

Chapter 7 Sec 88

Jordan's Lemma

Jordan's Lemma serves as an aid in evaluating integrals of the type:

$$\int_{-\infty}^{\infty} f(x)e^{iax} = \int_{-\infty}^{\infty} f(x) \cos(ax) + i \int_{-\infty}^{\infty} f(x) \sin(ax)$$

Jordan's Lemma

Suppose that

$$z = x + iy$$

- a) A function $f(z)$ is analytic at all points in the upper half plane $y \geq 0$ that are exterior to the a circle $|z| = R_0$ where R_0 can take on the value of zero.
- b) C_R denotes a semicircle $z = Re^{i\theta}$ ($0 \leq \theta \leq \pi$) where $R > R_0$
- c) For all points z on C_R , there is a positive constant M_R such that:

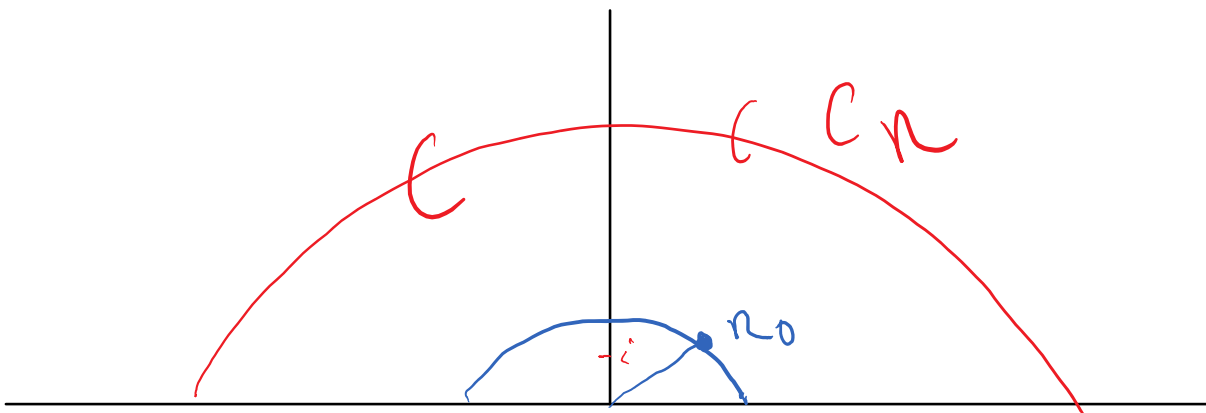
$$|f(z)| \leq M_R \quad \text{and} \quad \lim_{R \rightarrow \infty} M_R = 0$$

$$\frac{z}{z^2 + 1}$$

$$\frac{z}{z^2 + 1}$$

Then for every positive constant a ,

$$\lim_{R \rightarrow \infty} \int_{C_R} f(z)e^{iaz} dz = 0$$



What this theorem is giving us is a criteria on $f(z)$, $|f(z)| \leq M_R$ that tells us when the integral $\int_{C_R} f(z)e^{iaz} dz$ is zero rather than having to estimate the integral it's self. This means that for our formula:

$$\int_{-\infty}^{\infty} f(x)e^{iax} dx = 2\pi i \sum_{k=1}^n \text{Res}_{z=z_k} f(z)e^{iaz} - \lim_{R \rightarrow \infty} \int_{C_R} f(z)e^{iaz} dz$$

$$|f(z)| \leq M_R$$

$$\lim_{R \rightarrow \infty} M_R = 0$$

We just need to bound $f(z)$ instead of the the whole rightmost integral to show that that it is zero.

Chapter 7 Sec 88

$$\frac{z}{z^2+1}$$

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- C_R denotes a semicircle $z = Re^{i\theta} (0 \leq \theta \leq \pi)$ where $R > R_0$
- For all points z on C_R , there is a positive constant M_R such that:

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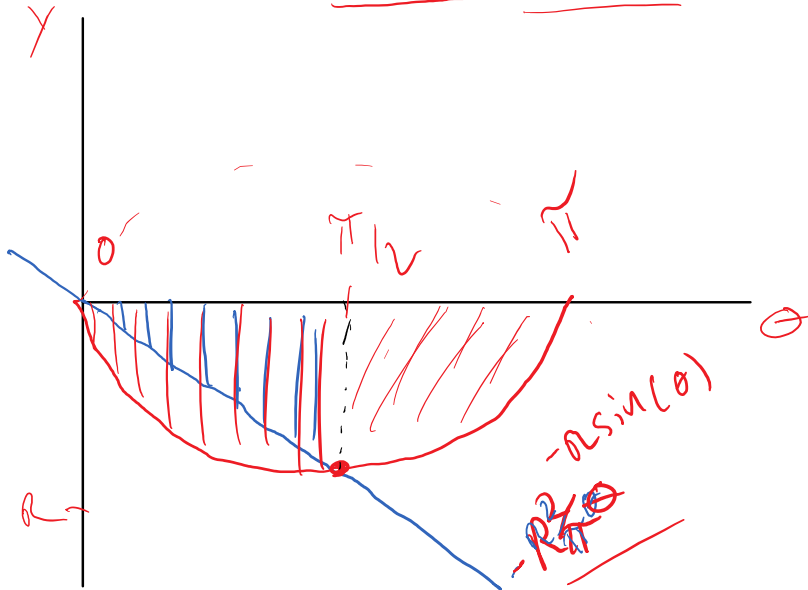
$$\lim_{R \rightarrow \infty} \int_{C_R} f(z) e^{iaz} dz = 0$$

Proof:

To prove this theorem we will first need to show Jordan's inequality

$$\int_0^\pi e^{-R \sin(\theta)} d\theta < \frac{\pi}{R}$$

Consider the graphs of $y = -R \sin(\theta)$, $y = -R \frac{2}{\pi} \theta$



Clearly from this we can say

$$\int_0^{\pi/2} e^{-R \sin(\theta)} d\theta \leq \int_0^{\pi/2} e^{-\frac{2R\theta}{\pi}} d\theta \leq \frac{\pi}{2R}$$

Now we just need to integrate and

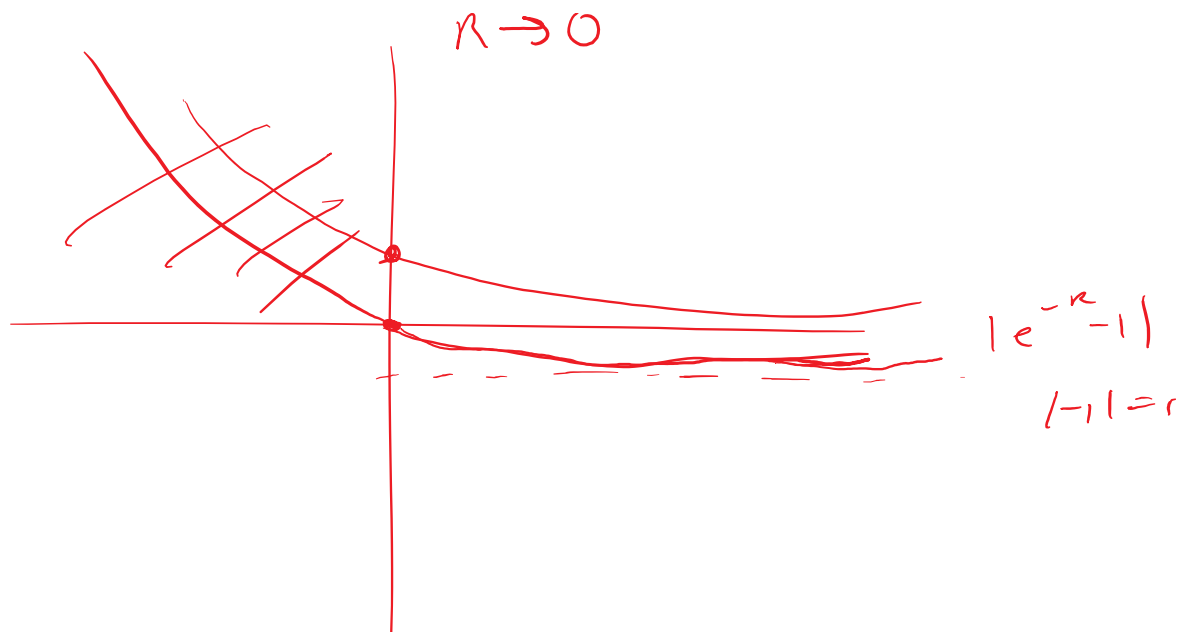
estimate $\int_0^{\pi/2} e^{-\frac{2R\theta}{\pi}} d\theta$ and use symmetry properties of sin around $x = \frac{\pi}{2}$

$$\int_0^{\pi/2} e^{-\frac{2R\theta}{\pi}} d\theta = \left. -\frac{\pi}{2R} e^{-\frac{2R\theta}{\pi}} \right|_0^{\pi/2} = -\frac{\pi}{2R} (e^{-\frac{2R \cdot \pi/2}{\pi}} - e^0)$$

$$= -\frac{\pi}{2R} (e^{-R} - 1) \leq \left| -\frac{\pi}{2R} (e^{-R} - 1) \right| \leq \frac{\pi}{2R} |e^{-R} - 1|$$

$$\lim_{R \rightarrow \infty} \left(\frac{\pi}{2R} \right) = 0$$

$$\int_0^{\pi/2} e^{-\frac{2R\theta}{\pi}} d\theta \leq \frac{\pi}{2R}$$



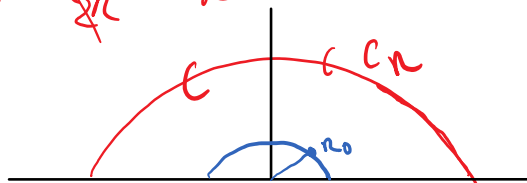
Chapter 7 Sec 88

So we now have that

$$\int_0^\pi e^{-R \sin(\theta)} d\theta < \frac{\pi}{R}$$

$$2 \int_0^{\pi/2} e^{-R \sin \theta} d\theta \leftarrow \frac{\pi}{R} = \frac{\pi}{R}$$

$$\int_0^\pi e^{-R \sin \theta} d\theta$$



So now let's just estimate

$$\int_{C_R} f(z) e^{iaz} dz$$

Integrating along the contour $z(\theta) = R e^{i\theta} \quad 0 \leq \theta \leq \pi$

$$z'(\theta) = R i e^{i\theta}$$

Keeping in mind that $|f(z)| \leq M_R$ and $\lim_{(R \rightarrow \infty)} M_R = 0$ and that $|e^{ia R e^{i\theta}}| \leq e^{-a R \sin(\theta)}$

$$\begin{aligned} |e^{ia R e^{i\theta}}| &= |e^{ia R (\cos \theta + i \sin \theta)}| = |e^{i a R \cos \theta - a R \sin \theta}| \\ &= |e^{i a R \cos \theta} e^{-a R \sin \theta}| = |e^{i a R \cos \theta}| |e^{-a R \sin \theta}| = e^{-a R \sin \theta} \end{aligned}$$

$$\int_{C_R} f(z) e^{iaz} dz = \int_0^\pi f(z(\theta)) e^{ia z(\theta)} z'(\theta) d\theta = \int_0^\pi f(R e^{i\theta}) e^{ia (R e^{i\theta})} R i e^{i\theta} d\theta$$

$$\leq \int_0^\pi |f(R e^{i\theta})| |e^{ia (R e^{i\theta})}| |R i e^{i\theta}| d\theta$$

$$\leq \int_0^\pi M_R e^{-a R \sin \theta} R d\theta = M_R R \int_0^\pi e^{-a R \sin \theta} d\theta$$

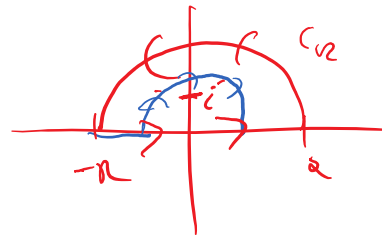
$$\leq M_R R \frac{\pi}{R} = M_R \pi$$

$$\lim_{R \rightarrow \infty} \int_{C_R} f(z) e^{iaz} dz \leq \lim_{R \rightarrow \infty} M_R \pi = 0$$

Chapter 7 Sec 88

6. Use Jordan's lemma to compute the principal value

$$PV \int_{-\infty}^{\infty} \frac{x e^{i\omega x}}{x^2 + 1} dx$$



where $\omega > 0$

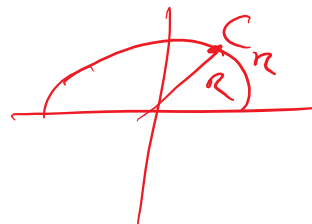
$$PV \int_{-\infty}^{\infty} \frac{x}{x^2 + 1} e^{i\omega x} dx = 2\pi i \sum \text{Res} \left(\frac{z}{z^2 + 1} e^{i\omega z} \right) - \lim_{R \rightarrow \infty} \int_{C_R} \frac{z}{z^2 + 1} e^{i\omega z} dz$$

$$\int_{-\infty}^{\infty} f(x) e^{iax}$$

$$f(z) = \frac{z}{z^2 + 1} \quad a = \omega$$

$$|f(z)| \leq M_R \quad \lim_{R \rightarrow \infty} M_R = 0$$

$$\text{Then } \lim_{R \rightarrow \infty} \int_{C_R} f(z) e^{i\omega z} dz = 0$$



$$\left| \frac{z}{z^2 + 1} \right| \leq \frac{|z|}{|z^2 + 1|} \leq \frac{R}{|z|^2 - 1} \leq \frac{R}{R^2 - 1}$$

$$\lim_{R \rightarrow \infty} \frac{R}{R^2 - 1} = 0 \quad \text{yes}$$

$$\Rightarrow \lim_{R \rightarrow \infty} \int_{C_R} f(z) e^{i\omega z} dz = 0$$

Chapter 7 Sec 88

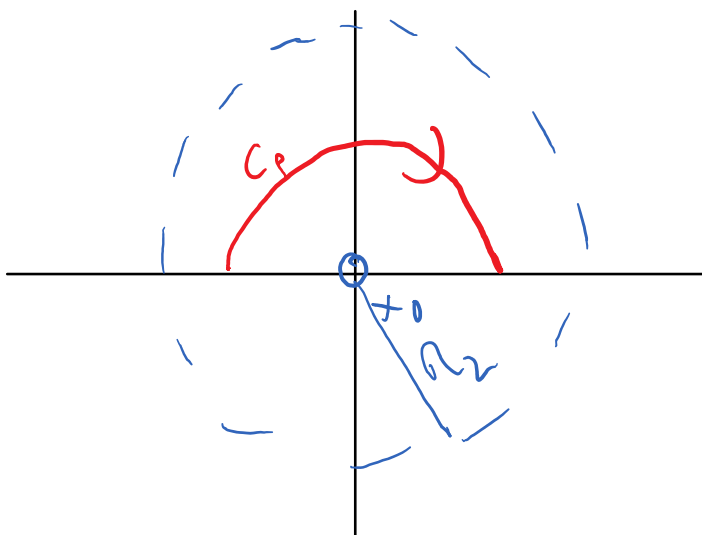
An Intended Path

Theorem Suppose that

- A function $f(z)$ has a simple pole at a point $z = x_0$ on the real axis, with a Laurent series representation in a punctured disk $0 < |z - x_0| < R_2$ and with residue B_0
- C_ρ denotes the upper half of a circle $|z - x_0| = \rho$, where $\rho < R_2$ and it is negatively oriented

Then

$$\lim_{\rho \rightarrow 0} \int_{C_\rho} f(z) dz = -B_0 \pi i$$



What this theorem allows us to do is to still be able to compute real improper integrals even when there is a singular point on the real axis!

Proof:

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n + \frac{B_0}{z - z_0}$$

We know that $f(z)$ has a laurent series, let's write it as $f(z) = \underline{g(z)} + \frac{B_0}{z - z_0}$ Where $g(z)$ is the taylor part. We know there are no other negative powers of $z - z_0$ since we are told $x_0 = z_0$ is a simple pole.

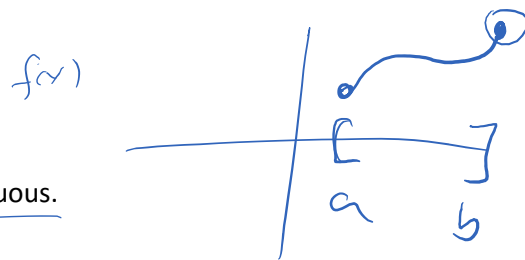
Then we can say that

$$\int_{C_\rho} f(z) = \int_{C_\rho} g(z) + \int_{C_\rho} \frac{B_0}{z - x_0} dz$$

Chapter 7 Sec 88

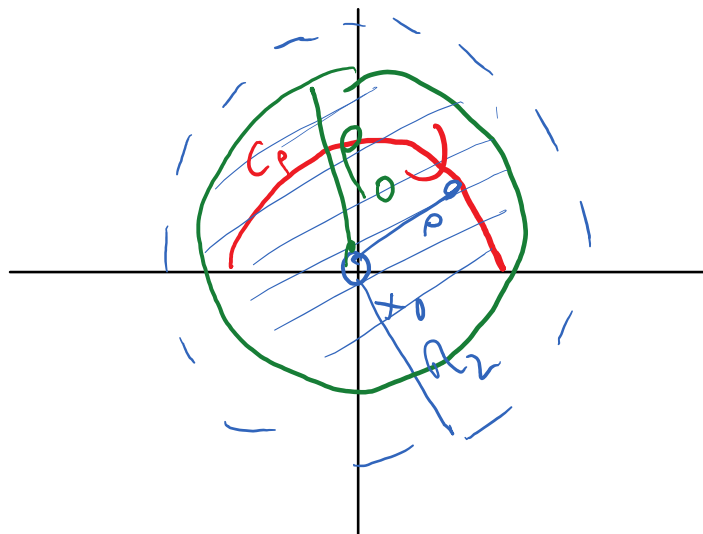
Since $g(z) = \sum_{n=0}^{\infty} a_n(z - x_0)^n$

Is a polynomial it is analytic and continuous.



We also have an old theorem which says that if a function like $g(z)$ is continuous on a closed disk then it must have a maximum value on that disk. We will choose some $\rho_0 < R_2$ and make a closed disk to work on.

Let M be the maximum value of g on our closed disk which contains the contour. Then $|g(z)| \leq M$



Estimating $\int_{C_\rho} g(z) dz$

$$\int_{C_\rho} g(z) dz \leq \int_{C_\rho} |g(z)| dz \leq \int_{C_\rho} M dz$$

$$= M \int_{C_\rho} dz = M \pi \rho$$

$$\lim_{\rho \rightarrow 0} M \pi \rho = 0 \Rightarrow \int_{C_\rho} g(z) dz = 0$$

Thus we can see that as $\rho \rightarrow 0$ does the integral of $g(z)$ that is

$$\lim_{\rho \rightarrow 0} \int_{C_\rho} g(z) = 0 \quad \text{and} \quad f(z) = \int g(z) + \int \frac{B_0}{z}$$

$$\lim_{\rho \rightarrow 0} \int_{C_\rho} g(z) = 0$$

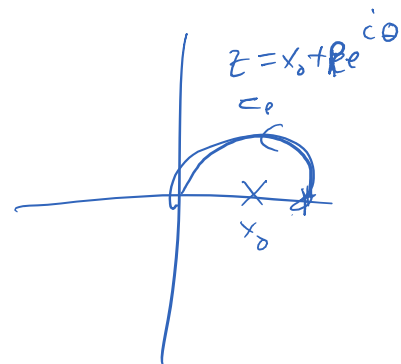
$$\lim_{\rho \rightarrow 0} \int_{C_\rho} f(z) = \int_{C_\rho} \cancel{g(z)} + \int_{C_\rho} \frac{B_0}{\cancel{z - z_0}}$$

Chapter 7 Sec 88

Now let's look at: $\int_{C_\rho} \frac{B_0}{z - x_0} dz = - \int_{-C_\rho} \frac{B_0}{z - x_0} dz$ Taking $-C_\rho$ orients us positively

Let's Parameterize C_ρ as $z = x_0 + \rho e^{i\theta}$ $0 \leq \theta \leq \pi$ and compute the integral

$$- \int_{C_\rho} \frac{B_0}{z - x_0} dz = - \int_0^\pi \frac{B_0}{x_0 + \rho e^{i\theta} - x_0} \rho i e^{i\theta} d\theta$$



$$\int_0^\pi \frac{B_0 \rho i e^{i\theta}}{\rho e^{i\theta}} d\theta = \int_0^\pi B_0 i d\theta = B_0 i \int_0^\pi d\theta = B_0 i \theta \Big|_0^\pi$$

$$= B_0 i (\pi - 0) = \boxed{-\pi i B_0}$$

So for any value of ρ $\int_{C_\rho} \frac{B_0}{z - x_0} dz = -\pi i B_0$ That is this integral has the constant value of $-\pi i$ for any ρ we take!

So if we take the limit of our original equality $\int_{C_\rho} f(z) = \int_{C_\rho} g(z) + \int_{C_\rho} \frac{B_0}{z - x_0} dz$

$$\lim_{\rho \rightarrow 0} \int_{C_\rho} f(z) = \lim_{\rho \rightarrow 0} \int_{C_\rho} g(z) - \lim_{\rho \rightarrow 0} \pi i B_0$$

$$\lim_{\rho \rightarrow 0} \int_{C_\rho} f(z) = -\pi i B_0$$

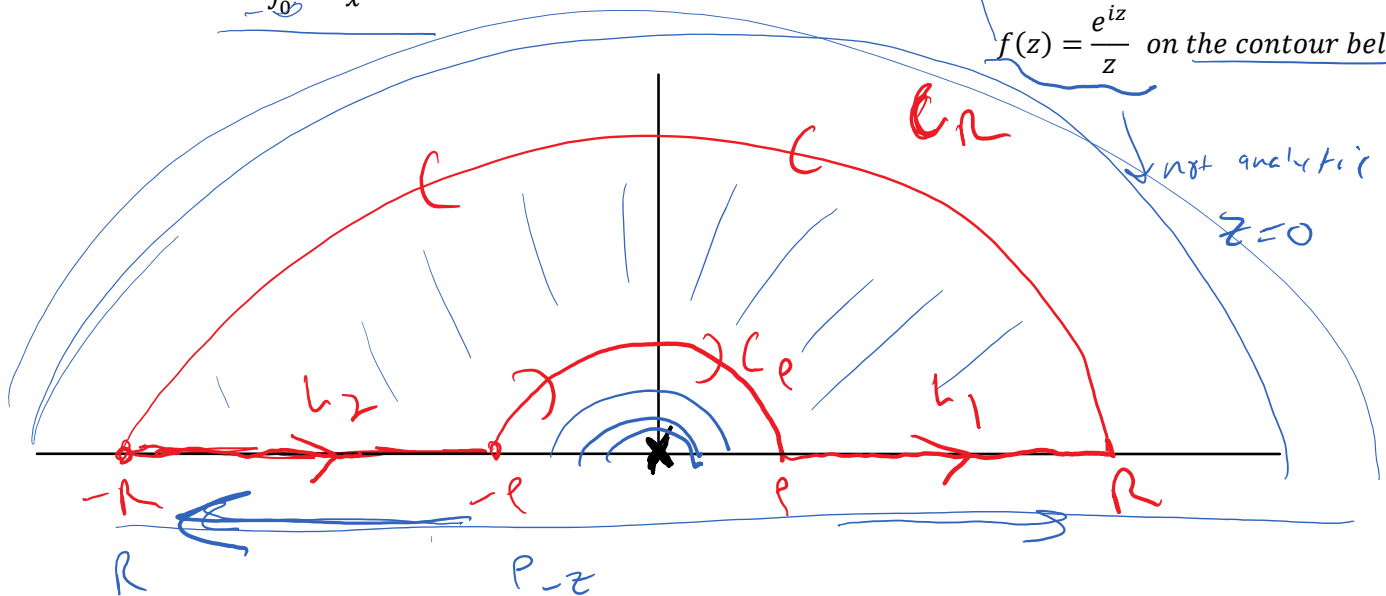
Chapter 7 Sec 88

Example: evaluate $\int_0^\infty \frac{\sin(x)}{x} dx = \pi/2$ This is called Dirichlet's integral.

$$f(z) = \int \frac{\cos(z)}{z} + \int \frac{\sin(z)}{z}$$

We will find it by integrating

$$f(z) = \frac{e^{iz}}{z} \text{ on the contour below}$$



The Cauchy Goursat theorem says that the integral $\frac{e^{iz}}{z}$ around the sum of these contours is zero

$$\int_{L_2} \frac{e^{iz}}{z} + \int_{C_\epsilon} \frac{e^{iz}}{z} + \int_{L_1} \frac{e^{iz}}{z} + \int_{C_R} \frac{e^{iz}}{z} = 0$$

Solving for the two integrals on L_1 and L_2 going along $-L_2$ and realizing they are just on the real axis gives.

$$\int_{L_1} \frac{e^{iz}}{z} + \int_{L_2} \frac{e^{iz}}{z} = - \int_{C_\epsilon} \frac{e^{iz}}{z} - \int_{C_R} \frac{e^{iz}}{z}$$

$$\int_{L_1} \frac{e^{iz}}{z} - \int_{-L_2} \frac{e^{iz}}{z} = - \int_{C_\epsilon} \frac{e^{iz}}{z} - \int_{C_R} \frac{e^{iz}}{z}$$

$$\int_{L_1} \frac{e^{iz}}{z} - \int_{L_2} \frac{e^{iz}}{z} = - \int_{C_\epsilon} \frac{e^{iz}}{z} - \int_{C_R} \frac{e^{iz}}{z}$$

$$\int \frac{e^{iz}}{z} - \int \frac{e^{-iz}}{z} = - \int_{C_\epsilon} \frac{e^{iz}}{z} - \int_{C_R} \frac{e^{iz}}{z}$$

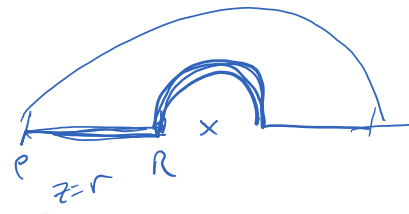
$$\sin(z) = \frac{e^{iz} - e^{-iz}}{2i}$$

$$\int_{\gamma_1} \frac{1}{z} - \int_{\gamma_2} \frac{e^{iz}}{z} dz = - \left(\int_{\gamma_1} \frac{e^{-iz}}{z} - \int_{\gamma_2} \frac{1}{z} \right)$$

$$\int_{\gamma_1} \frac{e^{iz}}{z} - \int_{\gamma_1} \frac{e^{-iz}}{z} = \int_{\gamma_1} \frac{e^{iz} - e^{-iz}}{z} dz = \int_{\gamma_1} 2i \frac{\sin(z)}{z} dz$$

Chapter 7 Sec 88

$$2i \int \frac{\sin(z)}{z} dz$$



So we now have that

$$2i \int_p^R \frac{\sin(r)}{r} dr = - \int_{C_p} \frac{e^{iz}}{z} dz - \int_{C_R} \frac{e^{iz}}{z} dz$$

$$\lim_{p \rightarrow 0} \int_{C_p} \frac{e^{iz}}{z} dz = 0$$

If we write out the Laurent Series for $\frac{e^{iz}}{z}$ using it's taylor series we can find the residue and see that it is a simple pole. (pole of order 1) satisfying our intended path theorem on C_p

$$\frac{e^{iz}}{z} = \frac{1}{z} e^{iz} = \frac{1}{z} \sum_{n=0}^{\infty} \frac{(iz)^n}{n!} = \sum_{n=0}^{\infty} \frac{i^n z^{n-1}}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{i^n z^{n-1}}{n!}$$

$$n=0 \quad \frac{i^0 z^{-1}}{0!} = \frac{1}{z}$$

$$n=1 \quad \frac{i^1 z^0}{1!} = i$$

$$n=2 \quad \frac{i^2 z^1}{2!} = -\frac{z}{2}$$

$$\Rightarrow \lim_{p \rightarrow 0} \int_{C_p} \frac{e^{iz}}{z} dz = -\pi i$$

Further we can examine $\left| \frac{1}{z} \right| \leq \frac{1}{R} = M_R$ and satisfies Jordan's Lemma

$$\int_{C_R} \frac{e^{iz}}{z} dz \quad f(z) = \frac{1}{z} \quad \left| \frac{1}{z} \right| \leq \frac{1}{R}$$

$$\Rightarrow \lim_{R \rightarrow \infty} \int_{C_R} \frac{e^{iz}}{z} dz = 0$$

As a result

$$\lim_{p \rightarrow 0} \lim_{R \rightarrow \infty} 2i \int_p^R \frac{\sin(r)}{r} dr = \lim_{p \rightarrow 0} \lim_{R \rightarrow \infty} \left(- \int_{C_p} \frac{e^{iz}}{z} dz \right) - \lim_{p \rightarrow 0} \lim_{R \rightarrow \infty} \int_{C_R} \frac{e^{iz}}{z} dz$$

$$2i \int_0^{\infty} \frac{\sin(r)}{r} dr = -(-\pi i) - 0$$

$$\frac{2i}{2i} \int_0^{\infty} \frac{\sin(r)}{r} dr = \frac{\pi i}{2i} \Rightarrow \int_0^{\infty} \frac{\sin(r)}{r} = \boxed{\frac{\pi}{2}}$$

