

Section 1.2 Integrals and General and Particular Solutions

Recall Last Time.

Now let's consider something we neglected, air resistance. Often air resistance is proportional to the speed of the object moving through the air. Let's suppose the acceleration due to air resistance on a tesla is $kv(t)$. Here k would be a small number like 0.001 or something. So we see that the faster the car goes the more air resistance we face. If this is the case how do we now relate the velocity function $v(t)$ to the acceleration?

$$\frac{dv}{dt} = a(t)$$

$$\frac{dv}{dt} = 30 - kv(t)$$

What is $v(t)$ given the equation we have written down? How can we verify our guess?

$$\frac{d}{dt} e^{kt} = \underline{ke^{kt}}$$

$$v(t) = \frac{1}{k}(ce^{-kt} - 30) \quad \text{RHS}$$

$$30 - k(ce^{kt} + 30/k)$$

$$30 - ke^{kt} + 30$$

$$v(t) = \frac{1}{k}(ce^{-kt} - 30) \quad \text{LHS}$$

$$\frac{dv}{dt} = ce^{-kt}$$

$$\frac{dv}{dt} = ke^{kt} = -ke^{kt}$$

$$\text{RHS}$$

$$30 - k(\frac{1}{k}ce^{-kt} - 30)$$

$$30 + ce^{-kt} - 30$$

$$ce^{-kt}$$

What is the value of C if we assume the Tesla started at rest?

$$v(t) = \frac{1}{k}(ce^{-kt} - 30)$$

$$v(0) = 0$$

$$v(0) = \frac{1}{k}(ce^{-k(0)} - 30) = 0$$

$$= \frac{-c}{k} + \frac{30}{k} = 0$$

$$\Rightarrow -c + 30 = 0$$

$$c = 30$$

Let's head over to Desmos to look at some of the possible solutions.

Section 1.2 Integrals and General and Particular Solutions

Some differential equations are in a form where we can directly integrate both sides of the equation to find the solution. This is the case when we have the derivative on one side and only functions of the independent variable on the other. That is when our differential equation is of the form:

$$\downarrow$$

$$\frac{d^{(n)}y}{dt^{(n)}} = f(t)$$

$$V = 30 - kv \quad \downarrow \quad g(t, v)$$

where x is the independent variable.

Example: Let's revisit the idea of a body going through constant acceleration a . If $v(t)$ is the velocity and $x(t)$ is the position we can write this one of two ways.

$$\rightarrow \frac{dv}{dt} = a \quad \text{or} \quad \rightarrow \frac{d^2x}{dt^2} = a$$

Note that the right hand side of each of these equations do not depend on v or x , the unknown functions in our differential equation. In this case $f(t) = a$ just a constant number. What do the solutions look like?

$$\int \frac{dv}{dt} dt = \int a dt$$

$$\rightarrow v(t) + C_1 = at + C_2$$

$$v(t) = at + (C_2 - C_1)$$

$$v(t) = at + C$$

$$\int \frac{d^2x}{dt^2} dt = \int a dt \rightarrow \frac{dx}{dt} = at + C_1$$

$$\int \frac{dx}{dt} = \int at + C_1$$

$$x(t) = \frac{at^2}{2} + C_1t + C_2$$

What information would we need to find the constants of integration?

$$v(a) = b$$

$$v(10) = 30$$

$$a(10) + C = 30$$

$$C = 30 - a(10)$$

or

$$x(a_1) = b_1$$

$$x(a_2) = b_2$$

$$\frac{dx}{dt}(a_3) = b_3$$

data for x and $\frac{dx}{dt}$

Two data for x

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This is in contrast to the equation we had for the Tesla example where we included a term for air resistance that depended on the velocity itself. If we had such a term our equations would be:

$$\frac{dv}{dt} = a - kv$$

$$\frac{d^2x}{dt^2} = a - k \frac{dx}{dt}$$

These equations have a form

$$\frac{d^{(n)}y}{dt^{(n)}} = f(t, y)$$

Where our right hand side no longer only depends on the independent variable. We will learn other methods for solving equations like this later on.

Example: Can we directly integrate both sides of the differential equation,

$$\frac{dy}{dt} = \sin(t) \cos(t) + \tanh(t) ?$$

$\sinh(t) = \frac{e^t - e^{-t}}{2}$
 $\cosh(t) = \frac{e^t + e^{-t}}{2}$
 $\tanh(t) = \frac{\sinh(t)}{\cosh(t)}$

$$y = \int \sin(t) \cos(t) dt + \int \tanh(t) dt$$

$u = \sin(t)$
 $\frac{du}{dt} = \cos(t) \Rightarrow dt = \frac{du}{\cos(t)}$
 $\int u \cos(t) \frac{du}{\cos(t)}$
 $\int u du$
 $= u^2/2$
 $= \frac{\sin^2(t)}{2}$

$\int \frac{\sinh(t)}{\cosh(t)} dt$
 $\downarrow$
 $\int \frac{1}{u} du$
 $= \ln|u| + C$
 $= \ln|\cosh(t)| + C$

$u = \cosh(t) \quad du = \sinh(t) dt$

$$\Rightarrow y = \frac{\sin^2(t)}{2} + \ln|\cosh(t)| + C$$

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Frustrated with your differential equations homework you have climbed to the top of the Phillips Hall a height of 100 ft (guessing) . In a fit of rage you throw your text book up and off the edge with an initial velocity of 5 ft/s . Find a function that gives the height of the book as a function of time knowing that the acceleration due to gravity is $32 \frac{ft}{s^2}$ in the downward direction. When will the book hit the ground? What is the velocity of the book when it hits the ground?

$$g = -32 \frac{ft}{s^2}$$

↓
a

$$\rightarrow \int \frac{dx}{dt^2} = -32$$

$$\int \frac{dx}{dt} = -32t + C_1$$

$$v(0) = 5$$

$$(1) \left. \frac{dx}{dt} \right|_0 = 5$$

$$(2) x(0) = 100$$

$$x(t) = -\frac{32t^2}{2} + C_1t + C_2$$

$$(1) \left. \frac{dx}{dt} \right|_{t=0} = -32(0) + C_1 = 5$$

$C_1 = 5$

$$x(t) = -16t^2 + 5t + C_2$$

$$(2) x(0) = -16(0^2) + 5(0) + C_2 = 100 \Rightarrow C_2 = 100$$

$$\Rightarrow x(t) = -16t^2 + 5t + 100$$

$$\left. \frac{dx}{dt} \right|_{t_1} = -32(t_1) + 5$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

↑
2a

$$ax^2 + bx + c = 0$$

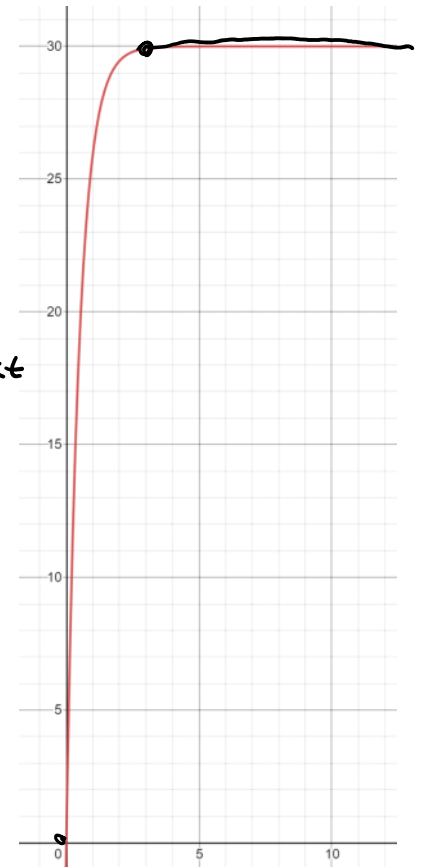
$$0 = \underbrace{-16}_{a}t^2 + \underbrace{5}_{b}t + \underbrace{100}_{c}$$

$$\rightarrow t = \frac{-5 \pm \sqrt{5^2 - 4(-16)(100)}}{2(-16)}$$

C_1 positive one

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Unlike an electric vehicle which applies instant torque as the pedal is pushed down, gasoline cars have an increase in torque as the RPM's of the engine increase. Suppose you are driving as a stunt person in the seemingly inevitable Fast and Furious 1000 movie. In your scene you are in a drag race for which the winner obtains exclusive rights to cruise the main drag in town. Suppose the acceleration of your car with the pedal to the metal is defined by: $a(t) = 30(1 - e^{-2t})$, shown to the right. What function gives your position on the road as a function of time assuming you drive in a straight line.



$$\frac{d^2x}{dt^2} = 30(1 - e^{-2t})$$

$$\int e^{kt} = \frac{1}{k}e^{kt}$$

$$\int \frac{d^2x}{dt^2} = \int 30 - 30e^{-2t}$$

$$\frac{dx}{dt} = 30t - 30\left(-\frac{1}{2}e^{-2t}\right) + C_1$$

$$\int \frac{dx}{dt} = \int 30t + 15e^{-2t} + C_1 \rightarrow v(0)=0 \rightarrow 30(0) + 15e^0 + C_1 = 0$$

$$15 + C_1 = 0$$

$$C_1 = -15$$

$$x(t) = \frac{30t^2}{2} + 15\left(-\frac{1}{2}\right)e^{-2t} + C_1 t + C_2$$

$$x(0) = 0$$

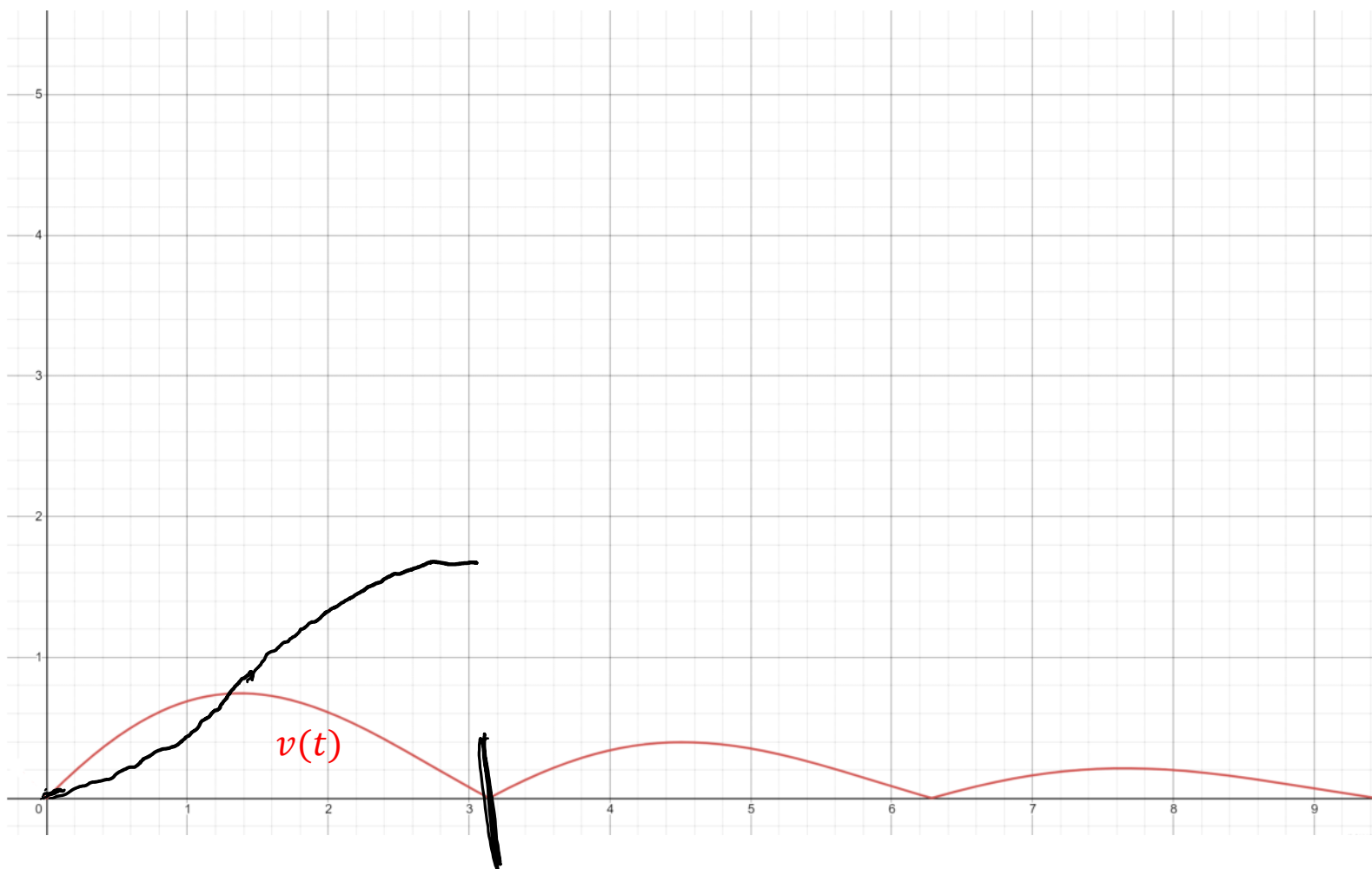
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$$-\frac{15}{2} + C_2 = 0$$

$$C_2 = \frac{15}{2}$$

Section 1.2 Integrals and General and Particular Solutions

Suppose a particle has the following velocity curve. Roughly sketch what the position function might look.



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LIATE

g ↓ 1 1 1 x
2 2 2 2 p

Example: Solve the following initial value problem.

$$\int \frac{dy}{dx} dx = \int x e^{-x} dx$$

$y =$

$$-x e^{-x} - \int -e^{-x} dx$$

$$-x e^{-x} + \int e^{-x} dx$$

$$y(x) = -x e^{-x} - e^{-x} + C$$

$$y(0) = -0 e^{-0} - e^{-0} + C = 1$$

$$-1 + C = 1 \Rightarrow C = 2$$

$$y(x) = -x e^{-x} - e^{-x} + 2$$

$$\frac{dy}{dx} = x e^{-x}; y(0) = 1$$

$$u = x$$

$$du = dx$$

$$dv = e^{-x}$$

$$v = -e^{-x}$$

$$\int u dv = uv - \int v du$$

Quick Recap.

Some differential equations are in a form where we can directly integrate both sides of the equation to find the solution. This is the case when we have the derivative on one side and only functions of the independent variable on the other. That is when our differential equation is of the form:

$$\frac{d^{(n)}y}{dt^{(n)}} = f(t)$$

where t is the independent variable.

Many physical situations do fall into this category, notably that of constant acceleration. To solve these types of DE's we simply,

1. Integrate both sides
2. Use any given initial data to find our unknown constants of integration.