

Section 2.1 Population Models

We have already looked at two population models so far specifically,

$$\frac{dP}{dt} = kP$$

The Natural Growth Equation

$$\frac{dP}{dt} = kP(M - P)$$

The Logistic Equation

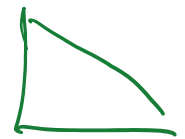
Both are more specific cases of the differential equation

$$\frac{dP}{dt} = (\beta(t) - \delta(t))P$$

Where $\beta(t)$ is a time dependent birth rate and $\delta(t)$ is a time dependent death rate.

For the natural growth equation β and δ are taken as constants.

$$\frac{dP}{dt} = (\beta - \delta)P \quad k = \beta - \delta \Rightarrow \frac{dP}{dt} = kP$$



For the logistic equation, the birth rate is taken to be a linear function of the population itself $\beta(t) = \beta_0 - \beta_1 P(t)$ and δ is still taken as constant. Note that the birth rate decreases as the population increases.

$$\begin{aligned} \frac{dP}{dt} &= (\beta_0 - \beta_1 P - \delta)P = (\beta_0 - \delta)P - P^2 \beta_1 \\ &= P(\beta_0 - \delta - P\beta_1) = \beta_1 P \left(\frac{\beta_0 - \delta}{\beta_1} - P \right) \\ \beta_1 &= k \quad M = \frac{\beta_0 - \delta}{\beta_1} \\ &= kP(M - P) \end{aligned}$$

Section 2.1 Population Models

Doomsday vs Extinction

The Earth is under attack! An alien species has sent an automated ship carrying an invasive species designed to get rid of all the humans before their arrival. The species itself reproduces quickly with a birth rate proportional to its population, $\beta(t) = kP$. The death rate however is constant. From our general population model we can see that the population is modeled by:

$$\left(\beta(t) - \delta \right) \frac{dP}{dt} = \underbrace{(kP - \delta)}_{\substack{\beta(t) - \delta \\ \sim}} P = \underbrace{kP^2 - \delta P}_{\sim} = \underbrace{kP \left(P - \frac{\delta}{k} \right)}_{\sim} = \underbrace{kP(P - M)}_{\sim}$$

Suppose $k=0.0004$ and that $\frac{\delta}{k} = M = 150$ find the general solution to the DE is there any hope for the Earth?

$$\frac{dP}{dt} = kP(P-M) \Rightarrow \frac{dP}{P(P-M)} = k dt$$

$$\int \frac{dP}{P(P-M)} = \int k dt = kt + C$$

↓

$$\frac{1}{P(P-M)} = \frac{A}{P} + \frac{B}{P-M} \Rightarrow 1 = \frac{A(P)(P-M)}{P} + \frac{B(P)(P-M)}{P-M}$$

$$\Rightarrow 1 = A(P-M) + BP \quad \text{choose } P=0 \Rightarrow 1 = A(0-M) + B(0)$$

$$\Rightarrow 1 = -MA \Rightarrow \boxed{A = -1/M}$$

choose $P=M$

$$1 = A(M-M) + BM \Rightarrow 1 = BM \Rightarrow \boxed{B = 1/M}$$

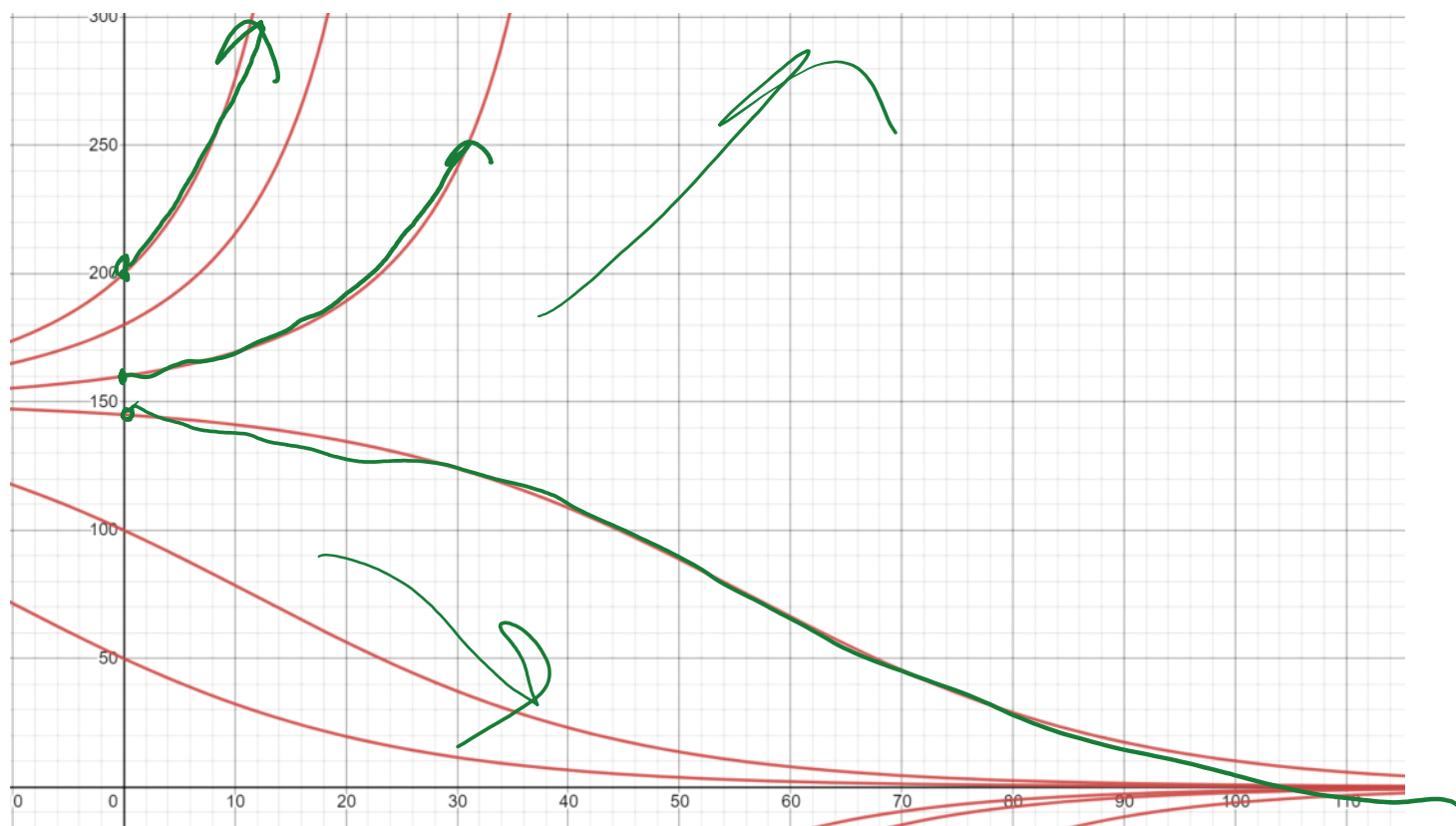
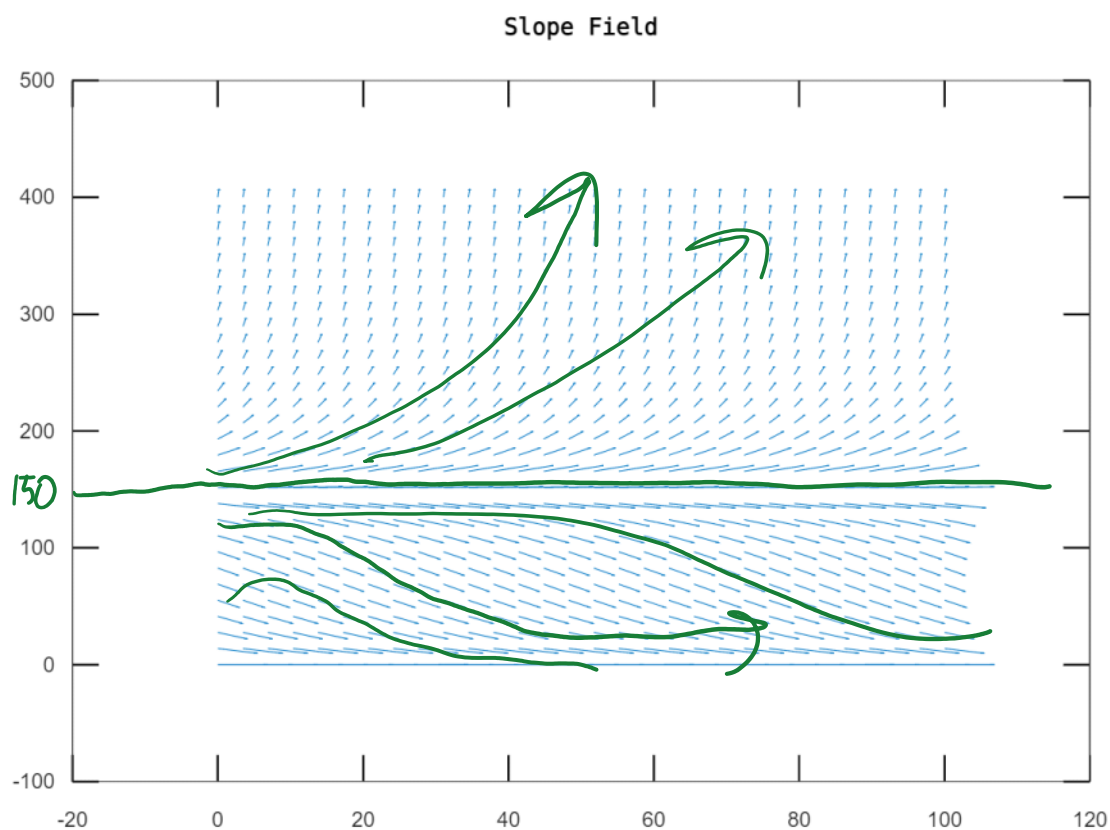
$$\int \frac{dP}{P(P-M)} = \int \frac{-1}{MP} + \frac{1}{M(P-M)} dP = \underline{-\frac{1}{M} \ln(P)} + \underline{\frac{1}{M} \ln(P-M)}$$

$$= \underline{\frac{1}{M} (\ln(P-M) - \ln(P))} = \underline{\frac{1}{M} \ln\left(\frac{P-M}{P}\right)} \leftarrow$$

$$\Rightarrow \frac{1}{M} \ln\left(\frac{P-M}{P}\right) = kt + C$$

Section 2.1 Population Models

So there is only hope if the initial population of the invaders is less than 150!



Section 2.1 Population Models

Birth and death rates can vary in time suppose a population is modeled by

$$\frac{dP}{dt} = (k + b \cos(2\pi t))P$$

So that the growth rate varies around a mean value k . What is the solution to this DE? How does it compare the natural growth equation $\frac{dP}{dt} = kP$?

$$\int \frac{dP}{P} = \int (k + b \cos(2\pi t)) dt$$

$$\ln(P) = kt + \frac{b}{2\pi} \sin(2\pi t) + C$$

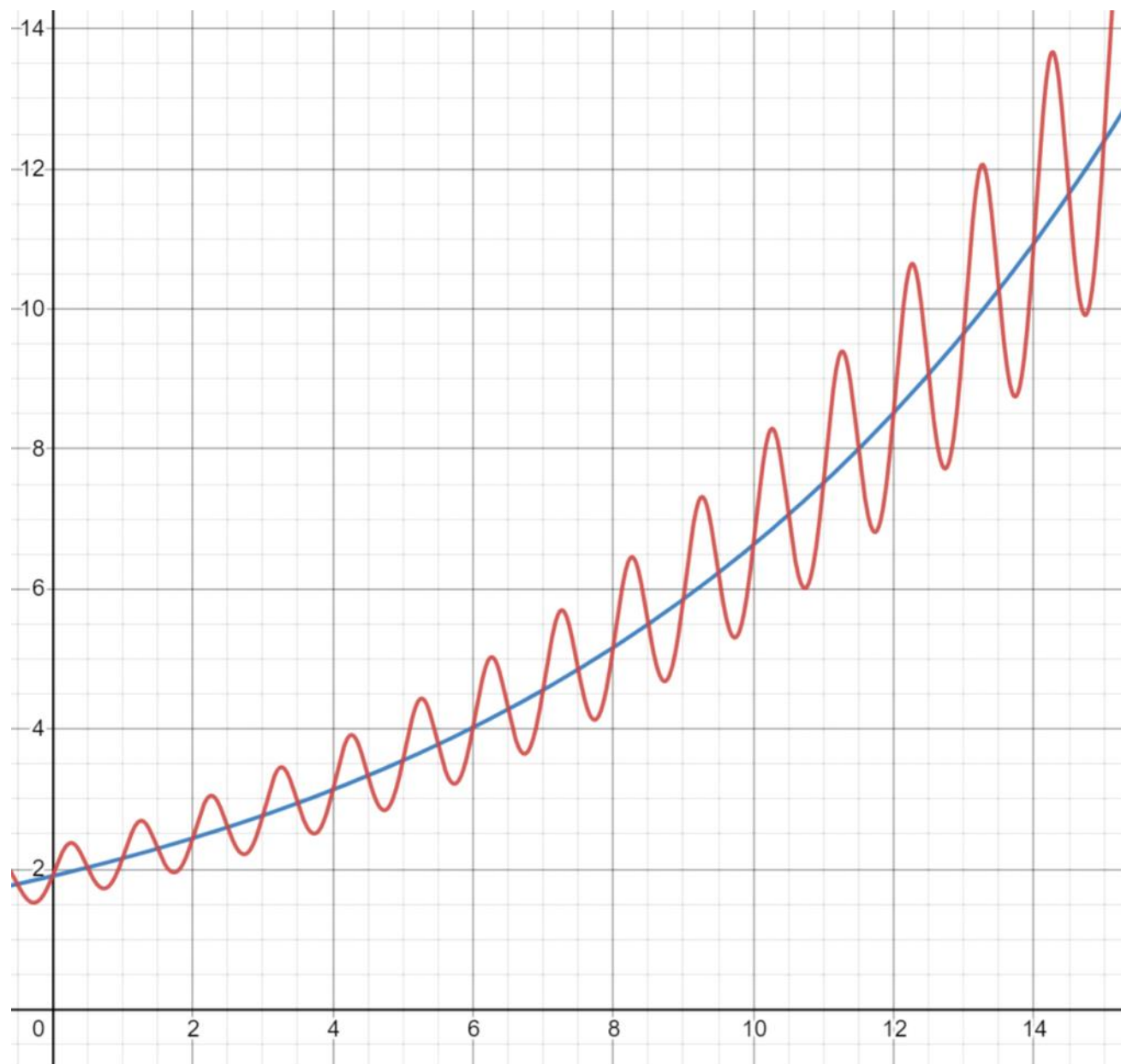
$$P = e^{kt + \frac{b}{2\pi} \sin(2\pi t) + C} = C e^{kt} e^{\frac{b}{2\pi} \sin(2\pi t)}$$

$$P(t) = \underline{C} \underline{e^{kt}} \underline{e^{\frac{b}{2\pi} \sin(2\pi t)}}$$

$$\frac{dP}{dt} = kP \Rightarrow P = C e^{kt}$$

Section 2.1 Population Models

So we can see that the solution oscillates around the regular exponential solution. Here I took $P_0 = 2, k = .125, b = 1.2$.



Section 2.1 Population Models

$$\frac{1}{m} \ln\left(\frac{P-m}{P}\right) = kt + C$$

$$\ln\left(\frac{P-m}{P}\right) = mkt + C \rightarrow e^{mkt+C} = e^{mkt} \boxed{e^C}$$

$$\frac{P-m}{P} = ce^{mkt}$$

$$P-m = Pce^{mkt}$$

$$P - Pce^{mkt} = m$$

$$P(1 - ce^{mkt}) = m$$

$$\boxed{P = \frac{m}{1 - ce^{mkt}}}$$

$$P(0) = P_0$$

$$P(0) = \frac{m}{1 - ce^0} = P_0$$

$$\frac{m}{1-c} = P_0 \Rightarrow \frac{1-c}{m} = \frac{1}{P_0} = \frac{m}{P_0} = 1-c$$

$$c = \boxed{1 - \frac{m}{P_0}}$$