

Section 1.1 Differential Equations and Mathematical Models.

What is a Differential Equation (DE)?

Differential Equation: An equation relating an unknown function and one or more of its derivatives. Recall that a derivative of a function is a rate of change of that function with respect to a variable that function depends on.

Let's make a differential equation!

A Tesla Model S Plaid can go from zero to 60 in 1.98 seconds. That gives us an acceleration of about 30mph/s. Let $v(t)$ be the velocity of a such a Tesla, how can we relate its velocity to its acceleration?

$$\frac{dv}{dt} = a \quad \frac{dv}{dt} = 30$$



What is $v(t)$ given the equation we have written down?

maybe $v(t) = 30t + C$

$$\frac{dv}{dt} = \frac{d}{dt}(30t + C) = 30$$

$v(0) = 0$ $v(0) = 30(0) + C = 0$ $C = 0$

Now let's consider something we neglected, air resistance. Often air resistance is proportional to the speed of the object moving through the air. Let's suppose the acceleration due to air resistance on a tesla is $kv(t)$. Here k would be a small number like 0.001 or something. So we see that the faster the car goes the more air resistance we face. If this is the case how do we now relate the velocity function $v(t)$ to the acceleration?

$$\frac{dv}{dt} = 30 - kv(t)$$

30 $\rightarrow$ kv

What is $v(t)$ given the equation we have written down? How can we verify our guess?

guess. $v(t) = -\frac{1}{k}(ce^{-kt} - 30)$ $\Rightarrow \frac{dv}{dt} = -\frac{-k}{k}ce^{-kt} = ce^{-kt}$

plug them in to $\frac{dv}{dt} = 30 - kv$

$$ce^{-kt} = 30 - k\left(-\frac{1}{k}(ce^{-kt} - 30)\right)$$

$$ce^{-kt} = 30 + (ce^{-kt} - 30)$$

$$ce^{-kt} = 30 + ce^{-kt} - 30$$

$$ce^{-kt} = ce^{-kt}$$

so if $v = -\frac{1}{k}(ce^{-kt} - 30)$ it solves the D.E. ✓

How to solve (we will cover later)

$$\frac{dv}{dt} = 30 - kv$$

$$\int \frac{dv}{30 - kv} = \int dt$$

$$-\frac{1}{k} \ln(30 - kv) = t + C$$

$$\ln(30 - kv) = -kt + C$$

$$30 - kv = e^{-kt + C}$$

$$30 - kv = ce^{-kt}$$

$$v = \frac{1}{k}(30 - ce^{-kt})$$

$$ce^{\downarrow kt} = ce^{kt} \quad \checkmark \quad \text{so in } x' = \dots \quad \text{it solves D.E.}$$

$$\left\{ \begin{array}{l} 30 - kv = \dots \\ 30 - kv = ce^{-kt} \end{array} \right.$$

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So with air resistance the velocity of the Tesla as it accelerates is $v(t) = -\frac{1}{k}(ce^{-kt} - 30)$ Lets suppose $k=0.04$ and $C=30$ so that $v(0)=0$



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What is another way we could write the physical situation of our Tesla down? Hint: $F=ma$

Let $x(t)$ be the position of the Tesla

$$\frac{dv}{dt} = 30 - kv(t)$$

$$\frac{d^2x}{dt^2} = 30 - k \frac{dx}{dt}$$

same
physical
model

$x(t)$ position

$$\frac{dx}{dt} = v$$

$$\frac{d^2x}{dt^2} = \frac{dv}{dt} = a$$

So we can write the same physics down as:

$$\frac{d^2x}{dt^2} = 30 - k \frac{dx}{dt}$$

Second order Differential Equation

Or

$$\frac{dv}{dt} = 30 - kv$$

First order differential equation

The order of a differential equation is defined by the highest derivative that shows up in it.

The most general form of an n -th order differential equation with independent variable x and unknown function $y(x)$ can be written as:

$$F(x, y, y', y'', \dots, y^n) = 0$$

where F is a real valued function of $n + 2$ variables.

Ex. Write the differential equation $y^{(4)} + \cos(x)y^{(3)} = \sin(x)y + e^x y''$ in the general form. What is the order of this differential equation?

$$y^{(4)} + \cos(x)y^{(3)} - \sin(x)y - e^x y'' = 0$$

$$F(x, y, y', y'', y''')$$

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Let's make another differential equation.... It has been estimated that a person with the delta variant of SARS-COV2 can infect between 5 or 9 other people compared to the original strain of around 2-3 other people. This rate of infecting other people is called the basic reproductive number of the virus (R_0). Let's choose $R_0 = 5$ for the delta variant and let's say the 5 people are infected over 2.5 days for two people per day. Let $P(t)$ be the number of people who have been infected, if each infected person infects 5 over 2.5 days others what is the rate of change $P(t)$?

$$\frac{dP}{dt} = k P(t)$$

$$\frac{5}{2.5} = 2$$

$$P(t) = C e^{kt}$$

$$\frac{dP}{dt} = C k e^{kt} = k (C e^{kt})$$

$\underbrace{C e^{kt}}_{P(t)}$

$$P(0) = C e^{k(0)} = C(1) = P_0$$

$$P(t) = P_0 e^{kt}$$

$$P(0) = P_0$$

$$P(1) = 2P_0 = P_0 e^{k(1)}$$

$$2 = e^k \rightarrow \ln(2) = \ln(e^k)$$

$$\ln(2) = k$$

How can we determine C?

$$P(t) = P_0 e^{\ln(2)t}$$

$$= P_0 (e^{\ln(2)})^t = P_0 2^t$$

What is a fundamental flaw with this particular model for the spread of SARS-COV2?

Assumes each infected person continues to spread even after they have recovered.

Assumes all people spread at the same rate.

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There are really 3 classes of the population when it comes to disease. Susceptible people, infected people, and recovered people who have presumed immunity from infection at least for a while. How might we model the rates of change of these three groups assuming recovered people have immunity forever?

$$\frac{dS}{dt} = -(\text{percent of susceptible people}) \times (\text{number of contacts per day}) \times (\text{number of infected people})$$

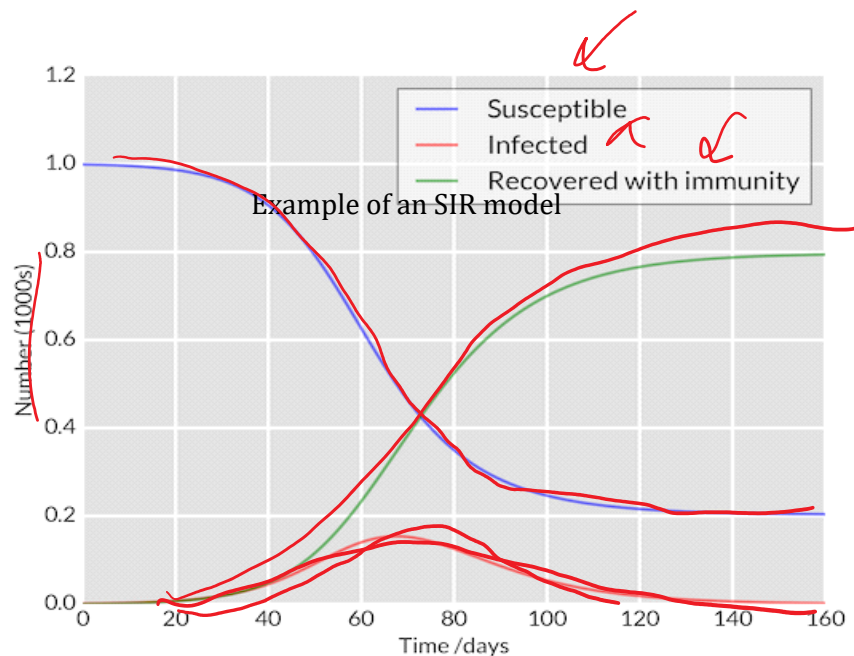
$$\frac{dR}{dt} = (\text{recovery rate}) \times (\text{number of infected people})$$

$$\frac{dI}{dt} = (\text{number of susceptible people}) - (\text{number of recovered people})$$

$$\frac{ds}{dt} = -\frac{S}{N} \beta I$$

$$\frac{dR}{dt} = k I$$

$$\frac{dI}{dt} = -\frac{S}{N} \beta I - k I$$



There are other models to including SEIR (Sick Exposed Infected and Recovered) These models can include time dependent rates of transmission as well to model lockdowns for example. One could add vaccinated groups, age classes for risk of death or severe infection and more.

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HW examples:

First, verify that $y(x)$ satisfies the given differential equation. Then, determine a value of the constant C so that $y(x)$ satisfies the given initial condition. Use a computer or graphing calculator to sketch several typical solutions of the given differential equation, and highlight the one that satisfies the given initial condition.

$$y' = y + 1; y(x) = C e^x - 1; y(0) = 7$$

Step 1. Take the needed derivatives

$$\frac{d}{dx}(C e^x - 1) = C e^x = y'$$

Step 2. Plug in the derivatives and function into the D.E.

$$\begin{aligned} y' &= y + 1 \\ \downarrow & \quad \downarrow \\ C e^x &= C e^x - 1 + 1 \\ C e^x &= C e^x \quad \checkmark \end{aligned}$$

To find C enforce the initial condition $y(0) = 7$

$$y(0) = C e^0 - 1 = 7$$

$$C(1) - 1 = 7$$

$$C = 8$$

so

$$\boxed{y = 8e^x - 1}$$

A function $y = g(x)$ is described below by some geometric property of its graph. Write a differential equation of the form $\frac{dy}{dx} = f(x, y)$ having the function g as its solution or as one of its solutions.

The slope of the graph of g at the point (x, y) is the sum of x and y .

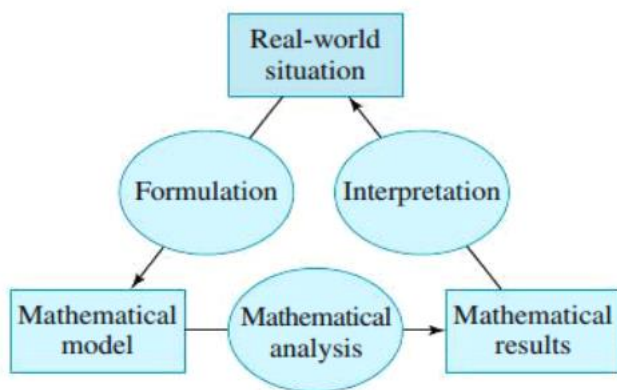
$$\frac{dy}{dx} = x + y$$

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Quick Recap.

What is a Differential Equation (DE)?

Differential Equation: An equation relating an unknown function and one or more of its derivatives. Recall that a derivative of a function is a rate of change of that function with respect to a variable that function depends on.



Differential Equations can be used to model the real world.

We can verify a function is a solution to a given differential equation by calculating the needed derivatives, substituting them in and checking that the equation is satisfied.

Example 8 Verify that the function $y(x) = 2x^{1/2} - x^{1/2} \ln x$ satisfies the differential equation

$$4x^2 y'' + y = 0$$

for all $x > 0$.

Solution First we compute the derivatives

$$y'(x) = -\frac{1}{2}x^{-1/2} \ln x \quad \text{and} \quad y''(x) = \frac{1}{4}x^{-3/2} \ln x - \frac{1}{2}x^{-3/2}.$$

Then substitution into Eq. (10) yields

$$4x^2 y'' + y = 4x^2 \left(\frac{1}{4}x^{-3/2} \ln x - \frac{1}{2}x^{-3/2} \right) + 2x^{1/2} - x^{1/2} \ln x = 0$$

if x is positive, so the differential equation is satisfied for all $x > 0$.

Because additive constants disappear and multiplicative constants keep multiplying when take derivatives.... To find particular solutions we need initial conditions.

Given the solution $y(x) = 1/(C - x)$ of the differential equation $dy/dx = y^2$ discussed in Example 7, solve the initial value problem

$$\frac{dy}{dx} = y^2, \quad y(1) = 2.$$

We need only find a value of C so that the solution $y(x) = 1/(C - x)$ satisfies the initial condition $y(1) = 2$. Substitution of the values $x = 1$ and $y = 2$ in the given solution yields

$$2 = y(1) = \frac{1}{C - 1},$$

so $2C - 2 = 1$, and hence $C = \frac{3}{2}$. With this value of C we obtain the desired solution

$$y(x) = \frac{1}{\frac{3}{2} - x} = \frac{2}{3 - 2x}.$$