

Section 1.4 Separable Equations and Applications

What is a separable differential equation?

The differential equation $\frac{dy}{dx} = f(x, y)$ is called separable if there are two functions $g(x), k(y)$ where g only depends on x and k only depends on y such that $f(x, y) = \underline{g(x)k(y)}$.

Example Which of the following differential equations are separable? Identify $g(x)$ and $k(y)$ if they are separable.

(1) $\frac{dy}{dx} = xy - 3y$

$\frac{dy}{dx} = \underbrace{(x-3)}_{g(x)} \underbrace{y}_{k(y)}$

(2) ~~$\frac{dy}{dx} = xy^2 - 3y$~~
 ~~$(xy-3)y$~~

(3) $\frac{dP}{dx} = \underbrace{kP(M-P)}_{k(P)}$
const
 $\downarrow$
 $g(x)$

(4) $\frac{dv}{dt} = 32 - kv$
 $\nearrow v$ is like y
 $\nwarrow$ const
 $g(x)=1$ $k(v)=32-kv$

(5) ~~$\frac{dy}{dx} = x^2 - y^2$~~
 ~~$(x+y)(x-y)$~~

(6) $\frac{dy}{dx} = \frac{x}{y}$
 $g(x)=x$ $k(y)=\frac{1}{y}$
 $\frac{x}{y} = g(x)k(y)$

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Solving Separable Differential Equations

First remember the chain rule! $\frac{d}{dx}H(y(x)) = H'(y)\frac{dy}{dx}$ if we let $h(y) = H'(y)$ we can write the chain rule as:

$$\frac{d}{dx}H(y(x)) = h(y)\frac{dy}{dx}$$

$k(y) = \frac{1}{h(y)} \quad h(y) = \frac{1}{k(y)}$

Let $\frac{dy}{dx} = g(x)k(y)$ how can we re-write this so that we can simply integrate both sides with respect x and end up with an equation we can solve for y with? Let $k(y) = 1/h(y)$...

$$\frac{dy}{dx} = g(x)k(y) \rightarrow \frac{dy}{dx} = g(x)\frac{1}{h(y)} \rightarrow h(y)\frac{dy}{dx} = g(x)$$

$$\int h(y)\frac{dy}{dx} dx = \int g(x) dx$$

$$\rightarrow H(y) = G(x) + C \rightarrow \text{solve for } y$$

$\int \frac{1}{k(y)} \frac{dy}{dx} dx$

So we have that:

$$\int h(y)\frac{dy}{dx} dx = \int g(x) dx$$

And we know that:

$$\frac{d}{dx}H(y(x)) = h(y)\frac{dy}{dx} \text{ so } \int h(y)\frac{dy}{dx} dx = H(y) + C \text{ and if } G'(x) = g(x) \text{ then } \int g(x) dx = G(x) + C$$

Lumping all the constants on one side we get the equation

$$\rightarrow H(y) = G(x) + C$$

Depending on the form of H we can then solve for y!

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Example: Find the general solution to the differential equation: $\frac{dy}{dx} = \tilde{k}y$

$$\begin{aligned} \frac{dy}{dx} &= \tilde{k}y & g(x) &= \tilde{k} & h(y) &= y \\ & & & \downarrow & & \\ & & h(y) &= \frac{1}{y} & & \\ h(y) \frac{dy}{dx} &= g(x) & & & & \\ \int \frac{1}{y} \frac{dy}{dx} dx &= \int \tilde{k} dx & & & & \\ \ln(y) &= \tilde{k}x + C & & & & \\ \downarrow & & & & & \\ H(y) &= G(x) + C & & & & \\ \text{solve for } y & & & & & \\ e^{\ln(y)} &= e^{\tilde{k}x + C} & & & & \end{aligned}$$

$$\begin{aligned} y &= e^{\tilde{k}x + C} \\ y &= e^{\tilde{k}x} e^C \\ y &= e^{\tilde{k}x} (C) \\ y &= Ce^{\tilde{k}x} \end{aligned}$$

$$\begin{aligned} \frac{1}{y} \frac{dy}{dx} &= \tilde{k}y \cdot \frac{1}{y} \\ \frac{1}{y} \frac{dy}{dx} &= \tilde{k} dx \\ \int \frac{1}{y} dy &= \int \tilde{k} dx \\ \ln(y) &= \tilde{k}x + C \\ y &= e^{\tilde{k}x + C} \\ y &= Ce^{\tilde{k}x} \end{aligned}$$

Example Find the particular solution to the differential equation $\frac{dy}{dx} = 2xy^2 + 3x^2y^2$, $y(1) = 1$

$$\begin{aligned} \frac{dy}{dx} &= 2xy^2 + 3x^2y^2 \\ \frac{dy}{dx} &= (2x + 3x^2)(y^2) \\ \frac{1}{y^2} dy &= (2x + 3x^2) dx \\ \int y^{-2} dy &= \int 2x + 3x^2 dx \\ \frac{y^{-1}}{-1} &= \frac{2x^2}{2} + \frac{3x^3}{3} + C \end{aligned}$$

$$\begin{aligned} -\frac{1}{y} &= x^2 + x^3 + C \\ y &= \frac{-1}{x^2 + x^3 + C} \\ y(1) &= \frac{-1}{1^2 + 1^3 + C} = 1 \\ &= \frac{-1}{2 + C} = 1 \\ -1 &= 2 + C \\ C &= -3 \end{aligned}$$

$$y(x) = \frac{-1}{x^2 + x^3 - 3}$$

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Implicit Solutions. There will be some D.E.'s for which we cannot directly solve for y .

Let $H(y) = G(x) + c$ be the result of using separation of variables on the differential equation $\frac{dy}{dx} = f(x, y)$ and suppose we can't or don't solve for y in terms of x we can then represent the solution implicitly with $H(y) - G(x) - C = 0$

If we name $K(x, y) = H(y) - G(x) - C = 0$ we call K the implicit solution of the differential equation.

Example: Solve the differential equation $\frac{dy}{dx} = \frac{-x}{y}$ $g(x) = -x$ $K(y) = \frac{1}{y}$

$$(y dx) \frac{dy}{dx} = \frac{-x}{y} (y dx)$$

$$y dy = -x dx$$

$$\int y dy = \int -x dx$$

$$\frac{y^2}{2} = -\frac{x^2}{2} + C$$

$$y^2 = 2(-\frac{x^2}{2} + C)$$

$$\rightarrow y = \pm \sqrt{2(-\frac{x^2}{2} + C)} \text{ explicit}$$

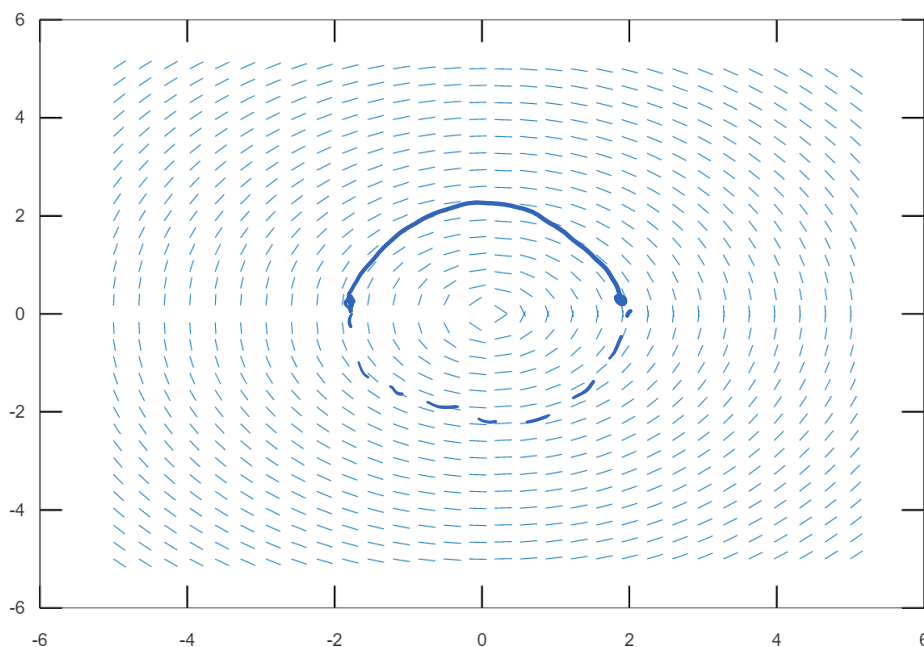
$$\frac{y^2}{2} + \frac{x^2}{2} + C = 0 \rightarrow y^2 + x^2 + C = 0$$

$$\boxed{x^2 + y^2 = C} \text{ Implicit}$$

(0,1)
 $y^2 + x^2 + 2C = 0$
 $1^2 + 0^2 = -2C$
 $C = -1/2$
 $y^2 + x^2 + 2(-1/2) = 0$
 $y^2 + x^2 - 1 = 0$
 $y^2 + x^2 = 1$
 $y^2 + x^2 + C = 0$
 $1^2 + 0^2 + C = 0$
 $1 + C = 0$
 $C = -1$

Let's also look at the slope field.

Slope Field

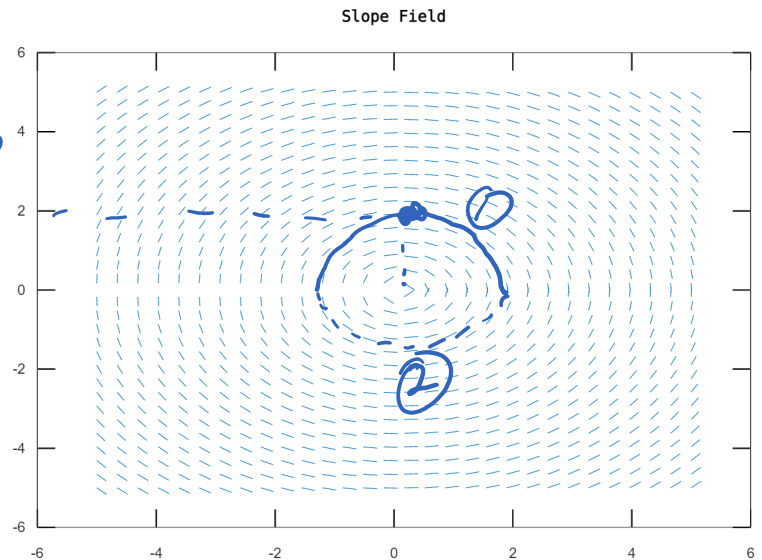


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In the last example we wrote down the implicit solution to $\frac{dy}{dx} = -\frac{x}{y}$ as $y^2 + x^2 - C = 0$ the equation of a circle. Suppose we are given the initial data $y(0) = 2$ what does this tell us about the particular solution?

$$\begin{aligned} y(0) &= 2 \\ y^2 + x^2 - C &= 0 \\ 2^2 + 0^2 - C &= 0 \\ 4 - C &= 0 \\ C &= 4 \\ x^2 + y^2 - 4 &= 0 \\ x^2 + y^2 &= 4 \\ \textcircled{1} \end{aligned}$$

$$\begin{aligned} y(0) &= -2 \\ y^2 + x^2 - C &= 0 \\ (-2)^2 + 0^2 - C &= 0 \\ 4 - C &= 0 \\ C &= 4 \\ x^2 + y^2 &= 4 \\ \textcircled{2} \end{aligned}$$



We must be careful though! We note that $\frac{dy}{dx} = -\frac{x}{y}$ is not defined at $y = 0$. This means that a solution to our D.E. cannot cross $y = 0$. So the picture of a full circle is not quite the correct one. Because of our derivative we need to restrict our solution to never cross the x-axis. So suppose we are given two initial data $y(0) = 2$ and $y(0) = -2$. How could we represent the solution to our differential equation?

For $y(0) = 2$ the set of all (x, y) that satisfy $x^2 + y^2 = 4$ with $y \geq 0$

Or

$$y = \sqrt{4 - x^2} \text{ on } [-2, 2]$$

For $y(0) = -2$ the set of all (x, y) that satisfy $x^2 + y^2 = 4$ with $y \leq 0$

Or

$$y = -\sqrt{4 - x^2} \text{ on } [-2, 2]$$

Also recall Theorem 1. In this case $f(x, y) = -\frac{x}{y}$ which is not continuous at $y = 0$ so we are not guaranteed a unique solution there, indeed if we had initial data $y(0) = 0$ we would have two solutions satisfying it!

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Some Applications

The natural growth equation. $\frac{dy}{dt} = ky$ We have seen this pop up a lot so far and that is because it can be used to model any phenomena for which the relevant quantity changes in proportion to that quantity. For example:

Population Growth: Suppose that b and d are the birth and death rates. Then the change in population P over the time Δt would be

$$\Delta P = b\Delta tP - d\Delta tP = (b - d)\Delta tP$$

$$\text{Then } \frac{\Delta P}{\Delta t} = (b - d)P$$

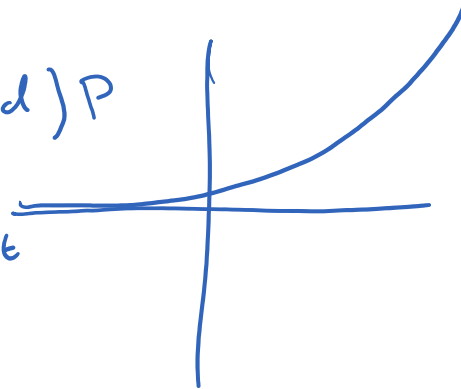
$$\text{And thus } \lim_{\Delta t \rightarrow 0} \frac{\Delta P}{\Delta t} = \frac{dP}{dt} = (b - d)P$$

$$\text{Or just } \frac{dP}{dt} = kP \quad k > 0$$

$$\lim_{\Delta t \rightarrow 0} (b - d)P = (b - d)P$$

$$\frac{dP}{dt} = (b - d)P$$

$$P(t) = e^{kt}$$



Radioactive decay

Suppose element A decays into Element B at a rate r over a time Δt and that B never turns into A. Then the change in the number of atoms, N , of element A over time Δt would be

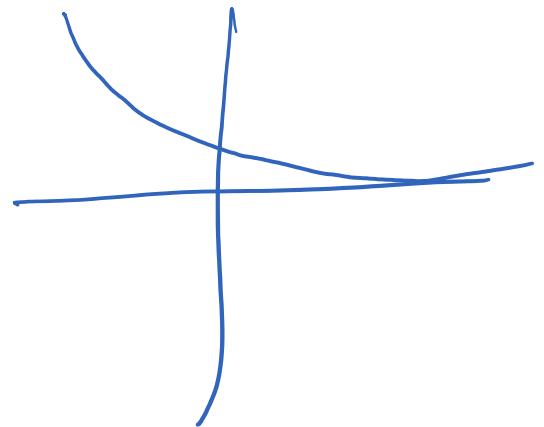
$$\Delta N = 0\Delta tN - r\Delta tN = (-r)\Delta tN$$

$$\text{Then } \frac{\Delta N}{\Delta t} = (-r)N$$

$$\text{And thus } \lim_{\Delta t \rightarrow 0} \frac{\Delta N}{\Delta t} = \frac{dN}{dt} = -rN$$

$$\text{Or just } \frac{dN}{dt} = -kN \quad k > 0$$

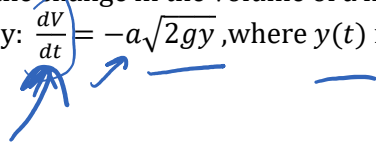
$$N(t) = e^{-kt}$$



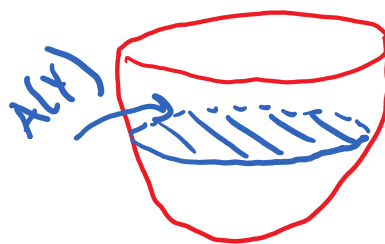
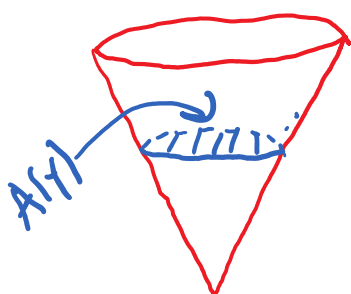
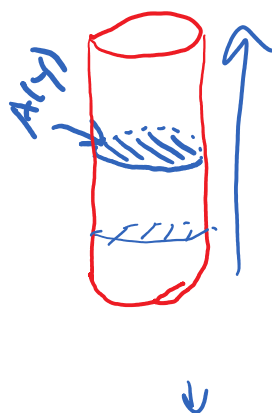
This equation shows up for compound interest, drug elimination in the body, viral spread and more! Whether something is growing or decaying just depends on the sign of the constant on the right hand side.

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Torricelli's law states that the change in the volume of a liquid like water in a tank that is being drained through a hole in the bottom is given by: $\frac{dV}{dt} = -a\sqrt{2gy}$, where $y(t)$ is depth of the liquid at time t .



Note that here we have two functions of time V and y and so we need to rewrite our DE in terms of just one of them in order to be able to solve it. Luckily integral calculus has us covered. Recall that we can find the volume of a solid by integrating along its center axis as a function of its cross-sectional area.



$$V = \int A(y) dy$$

$V = \int A(y) dy$ Remember that an integral is a sum and here we are adding up all the cross-sectional areas to get a volume! What happens when we take the derivative of this equation with respect to y

$$\frac{d}{dy} V = \frac{d}{dy} \left(\int A(y) dy \right)$$

$$\boxed{\frac{dV}{dy} = A(y)}$$

chain rule

$$\frac{dV}{dt} = \frac{dV}{dy} \frac{dy}{dt}$$

$$\frac{dV}{dt} = A(y) \frac{dy}{dt}$$

So we can write $\frac{dV}{dt} = A(y) \frac{dy}{dt}$, we can take this and replace the right hand side to have a DE just in terms of the function y !

$$\frac{dV}{dt} = -a\sqrt{2gy}$$

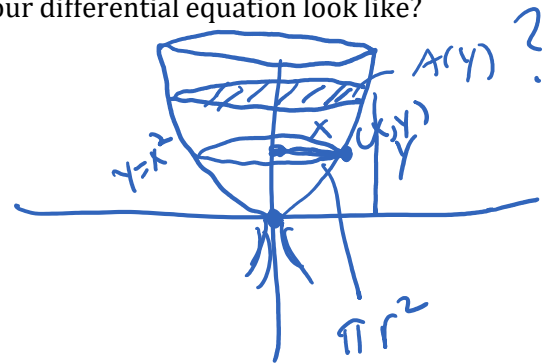
$$\boxed{A(y) \frac{dy}{dt} = -a\sqrt{2gy}}$$

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Let's suppose the tank was shaped like a parabola $y = x^2$. What would our differential equation look like?

$$A(y) \frac{dy}{dt} = -a\sqrt{2gy}$$

$$\pi y \frac{dy}{dt} = -a\sqrt{2gy}$$



$$\begin{aligned} y &= x^2 \\ x &= \sqrt{y} \\ A(y) &= \pi (\sqrt{y})^2 \\ &= \pi y \end{aligned}$$

What is the general solution?

$$\cancel{dt} \frac{\pi y \frac{dy}{dt}}{\sqrt{y}} = \frac{-a\sqrt{2gy}}{\sqrt{y}} dt$$

$$\int \pi (y)^{1/2} dy = \int -a\sqrt{2g} dt$$

$$\frac{2}{3} \pi y^{3/2} = (-a\sqrt{2g}t + C)^{3/2} \pi$$

$$y^{3/2} = \frac{-3}{2\pi} a\sqrt{2g}t + C$$

$$y(t) = \left(\frac{-3}{2\pi} a\sqrt{2g}t + C \right)^{2/3}$$

How could we get an equation for the volume from this?

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Let's solve the logistic equation $\frac{dy}{dt} = ky(M - y)$.

$$\cancel{dt} \frac{1}{y(m-y)} \frac{dy}{\cancel{dt}} = ky(M-y) \cancel{1/y(m-y)} dt$$

$$\rightarrow \int \frac{1}{y(m-y)} dy = \int k dt$$

$$= kt + C$$

want $\frac{1}{y(m-y)} = \frac{A}{y} + \frac{B}{(m-y)} \rightarrow$

$$\int \frac{1}{y(m-y)} dy = \int \frac{1}{my} dy + \int \frac{1}{m(m-y)} dy$$

$\downarrow u=m-y \quad du=-dy$

$$= \frac{1}{m} \ln(y) - \frac{1}{m} \ln(m-y) = kt + C$$

$$\frac{1}{m} \ln\left(\frac{y}{m-y}\right) = kt + C$$

$$\ln\left(\frac{y}{m-y}\right) = mkt + C$$

$$\frac{y}{m-y} = e^{mkt} \quad \text{solve for } y$$

$$1 = \frac{A(y)(m-y)}{y} + \frac{B(y)(m-y)}{(m-y)}$$

$$1 = A(m-y) + By$$

$$1 = A(m)$$

$$A = 1/m$$

$$1 = B(m)$$

$$B = 1/m$$

$$y = m$$

$$\leftarrow y=0$$

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$$\frac{y}{m-y} = ce^{mkt}$$

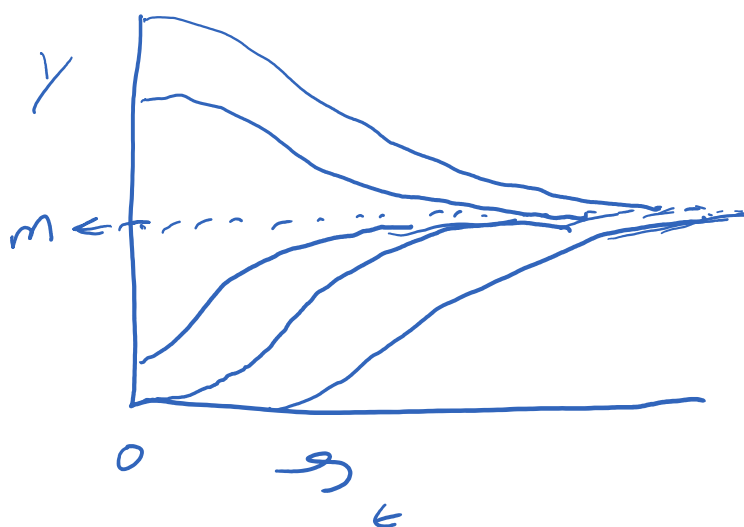
$$y = (ce^{mkt})(m-y)$$

$$y = cm e^{mkt} - yce^{mkt}$$

$$y + yce^{mkt} = cm e^{mkt}$$

$$y(1 + ce^{mkt}) = cm e^{mkt}$$

$$y(t) = \frac{cm e^{mkt}}{1 + ce^{mkt}}$$



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Just before midday the body of an apparent homicide victim is found in a room that is kept at a constant temperature of 70F. At 12 noon the temperature of the body is 80F and 1:00pm it is 75F. Assume that the temperature of the body at the time of death was 98.6F and that it has cooled in accord with Newton's law of cooling. What was the time of death?

Newton's law of Cooling is given by: $\frac{dT}{dt} = -k(T - A)$

$k(T - A)$ T temperature
 k cooling constant
 A ambient temp

$$\frac{dT}{dt} = -k(T - A)$$

$$\int \frac{dT}{T - A} = \int -k dt$$

$u = T - A$
 $du = dT$

$$= -kt + C$$

$$\ln(T - A) = -kt + C$$

$$T - A = Ce^{-kt}$$

$$T(t) = Ce^{-kt} + A$$

$$t = 0 \rightarrow 80 = T$$

$$T(0) = 80 \quad T(1) = 75$$

Find k

$$T(0) = 80 = Ce^{-k \cdot 0} + 70$$

$$T(1) = 75 = Ce^{-k} + 70$$

$$\int \frac{1}{u} du = \ln|u|$$

$$80 = C + 70$$

$$75 = Ce^{-k} + 70$$

$$10 = C$$

$$75 = Ce^{-k} + 70$$

$$5 = 10e^{-k}$$

$$\frac{1}{2} = e^{-k}$$

$$k = -\ln\left(\frac{1}{2}\right) = \ln(2)$$

$$T(t) = 10e^{-\ln(2)t} + 70$$

$$98.6 = 10e^{-\ln(2)t} + 70$$

$$28.6 = 10e^{-\ln(2)t}$$

$$2.86 = e^{-\ln(2)t}$$

$$\ln(2.86) = -\ln(2)t \rightarrow t = \frac{-\ln(2.86)}{\ln(2)}$$

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Radio Active Decay: Carbon extracted from ancient bone found in a cave has only $1/6$ the amount of C^{14} than samples from bones taken today. How old is the ancient bone? The Half-life of C^{14} is 5730 years.

$$\frac{dN}{dt} = -kN \quad k > 0$$

$$\frac{dN}{N} = -k dt$$

$$\ln(N) = -kt + C$$

$$N = e^{-kt + C}$$

$$N = Ce^{-kt}$$

$$N(0) = Ce^{-k(0)}$$

$$\frac{1}{2}N(0) = Ce^{-k(5730)}$$

$$\frac{1}{2} = e^{-5730k}$$

$$\ln(1/2) = -5730k$$

$$k = \frac{-\ln(1/2)}{5730} = \frac{\ln(2)}{5730}$$

$$N(t) = Ce^{-\frac{\ln(2)}{5730}t}$$

Ancient bone has $\frac{N_0}{6}$ amount C^{14} so when it was alive

$$6N_0 = C^{14}$$

$$N(t) = Ce^{-\frac{\ln(2)}{5730}t}$$

$$N(0) = N_0 = Ce^0$$

$$C = N_0 = e^{-\frac{\ln(2)}{5730}t}$$

$$N(t) = N_0 e^{-\frac{\ln(2)}{5730}t}$$

Find t when $N(t) = \frac{1}{6}N_0$

$$\frac{1}{6}N_0 = N_0 e^{-\frac{\ln(2)}{5730}t}$$

$$\ln(1/6) = -\frac{\ln(2)}{5730}t$$

$$t = -5730 \frac{\ln(1/6)}{\ln(2)} =$$

$$-14,811. \dots$$

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Crypto Currency is about the only way to earn above 1% interest on your savings. Algorand is a crypto currency that has a variable interest rate based on supply and demand for it ranging from 2% to 8%. Lets suppose the effective interest rate over a 10 year period was 6% compounded continuously. If you had originally bought 10,000 algo how many algo do you have after 10 years?

$$\frac{dA}{dt} = KA$$

$$K \geq 0$$

$$K = 0.06$$

$$A = Ce^{Kt}$$

$$A(0) = 10,000 = Ce^{K(0)}$$

$$A(t) = 10,000 e^{0.06t}$$

$$= 10,000 e^{0.06(10)} =$$

$$10,000 e^{0.6} = 18,221$$

$$A_0(1 + r/c)^t$$

$\lim_{c \rightarrow \infty} A_0(1 + r/c)^t = A_0 e^{rt}$

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To celebrate getting an A in Math 383 you decide to go skydiving jumping out of a plane at 8000ft. After you jump you fall for 15 seconds before opening your parachute. During your fall the drag due to air resistance causes an acceleration of $-kv$ where v is your velocity. Before the chute opens $k = 0.1$ and after you open the chute $k = 1.6$. How long were you in the air?

$$a(t) = -32 - \underline{kv}$$

$$\rightarrow \frac{dv}{dt} = -32 - kv = -(32 + kv)$$

$$\int \frac{dv}{32 + kv} = \int -dt$$

↓

$$u = 32 + kv$$

$$du = k dv \quad \int \frac{1}{u} du$$

$$\frac{1}{k} du = dv$$

$$\frac{1}{k} \ln(32 + kv) = -t + C$$

$$\ln(32 + kv) = -kt + C$$

$$32 + kv = Ce^{-kt}$$

$$v = \frac{1}{k} (Ce^{-kt} - 32)$$

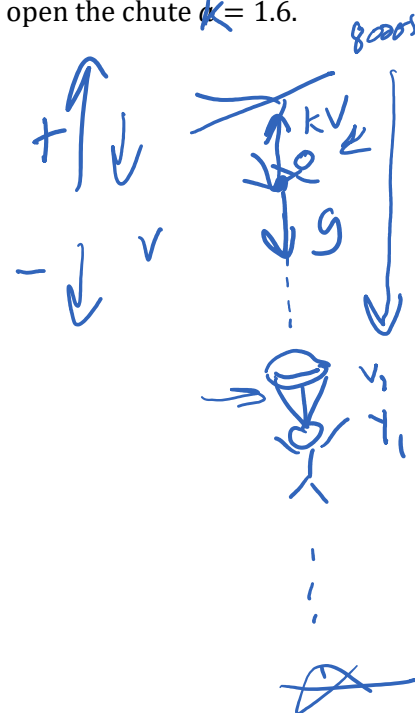
$$v(0) = v_0 = \frac{1}{k} (Ce^0 - 32)$$

$$kv_0 = C - 32$$

$$C = kv_0 + 32$$

$$v = \frac{1}{k} ((kv_0 + 32)e^{-kt} - 32)$$

$$v = \left(v_0 + \frac{32}{k}\right)e^{-kt} - \frac{32}{k}$$



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$$v = (v_0 + \frac{32}{k})e^{-kt} - \frac{32}{k} \quad (1)$$

$$\frac{dy}{dt} = (v_0 + \frac{32}{k})e^{-kt} - \frac{32}{k}$$

$$y = \int (v_0 + \frac{32}{k})e^{-kt} - \frac{32}{k} dt$$

$$y = -\frac{1}{k}(v_0 + \frac{32}{k})e^{-kt} - \frac{32}{k}t + C$$

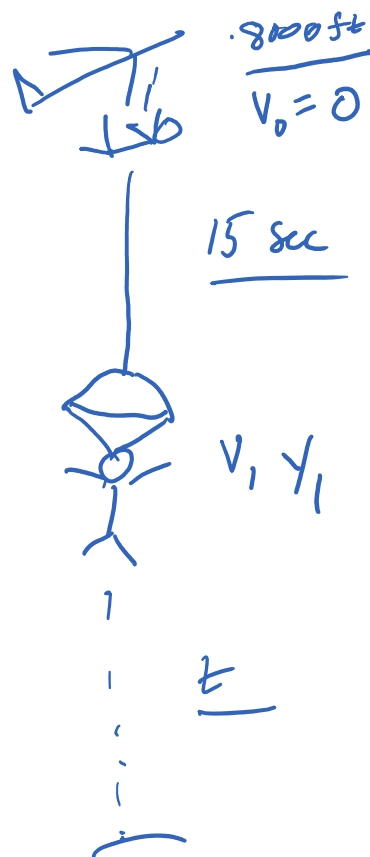
$$y(0) = y_0 = -\frac{1}{k}(v_0 + \frac{32}{k})e^0 - \frac{32}{k}(0) + C$$

$$y_0 = -\frac{1}{k}(v_0 + \frac{32}{k}) + C$$

$$C = y_0 + \frac{1}{k}(v_0 + \frac{32}{k})$$

$$y(t) = -\frac{1}{k}(v_0 + \frac{32}{k})e^{-kt} - \frac{32}{k}t + y_0 + \frac{1}{k}(v_0 + \frac{32}{k}) \quad (2)$$

plug in $v_0=0$, $y_0=8000$, $t=15$, $k=1$ into (1) (2)
that will give v_1, y_1 The velocity and Height
when you open the chute. Then take v_1, y_1
as init cond plug them in (2) with $k=1.6$
set $y=0$ and solve for t .



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A partial differential equation $\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}$ A different use of separation of variables.

$$u(x, t) = \phi(x) \theta(t)$$

$$\frac{\partial}{\partial t} \phi(x) \theta(t) = k \frac{\partial^2}{\partial x^2} \phi(x) \theta(t)$$

$$\frac{\phi(x) \theta'(t)}{\phi(x) \theta(t)} = k \frac{\phi''(x) \theta(t)}{\phi(x) \theta(t)}$$

$$\frac{\theta'(t)}{\theta(t)} = k \frac{\phi''(x)}{\phi(x)} = \text{const} = \lambda$$

$$\frac{\theta'(t)}{\theta(t)} = \lambda \quad k \frac{\phi''(x)}{\phi(x)} = \lambda$$

↓ ↓

$$\theta' = \lambda \theta$$

$$\theta(t) = C e^{\lambda t}$$

↓

$$\phi'' = \frac{1}{k} \lambda \phi$$

