

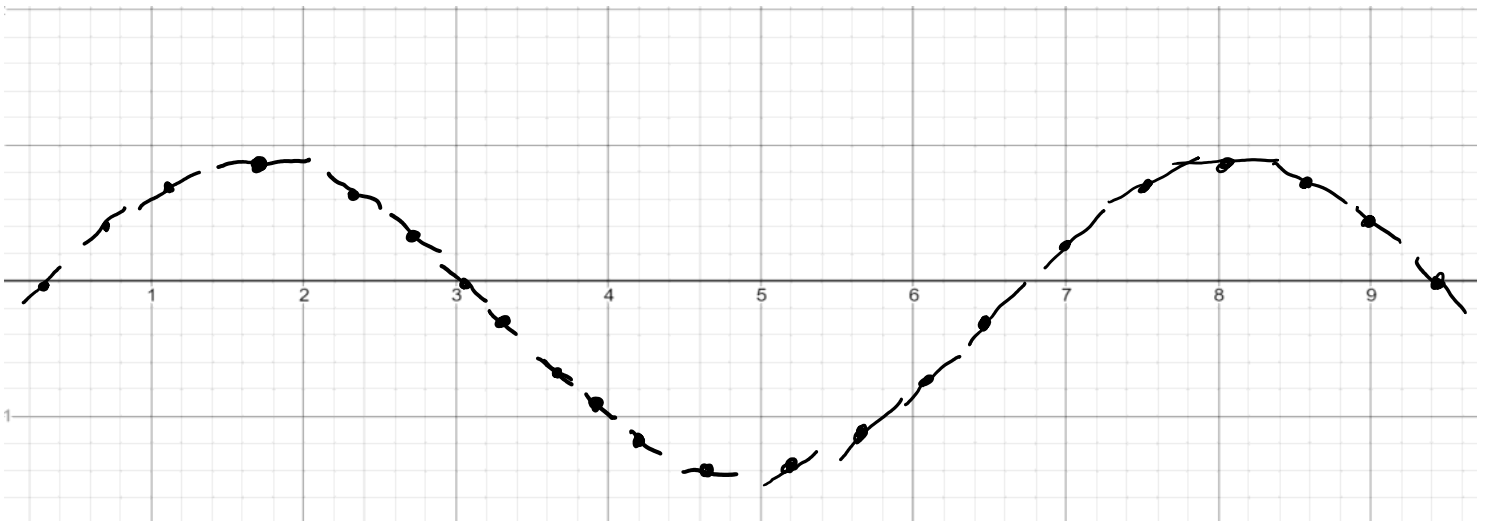
Section 1.3 Slope Fields and Solution Curves

In the last section we dealt with differential equations where the derivative only depended on the independent variable. In this section we will look at one way to understand differential equations where the derivative depends not only on the independent variable but also the unknown functions itself. In particular we will look at first order differential equations of the form:

$$\frac{dy}{dx} = f(x, y)$$

While some differential equations of this form can be solved exactly, (and we will learn methods soon how to do so) some are not so clear. In these cases we can still learn a lot about the system through graphical means.

Before we fully get into that let's play a guessing game. Below is a picture of a bunch of small tangent line segments. Any guesses as to what function they are all tangent to?



Section 1.3 Slope Fields and Solution Curves

Consider the differential equation

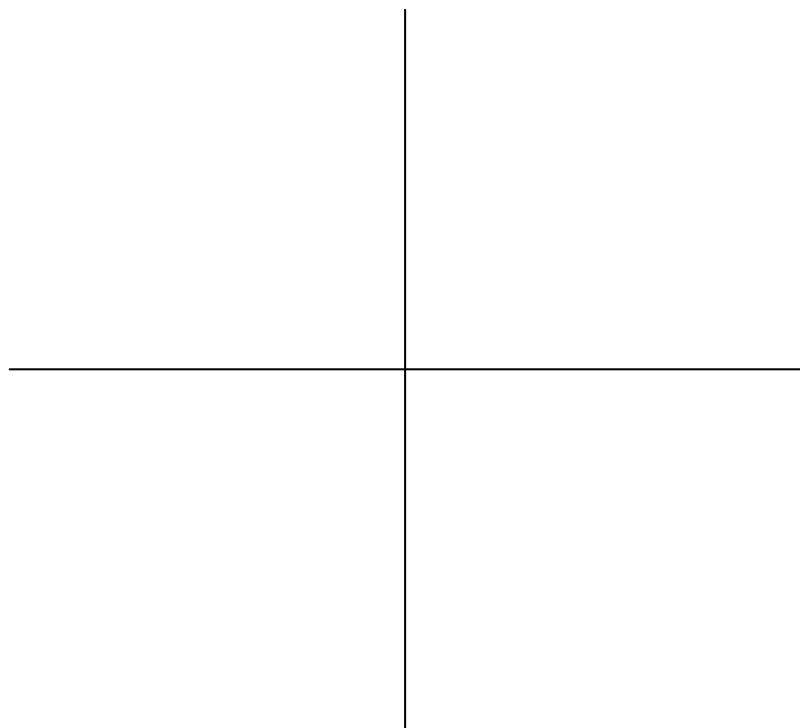
$$\frac{dy}{dx} = y - x$$

What does this equation tell us about $y(x)$ at the point $(2,4)$? What is it telling us about $y(x)$ at any point (x,y) in the plane?

How can we use the fact above to construct a graph that can help us visualize the general solution to the D.E?

Section 1.3 Slope Fields and Solution Curves

Let's construct a slope field for a differential equation we have looked at. $y' = ky$. Lets take $k = 1$



Let's sketch some solutions with $C = 1, C = 2, C = -1, C = -2$

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Plotting slope fields with a computer...

Some of the programs or codes you can use include Maple, Mathematica, Sage, Matlab, Octave, Python, R and more. Since your labs will be done primarily in Matlab we will use Octave. Octave is an opensource version of Matlab with almost all of the same commands! You can download Octave for free if you like but in class I will primarily use a free online version called Octave online. If you want to follow along head to <https://octave-online.net/>.

To make our slope fields we will use a matlab/octave function called quiver. `quiver(x,y,u,v)` plots the u,v components of a vector at the point (x,y) . However here we are not really plotting vectors but slopes of lines. One way we could proceed would be to put a unit vector at (x,y) that points in the same direction of the slope...

So the question is: if $\frac{dy}{dx} = f(x, y)$ gives us the slope of the tangent line at (x, y) what is a unitvector $\langle u, v \rangle$ that points in the same direction as that tangent line?

The code:

```
clear all                % clear the variables for good measure
close all               %close any open figures for good measure
funn = @(x, y)y;        % define function dy/dx=f(x, y)
[x, y] = meshgrid(-5:0.5:5); % intervals for x and y
slopes = funn(x, y);    % matrix of slope values
dy = slopes ./ sqrt(1 + slopes.^2); % normalize the line element..
dx = ones(length(dy)) ./ sqrt(1 + slopes.^2); % ...magnitudes for dy and dx
h = quiver(x, y, dx, dy, 0.5); % plot the direction field
```

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Let's look at some slope fields.

Like our Tesla we can also consider a falling body with air resistance.
Let's take $k = 0.16$ and look at $t \in [0, 25]$, $v \in [0, 400]$

$$\frac{dv}{dt} = 32 - kv$$

The logistic model models populations where the environment can only support a maximum population M
Let's take $k=0.004$ $M= 150$ and look at $t \in [0, 100]$, $P \in [0, 300]$

$$\frac{dP}{dt} = kP(M - P)$$

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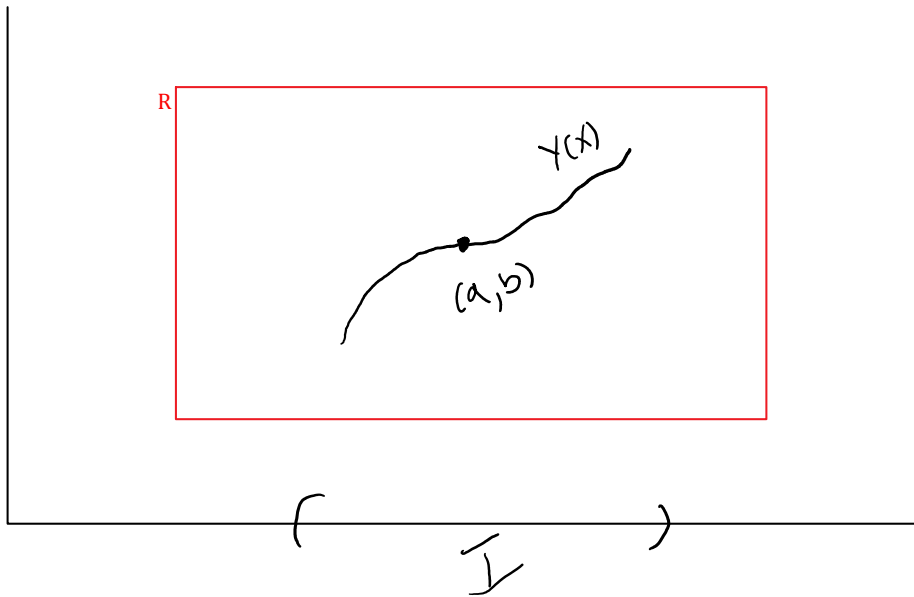
Existence and Uniqueness of a solution to a DE. Before going to all of the trouble of searching for a solution or analyzing a DE it would be nice to know if we can even find one! Luckily there is a theorem for that.

Theorem 1: Existence and Uniqueness of a solution to a DE

Suppose that both the function $f(x, y)$ and its partial derivative $D_y f(x, y)$ are continuous on some rectangle R in the x - y plane that contains the point (a, b) in its interior. Then for some open interval I containing a , the initial value problem

$$\frac{dy}{dx} = f(x, y), \quad y(a) = b$$

has one and only one solution that is defined on the interval I



Section 1.3 Slope Fields and Solution Curves

Consider the DE $x \frac{dy}{dx} = 2y$ and the following possible initial conditions (1) $y(-1) = 2$, (2) $y(0) = 0$, and (3) $y(0) = 4$.

Theorem 1: Existence and Uniqueness of a solution to a DE

Suppose that both the function $f(x, y)$ and its partial derivative $D_y f(x, y)$ are continuous on some rectangle R in the x - y plane that contains the point (a, b) in its interior. Then for some open interval I containing a , the initial value problem

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Using Theorem 1 determine whether or not there is a unique solution on some interval containing the initial data point given for each initial condition. If not explain why?

(1) $y(-1) = 2$

(2) $y(0) = 0$

(3) $y(0) = 4$