

Section 1.5 Linear First Order Equations

A linear first order differential equation is one of the form:

$$\frac{dy}{dx} + yP(x) = Q(x)$$

If we carefully examine the left hand side it almost looks like a product rule in that we see a sum involving y and the derivative of y , but clearly something is missing. However imagine if the left hand side was just a product rule after taking a derivative with respect to x . Then we could solve the DE by simply integrating both sides! What we will do is multiply both sides of the linear first order by something so that the left hand side becomes a product rule.

$$\int \frac{d}{dx}(y h) = \int \tilde{Q}(x)$$

$$\frac{d}{dx}fg = \frac{df}{dx}g + f\frac{dg}{dx}$$

Let's call the mystery function that we want $I(x)$.

$$\frac{d}{dx}fg = \frac{df}{dx}g + f\frac{dg}{dx}$$

$$\frac{dy}{dx} + yP(x) = Q(x)$$

$$\frac{dy}{dx} I(x) + yP(x)I(x) = Q(x)I(x)$$

$$\downarrow$$

$$\frac{dI}{dx}$$

So to get what we are after we need, $\frac{dI}{dx} = P(x)I(x)$

$$P(x)e^{f(x)} \Rightarrow P(x) = f'(x) \quad \frac{d}{dx} e^{f(x)} = e^{f(x)} \cdot f'(x) = f'(x)e^{f(x)}$$

$$\Rightarrow f(x) = \int P(x) \quad I = e^{f(x)}$$

$$\Rightarrow I = e^{\int P(x) dx}$$

$$\frac{d}{dx} y e^{f(x)}$$

$$\frac{dy}{dx} e^{f(x)} + y \frac{d}{dx} e^{f(x)} = Q e^{f(x)}$$

$$\frac{dy}{dx} e^{f(x)} + y e^{f(x)} \frac{d}{dx} f(x) = Q e^{f(x)}$$

$$\frac{dy}{dx} e^{f(x)} + y e^{f(x)} P(x) = Q e^{f(x)}$$

$$\frac{d}{dx} (y(x) I(x))$$

$$\rightarrow I = C$$

$$' \frac{d}{dx} (y(x) I(x))$$

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So if we multiply both sides of the equation

$$\frac{dy}{dx} + yP(x) = Q(x)$$

by

$$I(x) = e^{\int P(x) dx}$$

We can re-write the equation as:

$$\frac{d}{dx}(I(x)y(x)) = Q(x)I(x)$$

$$\int \frac{d}{dx}(I(x)y(x)) dx = \int Q(x)I(x) dx$$

$$I(x)y(x) + C = \int Q(x)I(x) dx$$

$$y(x) = \frac{C}{I(x)} + \frac{1}{I(x)} \int Q(x)I(x) dx$$

And solve by then just integrating both sides of the equation then solving for y.

Example: Remember this one? $\frac{dv}{dt} = 32 - kv$

$$\frac{dv}{dt} + kv = 32$$

$$P(t) = k$$

$$Q(t) = 32$$

$$I = e^{\int P(t) dt} = e^{\int k dt}$$

$$= e^{kt}$$

$$\frac{dv}{dt} e^{kt} + k e^{kt} v = 32 e^{kt}$$

$$\int \frac{d}{dt}(v e^{kt}) = \int 32 e^{kt}$$

$$v e^{kt} = 32 \frac{1}{k} e^{kt} + C$$

$$v = \frac{32}{k} \frac{e^{kt}}{e^{kt}} + \frac{C}{e^{kt}}$$

$$v = \frac{32}{k} + C e^{-kt}$$

$$\frac{d}{dt}(v e^{kt}) = \frac{dv}{dt} e^{kt} + v k e^{kt}$$

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Example: Find the General solution to the D.E. $\frac{dy}{dx} + y \cot(x) = \cos(x)$

Step 1. Find integrating factor:

$$I(x) = e^{\int P(x) dx} = e^{\int \cot(x) dx}$$

$$= e^{\ln|\sin(x)|} = \sin(x) = I(x)$$

$$\int \cot(x) = \int \frac{\cos(x)}{\sin(x)}$$

$$u = \sin(x)$$

$$du = \cos(x) dx$$

$$\int \frac{1}{u} du$$

$$= \ln(u)$$

$$= \ln|\sin(x)|$$

$$\frac{dy}{dx} \sin(x) + y \cot(x) \sin(x) = \sin(x) \cos(x)$$

$$- \frac{dy}{dx} \sin(x) + y \frac{\cos(x)}{\sin(x)} \sin(x) = \sin(x) \cos(x)$$

$$- \int \frac{d}{dx} (y \sin(x)) = \int \sin(x) \cos(x) \rightarrow$$

$$y \sin(x) = \frac{\sin^2(x)}{2} + C$$

$$y = \frac{\sin(x)}{2} + \frac{C}{\sin(x)}$$

$$y = \frac{1}{2} \sin(x) + C \csc(x)$$

$$\left\{ \begin{aligned} & \frac{d}{dx} (y \sin(x)) \\ &= \frac{dy}{dx} \sin(x) + y \cos(x) \end{aligned} \right.$$

$$u = \sin(x)$$

$$du = \cos(x) dx$$

$$\int u du = \frac{u^2}{2}$$

$$= \frac{\sin^2(x)}{2}$$

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$$y' + p(x)y = q(x)$$

Example: find the particular solution to the DE $(x^2 + 4)y' + 3xy = x, y(0) = 1$

$$\frac{(x^2+4)y'}{x^2+4} + \frac{3xy}{x^2+4} = \frac{x}{x^2+4}$$

$$p(x) = \frac{3x}{x^2+4} \quad q(x) = \frac{x}{x^2+4}$$

$$\rightarrow y' + y\left(\frac{3x}{x^2+4}\right) = \frac{x}{x^2+4}$$

① Find I.f

$$I = e^{\int p(x) dx} = e^{\int \frac{3x}{x^2+4} dx}$$

$$u = x^2 + 4 \\ du = 2x dx \rightarrow dx = \frac{du}{2x}$$

$$= \int \frac{3x \frac{du}{2x}}{u} = \int \frac{3}{2} \frac{1}{u} du = \frac{3}{2} \ln(u)$$

$$= e^{\frac{3}{2} \ln(x^2+4)} = e^{\ln((x^2+4)^{3/2})} = (x^2+4)^{3/2}$$

② multiply everything by I

$$y' (x^2+4)^{3/2} + y \left(\frac{3x}{x^2+4} \right) (x^2+4)^{3/2} = \frac{x}{(x^2+4)^{1/2}}$$

$$\frac{d}{dx} \left(y (x^2+4)^{3/2} \right) = x (x^2+4)^{1/2}$$

③ Integrate both sides

$$\int \frac{d}{dx} \left(y (x^2+4)^{3/2} \right) dx = \int x (x^2+4)^{1/2} dx$$

$$u = x^2 + 4 \\ \frac{1}{2} du = x dx$$

$$y (x^2+4)^{3/2} = \frac{1}{2} \int u^{1/2} du$$

$$y (x^2+4)^{3/2} = \frac{1}{2} \cdot \frac{2}{3} u^{3/2} + C$$

$$y (x^2+4)^{3/2} = \frac{1}{3} (x^2+4)^{3/2} + C$$

step ④ solve for y

$$y = \frac{1}{3} + \frac{C}{(x^2+4)^{3/2}}$$

$$y(0) = \frac{1}{3} + \frac{C}{(0^2+4)^{3/2}} = 1$$

$$C = (1 - \frac{1}{3}) 4^{3/2} \\ = (\frac{2}{3}) 8 = \frac{16}{3}$$

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Theorem: If the functions $P(x)$ and $Q(x)$ are continuous on an interval I containing the point x_0 then the differential equation

$$\frac{dy}{dx} + yP(x) = Q(x), \quad y(x_0) = y_0$$

Has a unique solution on the interval I .

Proof:

$$y(x) = \frac{C}{e^{\int P(x) dx}} + \frac{1}{e^{\int P(x) dx}} \int Q(x) e^{\int P(x) dx} dx$$

$$(1) \quad y(x) = C e^{-\int P(x) dx} + e^{-\int P(x) dx} \int Q(x) e^{\int P(x) dx} dx$$

$$(1) \text{ solves our DE } \frac{dy}{dx} + yP(x) = Q(x)$$

Since we assume $P(x), Q(x)$ are continuous on I

$\Rightarrow \int P(x) dx$ exists and $\int Q(x) e^{\int P(x) dx} dx$ exists on I

$\Rightarrow y(x)$ exists on I

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Solve the DE

(type in before notes)

$$\frac{1}{x^2} \left(x^2 \frac{dy}{dx} + xy \right) = \sin(x), y(1) = y_0$$

$$\frac{dy}{dx} + \frac{1}{x}y = \frac{\sin(x)}{x^2} \rightarrow \frac{dy}{dx} + yP(x) = Q(x)$$

$P(x) = \frac{1}{x} \quad Q(x) = \frac{\sin(x)}{x^2}$

$$I = e^{\int \frac{1}{x} dx} = e^{\ln(x)} = x$$

$$\frac{dy}{dx} x + \frac{x}{x} y = \frac{\sin(x)}{x^2} x$$

$$\frac{dy}{dx} x + y = \frac{\sin(x)}{x}$$

↓ same as

$$\int \frac{d}{dx}(yx) = \int \frac{\sin(x)}{x}$$

$$yx = y_0 + \int_1^x \frac{\sin(t)}{t} dt$$

$$y = \frac{y_0}{x} + \frac{1}{x} \int_1^x \frac{\sin(t)}{t} dt$$

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So we have

$$y(x) = \frac{1}{x} \left[y_0 + \int_1^x \frac{\sin(t)}{t} dt \right]$$

$$y(4) = \frac{1}{4} \left[y_0 + \int_1^4 \frac{\sin(t)}{t} dt \right]$$

Lets find $y(4)$ using Simpson's rule to approximate the otherwise intractable integral, how could you make a plot of the solution from $t = 0..10$

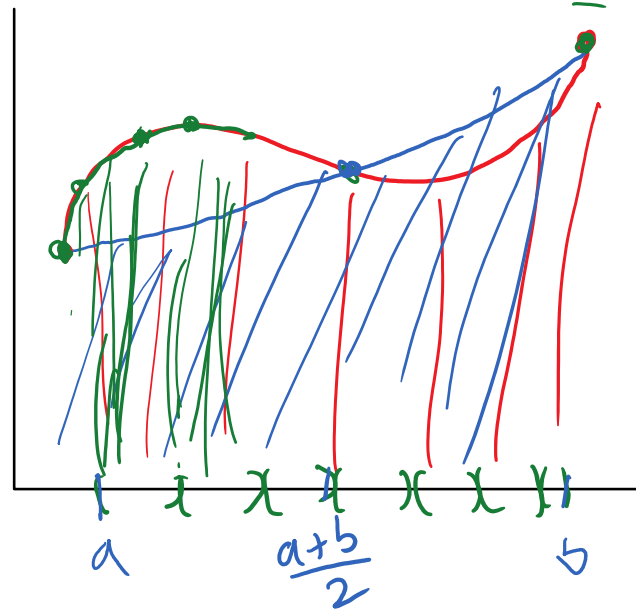
Simpsons rule: $\frac{(b-a)}{6} (f(a) + 4f(m) + f(b))$

It's easy to integrate a quadratic function so for function that might not be so easy to integrate we can approximate it with a parabola. Consider a function $f(x)$ on an interval (a, b) and the midpoint of that interval $m = \frac{a+b}{2}$. We can fit a quadratic function $q(x) = Ax^2 + Bx + C$ to go through all three points by solving the following system for the coefficients A, B, C

$$\begin{aligned} f(a) &= Aa^2 + Ba + C \\ f(m) &= Am^2 + Bm + C \\ f(b) &= Ab^2 + Bb + C \end{aligned}$$

Then we could integrate, however there is a nicer formula for the area under that quadratic.

$$\begin{aligned} \int_a^b Ax^2 + Bx + C dx &= \frac{A}{3}(b^3 - a^3) + \frac{B}{2}(b^2 - a^2) + C(b - a) = \frac{1}{6} (2A(b^3 - a^3) + 3B(b^2 - a^2) + 6C(b - a)) \\ &= \frac{b-a}{6} (2A(b^2 + ba + a^2) + 3B(b + a) + 6C) \\ &= \frac{b-a}{6} ((Aa^2 + Ba + C) + (Ab^2 + Bb + C) + Aa^2 + Ab^2 + 2Aab + 2Bb + 2Ba + 4C) \\ &= \frac{b-a}{6} ((Aa^2 + Ba + C) + (Ab^2 + Bb + C) + A(a^2 + b^2 + 2ab) + 2B(a + b) + 4C) \\ &= \frac{b-a}{6} ((Aa^2 + Ba + C) + (Ab^2 + Bb + C) + (A(a+b)^2 + 2B(a+b) + 4C)) \\ &= \frac{b-a}{6} \left(\underbrace{(Aa^2 + Ba + C)}_{q(a)} + \underbrace{(Ab^2 + Bb + C)}_{q(b)} + 4 \left(\underbrace{A \left(\frac{a+b}{2} \right)^2 + B \frac{(a+b)}{2} + C}_{q(m)} \right) \right) \\ &= \frac{(b-a)}{6} (f(a) + 4f(m) + f(b)) \end{aligned}$$



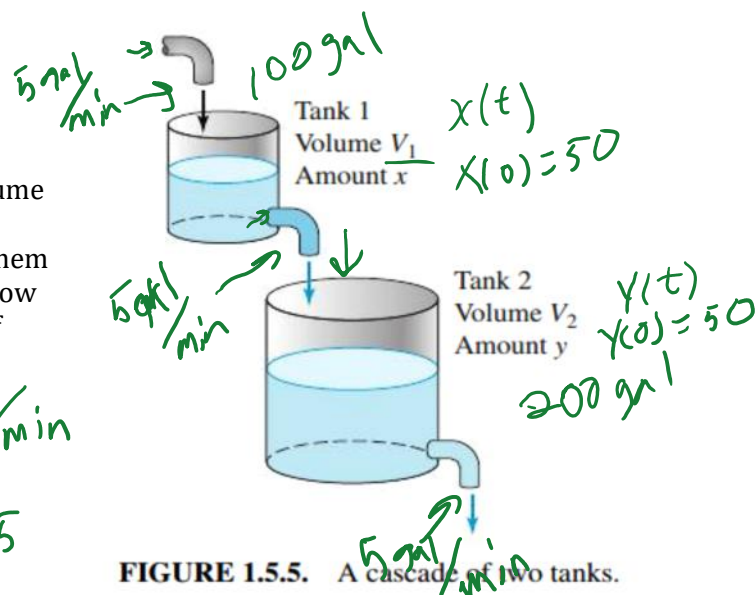
In octave or matlab:

```
N=100 %number of sub intervals
a=1; % start point
b=4 %end point
y0=2; %initial value
f= @(x)sin(x)/x %function we are integrating
x=linspace(a,b,N) % partition interval into N peices
sum=0; % initialize sum
for i=2:N % use simpson's rule on each sub interval and add them up
    sum=sum+((x(i)-x(i-1))/6)*(f(x(i-1))+4*f((x(i)+x(i-1))/2)+f(x(i)))
end;
sum=(y0+sum)./b % add y0 and divide by "x" which is our endpoint
```

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Buckets of fun!

Consider two tanks one of volume $V_1 = 100 \text{ gal}$ and one of volume $V_2 = 200 \text{ gal}$ each with amounts of salt $x(t), y(t)$ in them respectively. Suppose both tanks start with 50 lbs of salt in them each and that fresh water is flowing in to tank 1. Assume all flow rates for each spout are 5 gallons per min. Find the amount of salt $x(t)$ in tank 1 and $y(t)$ in tank 2 as functions of time.



Hint: Think about (rate in) - (rate out)

$$\frac{dx}{dt} = 0 - \frac{x(t)}{V_1} \cdot 5 \rightarrow \frac{dx}{dt} = -\frac{x}{100} \cdot 5$$

$$\Rightarrow \boxed{\frac{dx}{dt} = -\frac{x}{20}} \quad (1)$$

$$\frac{dy}{dt} = \frac{x}{V_1} \cdot 5 - \frac{y}{V_2} \cdot 5 \Rightarrow \frac{dy}{dt} = \frac{x}{20} - \frac{y}{40} \cdot 5$$

$$\Rightarrow \boxed{\frac{dy}{dt} = \frac{x}{20} - \frac{y}{40}} \quad (2)$$

$$\textcircled{1} \quad \frac{dx}{dt} + \frac{x}{20} = 0 \rightarrow \begin{matrix} p(t) = 1/20 \\ q(t) = 0 \end{matrix} \quad I(t) = e^{\int p(t) dt} = e^{\int 1/20 dt} = e^{1/20 t}$$

$$\frac{dx}{dt} e^{1/20 t} + x \frac{1}{20} e^{1/20 t} = 0 \Rightarrow \frac{d}{dt} (x e^{1/20 t}) = 0 \Rightarrow x e^{1/20 t} = C$$

$$\Rightarrow x(t) = C e^{-1/20 t}, \quad x(0) = 50 = C e^0 \Rightarrow C = 50$$

$$\boxed{x(t) = 50 e^{-1/20 t}}$$

$$\textcircled{2} \quad \frac{dy}{dt} = \frac{x}{20} - \frac{y}{40} = \frac{50 e^{-1/20 t}}{20} - \frac{y}{40}$$

$$= \frac{dy}{dt} + \frac{y}{40} = \frac{5}{2} e^{-1/20 t}$$

$$p(t) = 1/40$$

$$Q(t) = \frac{5}{2} e^{-1/20 t}$$

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$$= \frac{dy}{dt} + \frac{y}{40} = \frac{5}{2} e^{-1/20 t} \quad P(t) = 1/40 \quad Q(t) = \frac{5}{2} e^{-1/20 t}$$

$$I(t) = e^{\int P(t) dt} = e^{\int 1/40 dt} = e^{1/40 t}$$

$$\frac{dy}{dt} e^{1/40 t} + \frac{y}{40} e^{1/40 t} = \frac{5}{2} e^{-1/20 t} e^{1/40 t} \rightarrow \frac{5}{2} e^{-1/40 t}$$

$$\frac{d}{dt} (y e^{1/40 t}) = \frac{5}{2} e^{-1/40 t}$$

$$y e^{1/40 t} = \frac{5}{2} \int e^{-1/40 t} dt = \frac{5}{2} (-40 e^{-1/40 t}) + C$$

$$y e^{1/40 t} = -100 e^{-1/40 t} + C$$

$$y(t) = \frac{-100 e^{-1/40 t}}{e^{1/40 t}} + C e^{1/40 t}$$

$$y(t) = -100 e^{(-1/20)t} + C e^{-1/40 t}$$

$$y(0) = 50 = -100 e^0 + C e^0 \Rightarrow 50 = -100 + C$$

$$C = 150$$

$$y(t) = -100 e^{-1/20 t} + 150 e^{-1/40 t}$$

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A roller coaster is made of steel and is designed to withstand the contraction and expansion of the steel with the temperature changes of the day up to some critical amplitude. Suppose the ambient temperature during the theme parks open season goes as $A(t) = a_0 + a_1 \cos(\omega t) + b_1 \sin(\omega t)$, where ω is the frequency of oscillation. If we want the temperature to oscillate every 24 hours we want a period of about $6\pi \approx 24$ so we might take $\omega = \frac{2\pi}{24} = \frac{\pi}{12}$

What differential equation could we use to model the temperature of the steel holding the coaster up?

$$\frac{dT}{dt} = K(A - T)$$

$$\frac{dT}{dt} = k(a_0 + a_1 \cos(\omega t) + b_1 \sin(\omega t) - T)$$

What is the general solution?

$$\frac{dT}{dt} + kT = k(a_0 + a_1 \cos(\omega t) + b_1 \sin(\omega t))$$

$$P(t) = k \quad Q(t) = k a_0 + a_1 \cos(\omega t) + b_1 \sin(\omega t)$$

$$I = e^{\int k dt} = e^{kt}$$

$$\frac{dT}{dt} e^{kt} + kT e^{kt} = k(a_0 + a_1 \cos(\omega t) + b_1 \sin(\omega t)) e^{kt}$$

$$\int \frac{d}{dt} (T e^{kt}) = \int k a_0 e^{kt} + k a_1 \cos(\omega t) e^{kt} + k b_1 \sin(\omega t) e^{kt}$$

$$T e^{kt} = \frac{k a_0}{k} e^{kt} + \int k a_1 \cos(\omega t) e^{kt} + \int k b_1 \sin(\omega t) e^{kt}$$

$$\begin{aligned} u &= \cos(\omega t) & du &= -\sin(\omega t) \\ v &= e^{\omega t} & dv &= \omega e^{\omega t} \end{aligned}$$

$$\frac{1}{\omega} e^{kt} \cos(\omega t) - \int \frac{1}{\omega} e^{kt} (-\sin(\omega t))$$

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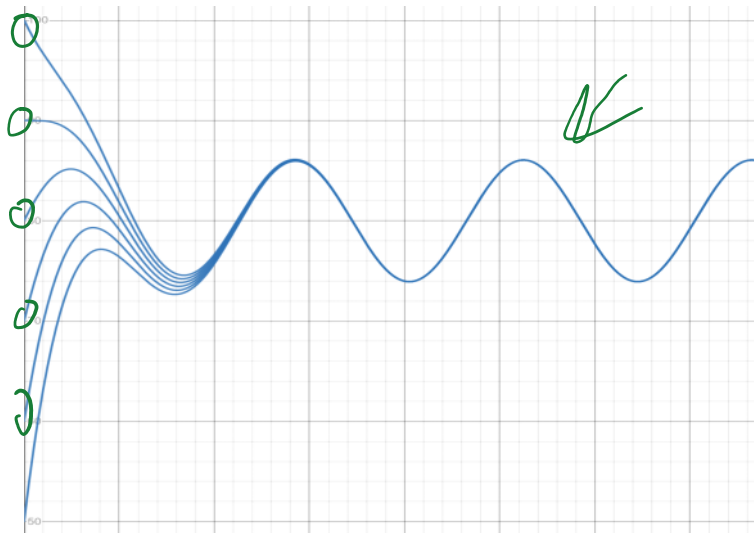
Let $a_0 = 80, a_1 = -5, b_1 = 5\sqrt{3}, \omega = \frac{\pi}{12}$,

Plugging in the constants we get

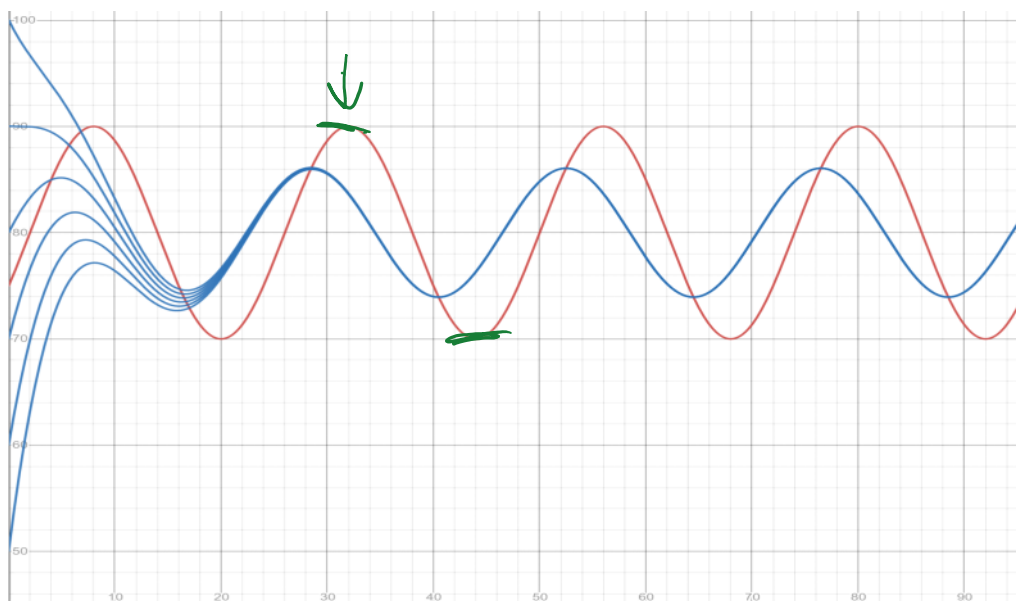
$$u(t) = 80 + e^{-kt}(u_0 - 82.3) + 2.33 \cos\left(\frac{\pi t}{12}\right) - 5.60 \sin\left(\frac{\pi t}{12}\right)$$

Let's plot this in Desmos and add some sliders for u_0 and k

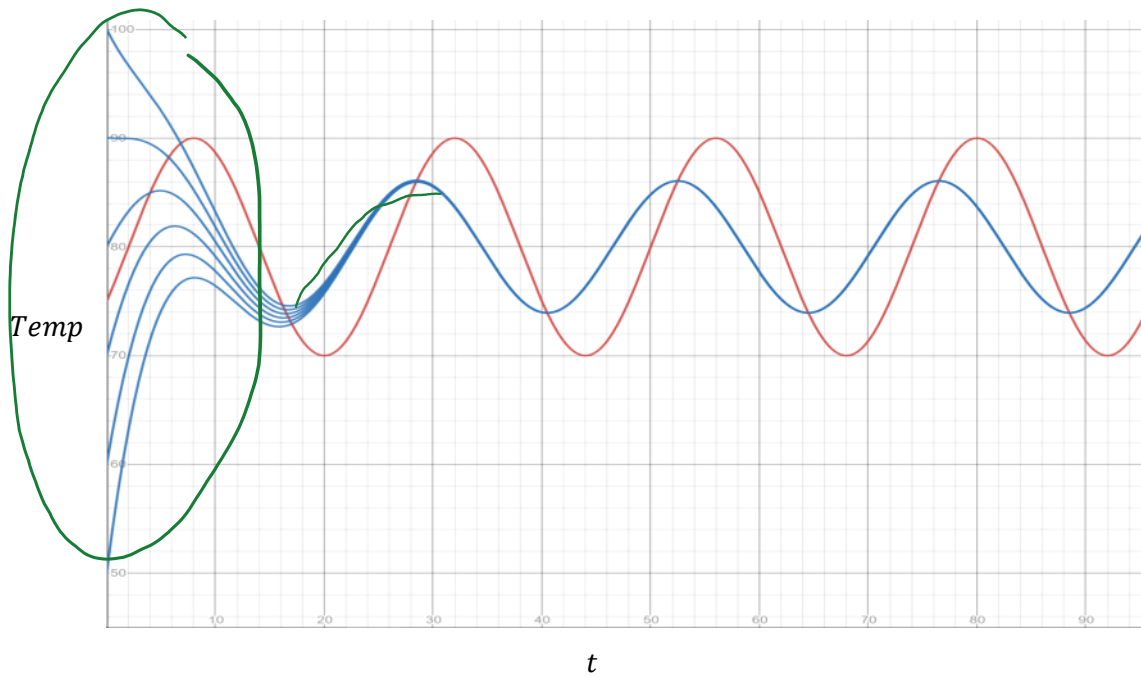
So we notice that whatever initial conditions we use we end up at the same solution eventually... Why is that?



Let's see how this curve compares to that of the outside temp itself...



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So we can see that the temperature of the steel has smaller amplitude of oscillation than the outside temperature, and we can calculate what that amplitude is if we know how the ambient temperature will change. If we knew the maximum oscillation allowed for the temperature of the steel we could determine under which conditions it would be safe to operate the coaster!