

Chapter 7 Sec 85-87

Applications of Residues

One of the best uses of Residues is to calculate contour integrals and one of the best uses of contour integrals is to calculate improper integrals in the real number system.

Sometimes we have to deal with integrals with infinite bounds something like:

$$\int_0^{\infty} f(x) dx = \lim_{R \rightarrow \infty} \int_0^R f(x) dx = \lim_{R \rightarrow \infty} F(R) - F(0)$$

Or perhaps both bounds are infinite in which case

$$\int_{-\infty}^{\infty} f(x) dx = \lim_{R \rightarrow \infty} \int_{-R}^0 f(x) dx + \lim_{R \rightarrow \infty} \int_0^R f(x) dx$$

If the limits above converge we say that the integrals do as well.

Another definition for improper integrals is what we call the Cauchy Principle Value of an integral.

$$PV \int_{-\infty}^{\infty} f(x) dx = \lim_{R \rightarrow \infty} \int_{-R}^R f(x) dx$$

$$\rightarrow \int_{-R}^0 + \int_0^R$$

At first glance these may seem like the same thing but there can be some subtleties.

Let's let $f(x) = x$ and evaluate $\int_{-\infty}^{\infty} x dx$ with both definitions to see what can happen.

Regular Calculus:

$$\int_{-\infty}^{\infty} x dx = \lim_{R \rightarrow \infty} \int_{-R}^0 x dx + \lim_{R \rightarrow \infty} \int_0^R x dx$$

$$\lim_{R \rightarrow \infty} \left. \frac{x^2}{2} \right|_{-R}^0 + \lim_{R \rightarrow \infty} \left. \frac{x^2}{2} \right|_0^R$$

$$\lim_{R \rightarrow \infty} \left(0 - \frac{(-R)^2}{2} \right) + \lim_{R \rightarrow \infty} \left(\frac{R^2}{2} - 0 \right)$$

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Principle Value:

$$PV \int_{-\infty}^{\infty} x dx = \lim_{R \rightarrow \infty} \int_{-R}^R x dx$$

$$\lim_{R \rightarrow \infty} \left. \frac{x^2}{2} \right|_{-R}^R = \lim_{R \rightarrow \infty} \left(\frac{R^2}{2} - \frac{(-R)^2}{2} \right)$$

$$= \lim_{R \rightarrow \infty} \left(\frac{R^2}{2} - \frac{R^2}{2} \right) = \lim_{R \rightarrow \infty} (0) = 0$$

So they can indeed be different However, if $\int_{-\infty}^{\infty} f(x) dx = \lim_{R \rightarrow \infty} \int_{-R}^0 f(x) dx + \lim_{R \rightarrow \infty} \int_0^R f(x) dx$ converges the so does $PV \int_{-\infty}^{\infty} f(x) dx = \lim_{R \rightarrow \infty} \int_{-R}^R f(x) dx$ and they will have the same value. This comes down to when we can actually say That the limit of a sum is the sum of the limits, that is true when both limits converge.

$$\lim_{x \rightarrow \infty} f(x) + g(x) = \lim_{x \rightarrow \infty} f(x) + \lim_{x \rightarrow \infty} g(x)$$

only if both limits exist

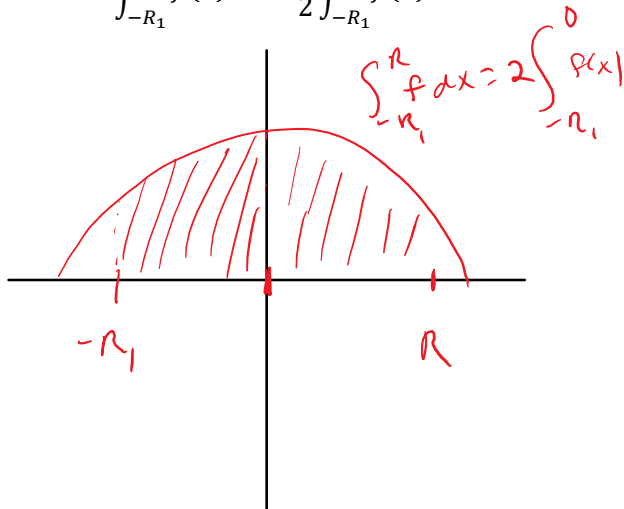
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If $f(x)$ happens to be an even function unlike our $f(x) = x$ which was odd we get a few identities which can be helpful.

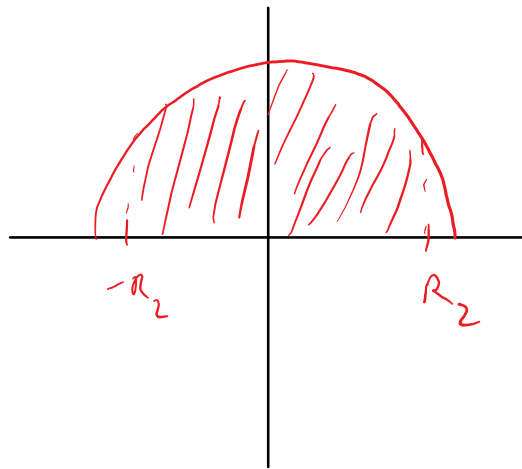
If $f(x) = f(-x)$ on $(-\infty < x < \infty)$ then

even

$$\int_{-R_1}^0 f(x) dx = \frac{1}{2} \int_{-R_1}^{R_1} f(x) dx$$



$$\int_0^{R_2} f(x) dx = \frac{1}{2} \int_{-R_2}^{R_2} f(x) dx$$



Thus we can say:

$$\int_{-R_1}^0 f(x) dx + \int_0^{R_2} f(x) dx = \frac{1}{2} \int_{-R_1}^{R_1} f(x) dx + \frac{1}{2} \int_{-R_2}^{R_2} f(x) dx$$

Since they are equal if the limit of one of the sides exists both do, the ones on the left are principle values when we apply limits

$$\begin{aligned} \int_{-\infty}^{\infty} f(x) dx &= \lim_{R \rightarrow \infty} \int_{-R}^0 f(x) dx + \lim_{R \rightarrow \infty} \int_0^R f(x) dx = \lim_{R \rightarrow \infty} \left(\frac{1}{2} \int_{-R}^R f(x) dx + \frac{1}{2} \int_{-R}^R f(x) dx \right) \\ &= \lim_{R \rightarrow \infty} \int_{-R}^R f(x) dx \\ &= PV: \int_{-\infty}^{\infty} f(x) dx \end{aligned}$$

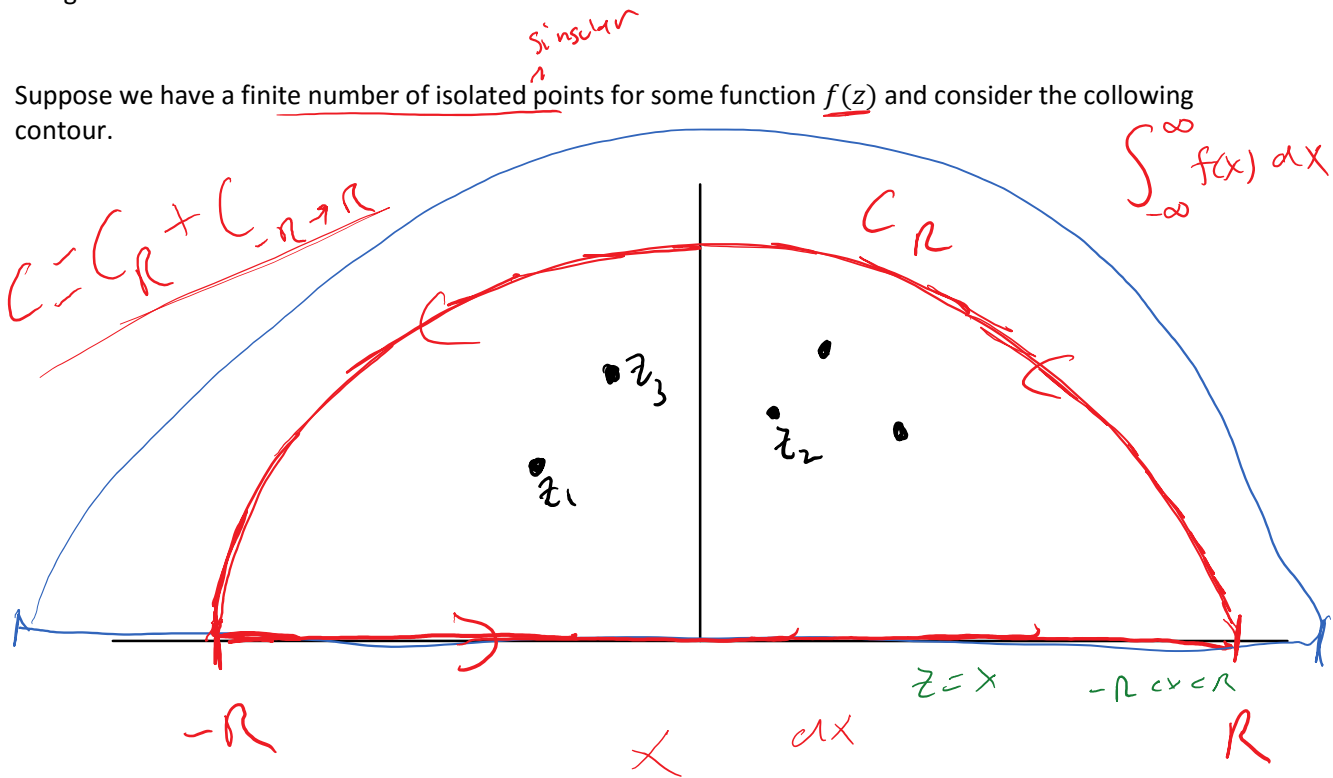
So if $f(x)$ is even then we will always have the equality

$$\int_{-\infty}^{\infty} f(x) dx = PV \int_{-\infty}^{\infty} f(x) dx$$

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With those preliminaries out of the way, let's see how we can apply some of our complex analysis to a real valued infinite integral.

Suppose we have a finite number of isolated points for some function $f(z)$ and consider the following contour.



From what we know about contour integrals we can say:

$$\int_C f(z) dz = 2\pi i \sum_{k=1}^n \text{Res}_{z=z_k} f(z)$$

$z = x + iy$
 $\nearrow z = x$

Further we can split the integral into two along the top and real axis

$$\int_C f(z) dz = \int_{C_R} f(z) dz + \int_{-R}^R f(z) dz$$

Note though that from $-R$ to R $f(z) = f(x)$ since there is no imaginary part there for $z = x + iy$ So we can then say...

$y=0$
 $f(x+iy) = f(x)$

$$2\pi i \sum_{k=1}^n \text{Res}_{z=z_k} f(z) = \int_C f(z) dz = \int_{C_R} f(z) dz + \int_{-R}^R f(x) dx$$

$$\int_{-R}^R f(x) dx = 2\pi i \sum_{k=1}^n \text{Res}_{z=z_k} f(z) - \int_{C_R} f(z) dz$$

$$\int_{\text{re}} f(z) dz = \int_{-R}^R f(z(x)) z'(x) dx = \int_{-R}^R f(x) dx$$

$z=x \quad z'=1 \quad z=x$

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So for a contour like this one we have that

$$\int_{-R}^R f(x) dx = 2\pi i \sum_{k=1}^n \text{Res}_{z=z_k} f(z) - \int_{C_R} f(z) dz$$

Taking our infinite limits we can see

$$\lim_{R \rightarrow \infty} \int_{-R}^R f(x) dx = \lim_{R \rightarrow \infty} 2\pi i \sum_{k=1}^n \text{Res}_{z=z_k} f(z) - \lim_{R \rightarrow \infty} \int_{C_R} f(z) dz$$

↓

$$\text{P.V.} \int_{-\infty}^{\infty} f(x) dx = 2\pi i \sum_{k=1}^h \text{Res}_{z=z_k} f(z) - \lim_{R \rightarrow \infty} \int_{C_R} f(z) dz$$

So if $\lim_{R \rightarrow \infty} \int_{C_R} f(z) dz = 0$ well then

$$\lim_{R \rightarrow \infty} \int_{-R}^R f(x) dx = \text{PV} \int_{-\infty}^{\infty} f(x) dx = 2\pi i \sum_{k=1}^n \text{Res}_{z=z_k} f(z)$$

Incidentally if **f is even** then the principal value is the same as that of the improper integral and we can just say that

even f
↓

$$\int_{-\infty}^{\infty} f(x) dx = 2\pi i \sum_{k=1}^n \text{Res}_{z=z_k} f(z)$$

And

$$\int_0^{\infty} f(x) dx = \frac{1}{2} \int_{-\infty}^{\infty} f(x) dx = \frac{1}{2} 2\pi i \sum \text{Res}$$

$$\int_0^{\infty} f(x) dx = \pi i \sum_{k=1}^n \text{Res}_{z=z_k} f(z)$$

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Example:

$$\int_0^{\infty} \frac{dx}{x^6+1}$$

$$f(x) = \frac{1}{x^6+1}$$

$$\int_0^{\infty} \frac{dx}{x^6+1} = \frac{1}{2} \int_{-\infty}^{\infty} \frac{dx}{x^6+1}$$

$$f(z) = \frac{1}{z^6+1}$$

$$p=1 \quad q=z^6+1$$

Theorem: Let two function p and q be analytic at a point z_0 . If

Then z_0 is a simple pole of $\frac{p(z)}{q(z)}$ and $p(z_0) \neq 0, q(z_0) = 0$ and $q'(z_0) \neq 0$

$$\text{Res}_{z=z_0} \frac{p(z)}{q(z)} = \frac{p(z_0)}{q'(z_0)}$$

$$2\pi i R$$

$$f(-x) = f(x)$$

$$\frac{1}{(-x)^6+1} = \frac{1}{x^6+1}$$

$$f(z) = \frac{1}{z^6+1}$$

find out where $f(z)$ is not analytic

$$z^6+1=0 \quad z^6=-1 \quad z=(-1)^{1/6}$$

$$(-1) = e^{i\pi} \quad (-1)^{1/6} = e^{i\frac{\pi+2k\pi}{6}} \quad k=0, \dots, 5$$

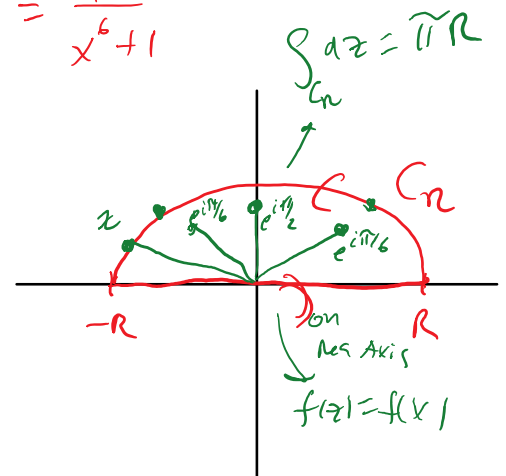
$k=0$	$k=1$	$k=2$
$e^{i\pi/6}$	$e^{i\pi/2}$	$e^{i5\pi/6}$

$$k=3 \quad e^{i7\pi/6}$$

$$k=4 \quad e^{i9\pi/6}$$

$$k=5 \quad e^{i11\pi/6}$$

$$\frac{\pi+2(3)\pi}{6} = \frac{\pi+6\pi}{6} = \frac{7\pi}{6}$$



$$\int_{-\infty}^{\infty} \frac{1}{x^6+1} dx = 2\pi i \left(\text{Res}_{z=e^{i\pi/6}} + \text{Res}_{z=e^{i\pi/2}} + \text{Res}_{z=e^{i5\pi/6}} \right) - \lim_{R \rightarrow \infty} \int_{C_n} \frac{1}{z^6+1} dz$$

$$\lim_{R \rightarrow \infty} \left| \int_{C_R} \frac{1}{z^6+1} dz \right| \leq \lim_{R \rightarrow \infty} \int_{C_R} \left| \frac{1}{z^6+1} \right| dz \leq \lim_{R \rightarrow \infty} \int_{C_R} \frac{1}{|z^6+1|} dz$$

Reverse triangle inequality $|x+y| \geq ||x|-|y||$

$$\leq \lim_{R \rightarrow \infty} \int_{C_R} \frac{1}{||z^6|-1|} dz = \lim_{R \rightarrow \infty} \int_{C_R} \frac{1}{|R^6-1|} dz = \lim_{R \rightarrow \infty} \int_{C_R} \frac{1}{R^6-1} dz$$

$$= \lim_{R \rightarrow \infty} \frac{1}{R^6-1} \int_{C_R} dz = \lim_{R \rightarrow \infty} \frac{1}{R^6-1} \pi R = \lim_{R \rightarrow \infty} \frac{\pi R}{R^6-1} \left(\frac{1}{R^6} \right)$$

$$= \lim_{R \rightarrow \infty} \frac{\pi/R^5}{1 - 1/R^6} = \frac{0}{1-0} = 0$$

Ex 1 cont

$$p(z) = 1 \quad q(z) = z^6 + 1 \quad q'(z) = 6z^5$$

$$\operatorname{Res}_{z=z_0} \frac{p(z_0)}{q'(z_0) q(z_0)} = \frac{1}{\left. \operatorname{Res}_{e^{i\pi/6}} \frac{1}{6(e^{i\pi/6})^5} \right| \left. \operatorname{Res}_i \frac{1}{6(i)^5} \right| \left. \operatorname{Res}_{e^{i5\pi/6}} \frac{1}{6(e^{i5\pi/6})^5} \right|}$$

$$\frac{1}{2} \int_{-\infty}^{\infty} \frac{1}{x^6 + 1} dx = \pi i \left(\frac{1}{6(e^{i\pi/6})^5} + \frac{1}{6(i)^5} + \frac{1}{6(e^{i5\pi/6})^5} \right)$$

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Theorem: Let two function p and q be analytic at a point z_0 . If

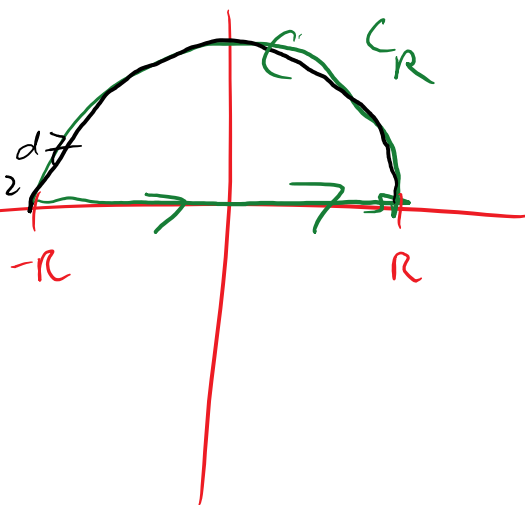
Then z_0 is a simple pole of $\frac{p(z)}{q(z)}$ and $\underline{p(z_0) \neq 0}$, $\underline{q(z_0) = 0}$ and $\underline{q'(z_0) \neq 0}$

$$\text{Res}_{z=z_0} \frac{p(z)}{q(z)} = \frac{p(z_0)}{q'(z_0)}$$

4. Use the residue theorem to evaluate the improper integral

$$\int_{-\infty}^{\infty} \frac{1}{x^2 + 2x + 2} dx$$

$$\int_{-\infty}^{\infty} \frac{1}{x^2 + 2x + 2} = 2\pi i \left\{ \text{Res} \frac{1}{z^2 + 2z + 2} \right\} - \lim_{R \rightarrow \infty} \int_{C_R} \frac{1}{z^2 + 2z + 2} dz$$



Show $\lim_{R \rightarrow \infty} \int_{C_R} \frac{1}{z^2 + 2z + 2} dz = 0$

$$\lim_{R \rightarrow \infty} \int_{C_R} \frac{1}{z^2 + 2z + 2} dz \leq \lim_{R \rightarrow \infty} \int_{C_R} \left| \frac{1}{z^2 + 2z + 2} \right| dz \quad \left(\frac{2}{2} = 1 \right)$$

$|z|^2 = 1$

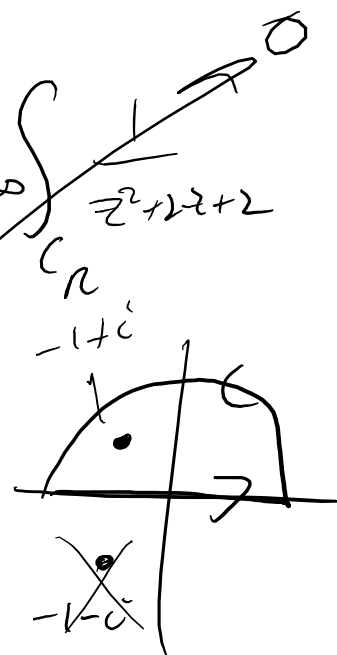
$$= \lim_{R \rightarrow \infty} \int_{C_R} \frac{1}{\underbrace{(z^2 + 2z + 1)}_{(z+1)^2} - 1 + 2} dz = \lim_{R \rightarrow \infty} \int_{C_R} \frac{1}{|(z+1)^2 + 1|} dz$$

$$\leq \lim_{R \rightarrow \infty} \int_{C_R} \frac{1}{||z+1|^2 - 1|} dz \leq \lim_{R \rightarrow \infty} \int_{C_R} \frac{1}{|z^2| - 1} dz$$

$$= \lim_{R \rightarrow \infty} \int_{C_R} \frac{1}{R^2 - 1} dz = \lim_{R \rightarrow \infty} \frac{1}{R^2 - 1} \int_{C_R} dz$$

$$= \lim_{R \rightarrow \infty} \frac{1}{R^2 - 1} 2\pi R \sim \lim_{R \rightarrow \infty} \frac{1}{R} = 0$$

$$\int_{-\infty}^{\infty} \frac{1}{x^2+2x+2} dx = 2\pi i \sum_{\text{Res}} \frac{1}{z^2+2z+2} - \lim_{R \rightarrow \infty} \int_{C_R} \frac{1}{z^2+2z+2}$$



$$x^2+2x+2=0 \quad \rightarrow \quad (x+1)^2 = -1$$

$$(x+1)^2 + 1 = 0$$

$$x+1 = \pm i$$

$$x = -1 \pm i$$

$$x = -1+i \quad -1-i$$

$$p = 1 \quad p(-1+i) \neq 0$$

$$q = x^2+2x+2 \quad q' = 2x+2$$

$$q(-1+i) = 0$$

$$2(-1+i) + 2 = 2i \neq 0$$

$$\text{Res}_{z=-1+i} \frac{1}{z^2+2z+2} = \frac{p(-1+i)}{q'(-1+i)} = \frac{1}{2i}$$

$$\int_{-\infty}^{\infty} \frac{1}{x^2+2x+2} dx = 2\pi i \frac{1}{2i} = \boxed{\pi}$$

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Application to Fourier Analysis

Often when analyzing a signal or solving a PDE Fourier transforms or integrals of the type

$$\int_{-\infty}^{\infty} f(x) \sin(ax) dx$$

$$\int_{-\infty}^{\infty} f(x) \cos(ax) dx$$

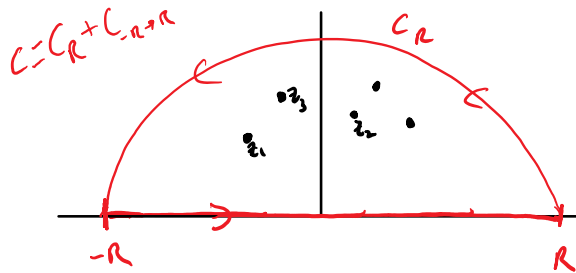
Tend to pop up all over the place. To this end we can use complex analysis to help us evaluate some of these integrals.

We have to be a bit clever though if you recall from long ago

$$|\sin(z)|^2 = \sin^2(x) + \sinh^2(y)$$

$$|\cos(z)|^2 = \cos^2(x) + \sinh^2(y)$$

So for our contour

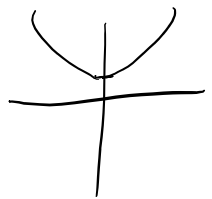


$$\text{Where } \lim_{R \rightarrow \infty} \int_{-R}^R f(x) \sin(ax) dx = \lim_{R \rightarrow \infty} 2\pi i \sum_{k=1}^n \text{Res}_{z=z_k} f(z) \sin(ax) - \lim_{R \rightarrow \infty} \int_{C_R} f(z) \sin(ax) dz$$

We will likely be unable to show that $\lim_{R \rightarrow \infty} \int_{C_R} f(z) \sin(ax) dz = 0$

$$\lim_{R \rightarrow \infty} \int_{C_R} |f(z) \sin(az)| dz = \lim_{R \rightarrow \infty} \int_{C_R} |f(z)| |\sin(az)| dz$$

$$= \lim_{R \rightarrow \infty} \int_{C_R} |f(z)| \sqrt{\sin^2(x) + \sinh^2(y)} dz$$



$$\sinh(y) = \frac{e^y - e^{-y}}{2}$$

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The trick that we will use will be to instead integrate

$$\int f(x) e^{iax} = \int f(x) (\cos(ax) + i \sin(ax)) dx$$

$$\int_{-\infty}^{\infty} f(x) e^{iax} = \int_{-\infty}^{\infty} f(x) \cos(ax) + i \int_{-\infty}^{\infty} f(x) \sin(ax)$$

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And then just take the real or imaginary part depending on whether we are after

$$\int_{-\infty}^{\infty} f(x) \cos(ax) dx$$

or

$$\int_{-\infty}^{\infty} f(x) \sin(ax) dx$$

This will work nicely since $e^{iaz} = e^{ia(x+iy)} = e^{iax} e^{-ay}$

$$iax + i\pi iy$$

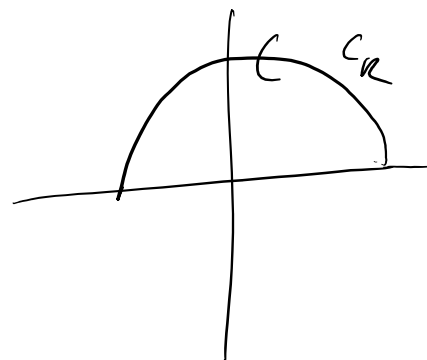
So when we consider $\lim_{R \rightarrow \infty} \int_{C_R} f(z) e^{iaz} dz$

$$\leq \lim_{R \rightarrow \infty} \int |f(z) e^{iax} e^{-ay}| =$$

$$\leq \lim_{R \rightarrow \infty} \int_{C_R} |f(z)| |e^{iax}| |e^{-ay}| dz$$

$$\downarrow$$

$$e^{-ay} \rightarrow 0$$



Theorem: Let z_0 be an isolated singular point of a function $f(z)$ the following two statements are equivalent.

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1. z_0 is a pole of order m
2. $f(z)$ can be written in the form

$$f(z) = \frac{\phi(z)}{(z-z_0)^m} \quad m = 1, 2, \dots$$

Where $\phi(z)$ is analytic and nonzero at z_0

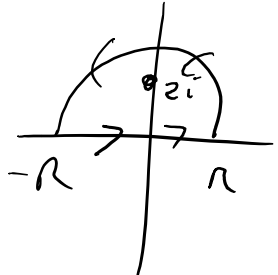
Example: Let $f(x) = \frac{1}{(x^2+4)^2}$ calculate $\int_{-\infty}^{\infty} f(x) \cos(2x) dx = \int_{-\infty}^{\infty} \frac{\cos(2x)}{(x^2+4)^2} dx$

More over if statements 1,2 are true

$$\text{Res}_{z=z_0} f(z) = \phi(z_0) \quad \text{When } m = 1$$

$$\text{Res}_{z=z_0} f(z) = \frac{\phi^{(m-1)}(z_0)}{(m-1)!} \quad \text{When } m = 2, 3, \dots$$

$$\text{Re} \left[\int_{-\infty}^{\infty} \frac{e^{2ix}}{(x^2+4)^2} dx \right] = \int_{-\infty}^{\infty} \frac{\cos(2x)}{(x^2+4)^2} dx$$

$$\int_{-\infty}^{\infty} \frac{e^{2ix}}{(x^2+4)^2} dx = 2\pi i \left\{ \text{Res}_{z=2i} \frac{e^{2iz}}{(z^2+4)^2} \right\} \quad \lim_{R \rightarrow \infty} \int_{C_R} \frac{e^{2iz}}{(z^2+4)^2} dz$$


find residues $\frac{e^{2iz}}{(z^2+4)^2}$ $z^2+4=0$ $z^2=-4$ $z = \pm 2i$

$$\text{Res}_{z=2i} \frac{e^{2iz}}{(z^2+4)^2} = f(z) = \frac{e^{2iz}}{((z+2i)(z-2i))^2} = \frac{e^{2iz}}{(z+2i)^2(z-2i)^2}$$

$m =$

$$\text{Res}_{z=2i} \frac{\phi^{(m-1)}(z_0)}{(m-1)!}$$

$$= \frac{e^{2iz}}{(z+2i)^2} \left\{ \phi(z) \right\} \frac{1}{(z-2i)^2} \leftarrow m=2$$

$\underbrace{\hspace{2cm}}_{z-z_0}$

$$= \frac{\phi'(z_0)}{1} \quad \frac{d}{dz} \frac{e^{2iz}}{(z+2i)^2} = \frac{2ie^{2iz}(z+2i)^2 - 2(z+2i)e^{2iz}}{(z+2i)^4}$$

$$\phi'(2i) = \frac{2ie^{2i(2i)}(2i+2i)^2 - 2(2i+2i)e^{2i(2i)}}{(2i+2i)^4}$$

$$2ie^{-4}(-16) - 2(4i)e^{-4} = -40ie^{-4}$$

$$\frac{2ie^{-4}(-16) - 2(4i)e^{-4}}{256} = \frac{-40ie^{-4}}{256}$$

$$\begin{aligned} \operatorname{Re} \left(\int_{-\infty}^{\infty} \frac{e^{i2x}}{(x^2+4)^2} dx \right) &= 2\pi i \left(-\frac{40e^{-4}}{256} i \right) \\ &= \boxed{\frac{40\pi e^{-4}}{128}} + 0i \end{aligned}$$

$$\int_{-\infty}^{\infty} f(x) \sin(2x) dx = 0$$

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5. Use the residue theorem to evaluate the Fourier transform

$$\mathcal{F}\left(\frac{x}{x^4+1}\right) = \int_{-\infty}^{\infty} \frac{x}{x^4+1} e^{i\omega x} dx$$