_0 then $f(z) \neq 0$ in the neighborhood of $0 < |z - z_0| < \epsilon$

$f(z) = (z - z_0)^m g(z)$

$f(z) \neq 0$



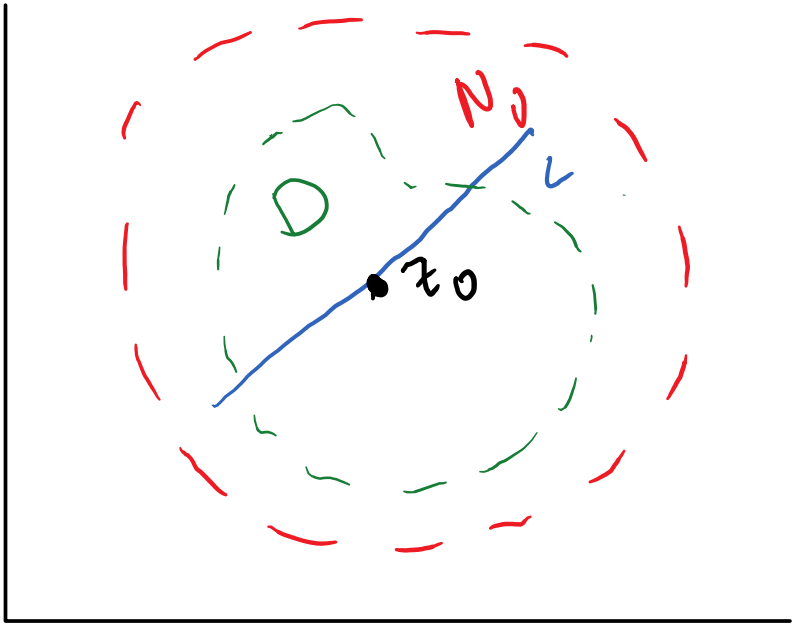
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Another Theorem :)

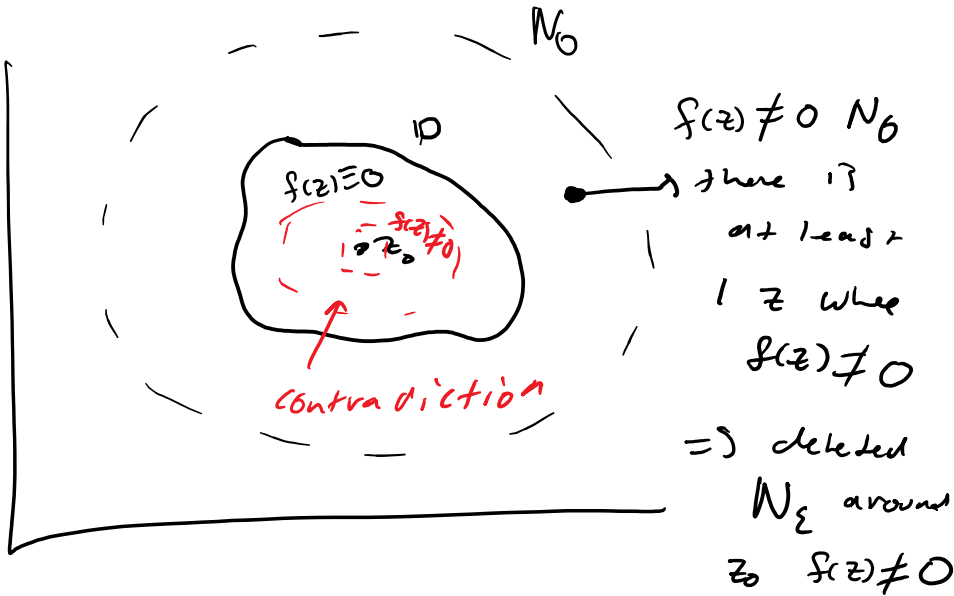
Theorem: Given a function f and a point z_0 suppose that

- 1. f is analytic throughout a neighborhood N_0 of z_0 *containing*
- 2. $f(z) = 0$ at each point z of a domain D or line segment L containing z_0

Then $f(z) = 0$ in N_0 that is, $f(z)$ is identically equal to zero throughout N_0



Proof: Assume not, Then according to the previous theorem there is a deleted neighborhood around z_0 where $f(z) \neq 0$. Of course this contradicts the the assumption that $f(z)$ is zero throughout D This means it indeed need to be identically zero in N_0



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Zeros and Poles

There are some connections between zeros and poles of functions which we will explore below.

Theorem: Suppose that

- a) two function p and q are analytic at a point z_0
- b) $p(z_0) \neq 0$ and q has a zero of order m at z_0

then the quotient $\frac{p(z)}{q(z)}$ has a pole of order m at z_0

$$\frac{z}{z-1} + \frac{1}{z-1}$$

Theorem: Let two function p and q be analytic at a point z_0 . If

$$p(z_0) \neq 0, q(z_0) = 0 \text{ and } q'(z_0) \neq 0$$

Then z_0 is a simple pole of $\frac{p(z)}{q(z)}$ and

$$\operatorname{Res}_{z=z_0} \frac{p(z)}{q(z)} = \frac{p(z_0)}{q'(z_0)}$$

Homework Example : Show that $z = 0$ is a simple pole of $f(z) = \csc(z) = \frac{1}{\sin(z)}$ and that the residue at $z_0 = 0$ is 1.

$$f(z) = \frac{p(z)}{q(z)} = \frac{p(z)=1}{q(z)=\sin(z)}$$

$$p(0) \neq 0 \checkmark$$

$$q(0) = 0$$

$$\sin(0) = 0 \checkmark$$

$$q'(z) = \cos(z)$$

$$q'(0) = \cos(0) = 1 \neq 0 \checkmark$$

$\Rightarrow z=0$ is a simple pole

$$\operatorname{Res}_{z=0} \frac{p(0)}{q'(0)} = \frac{1}{\cos(0)} = \frac{1}{1} = 1$$