

Section 2.2 Equilibrium Solutions and Stability

Equilibrium:

a state of physical balance.

"I stumbled over a rock and recovered my equilibrium"

a calm state of mind.

"his intensity could unsettle his equilibrium"

a state in which a process and its reverse are occurring at equal rates so that no overall change is taking place.

"ice is in equilibrium with water"

a situation in which supply and demand are matched and prices stable.

"the market is in equilibrium"

With respect to Ordinary Differential Equations we say we are in equilibrium when our derivative is zero, that is:

$$\frac{dx}{dt}(t) = 0 \quad \text{or} \quad \frac{dy}{dx}(x) = 0$$

This is because derivatives are rates of change, when or where they are zero our function is not changing. In particular we talk about equilibrium solutions of Autonomous differential equations.

An Autonomous differential equation is one of the form:

$$\frac{dx}{dt} = f(x)$$

That is, one for which the derivative can be written as an expression of only the unknown function.

Ex Which of the following ODE's are autonomous?

$$\frac{dx}{dt} = x^2 + 2x + \cos(x)$$

$$\frac{dy}{dx} = y \cos(x) + y^2$$

$$\frac{dx}{dt} = x \cos(t) + 4x$$

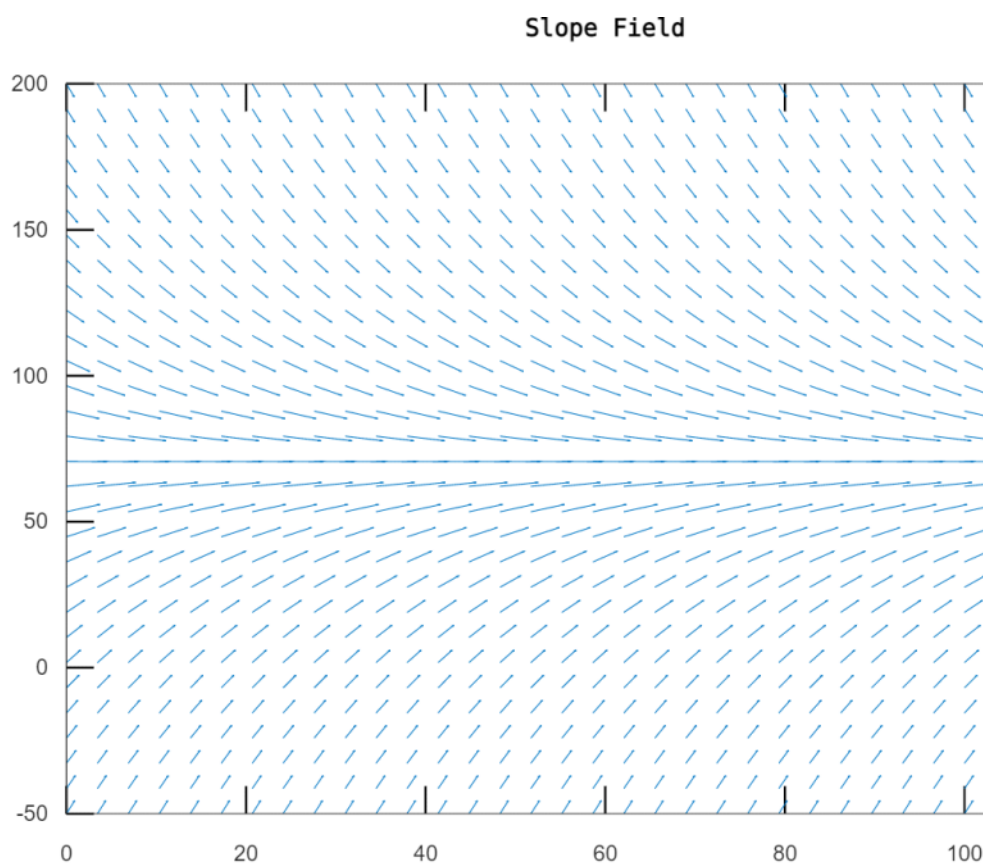
The equilibrium solution of an autonomous first order ODE is a constant value c such that when $x(t) = c$

$$\frac{dx}{dt} = f(c) = 0.$$

The point c is also referred to as a critical point.

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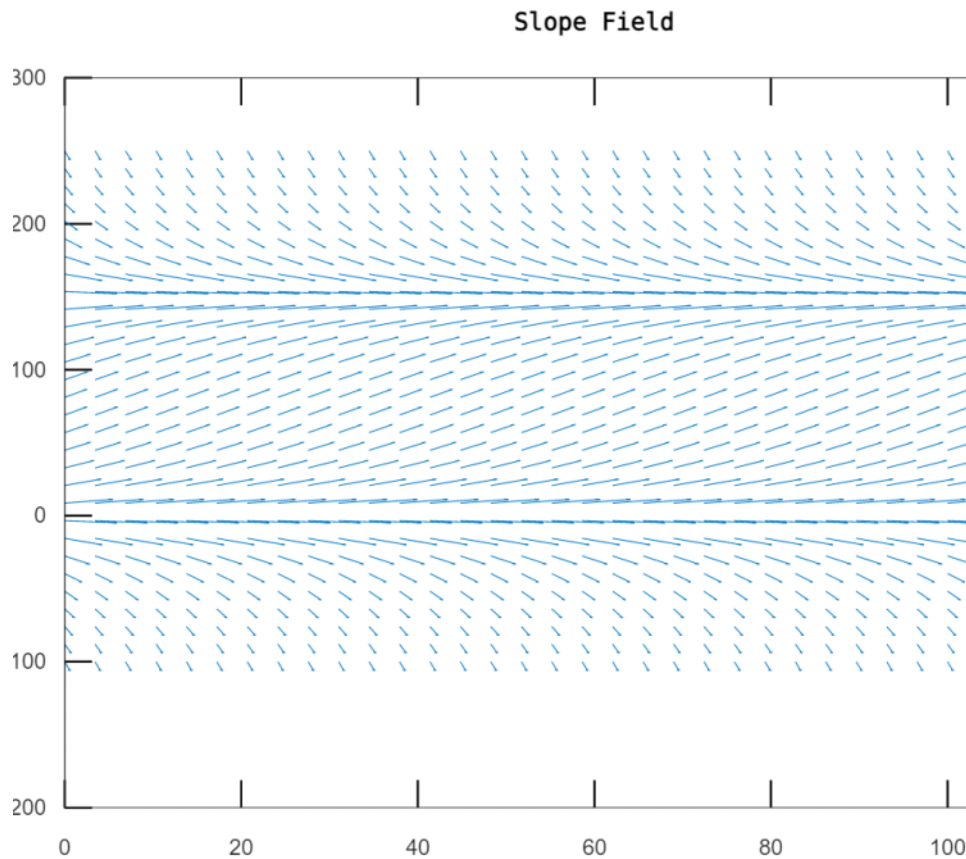
Example: What is the equilibrium solution to the ODE $\frac{dT}{dt} = k(A - T)$ where k, A are constant?



Note that the solution to this ODE is $x(t) = A + (x_0 - A)e^{-kt}$ what happens when we take the $\lim_{t \rightarrow \infty} x(t)$

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Let's consider again the logistic equation $\frac{dx}{dt} = kx(M - x)$ what are our equilibrium solutions?



In this case our solution is given by

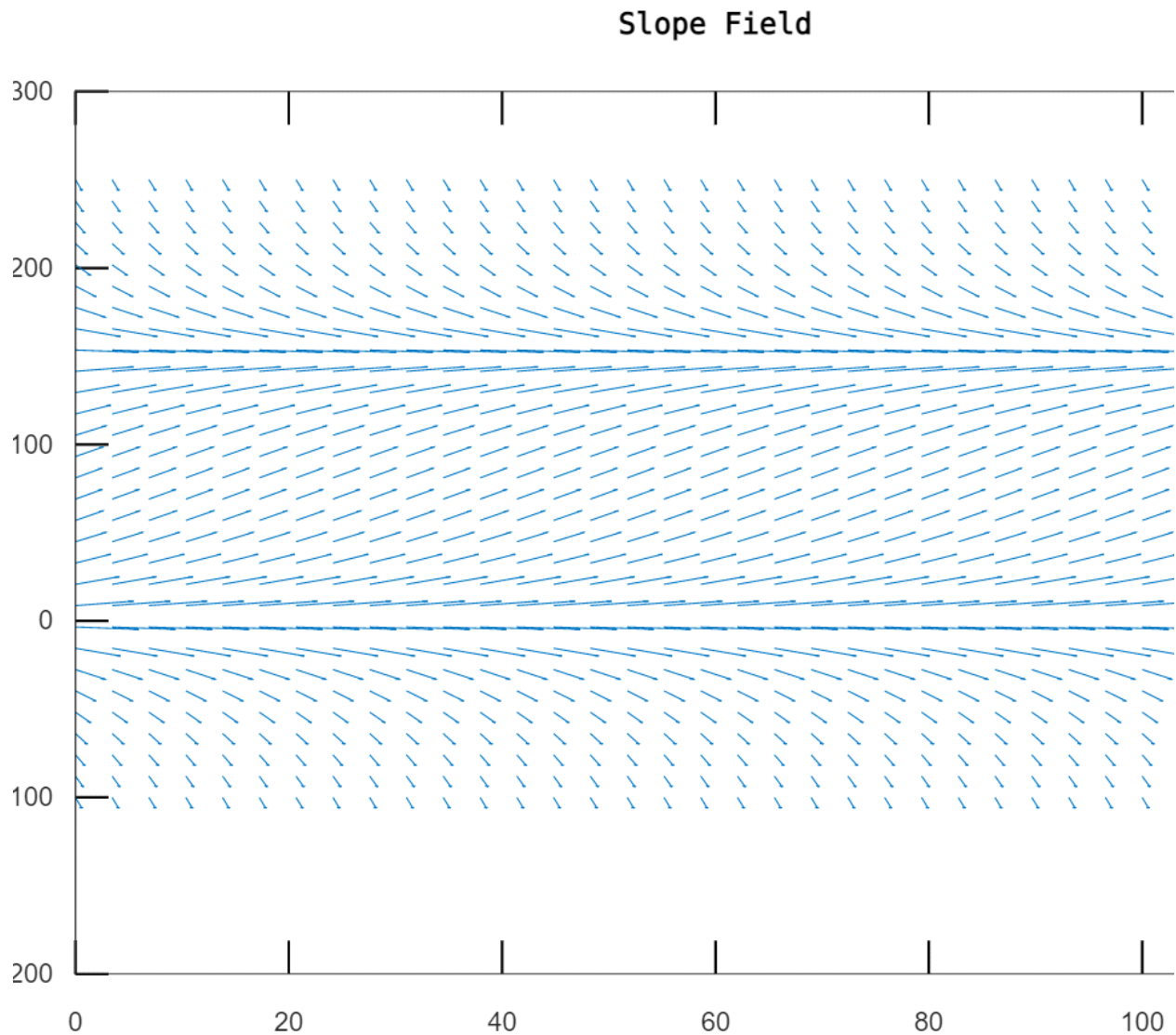
$$x(t) = \frac{Mx_0}{x_0 + (M - x_0)e^{-kMt}}$$

What is the $\lim_{t \rightarrow \infty} x(t)$ when $x_0 < 0$, $x_0 > 0$, and $x_0 > M$?

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Stable and unstable equilibrium solutions.

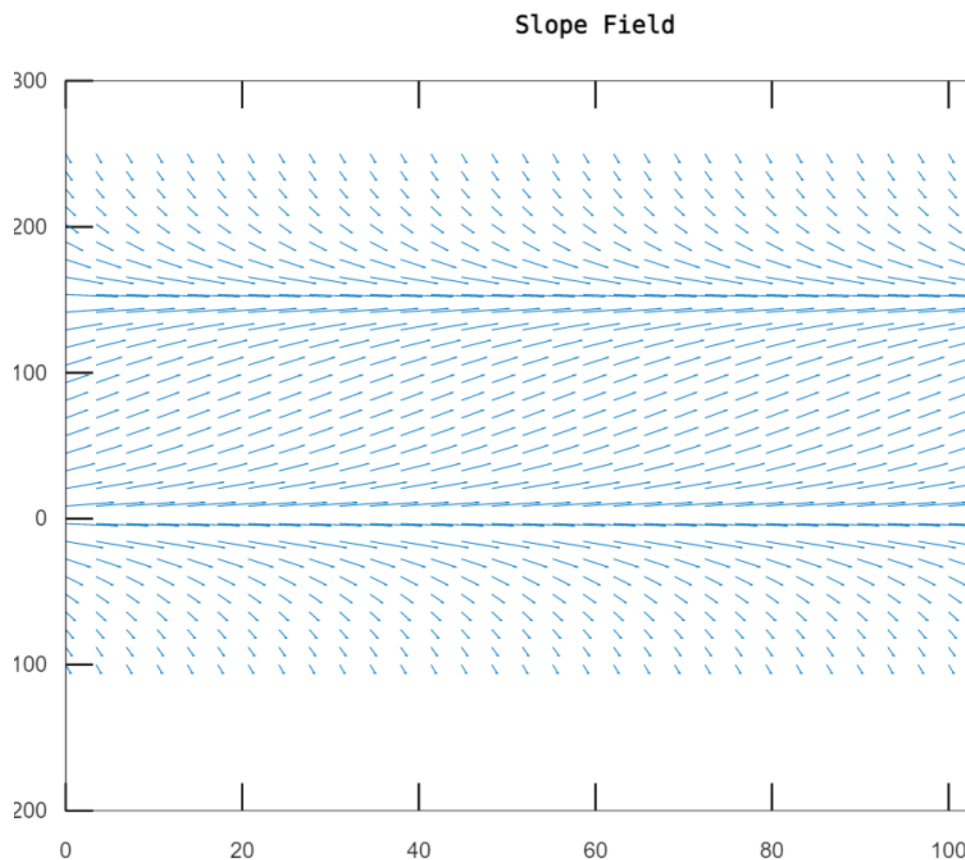
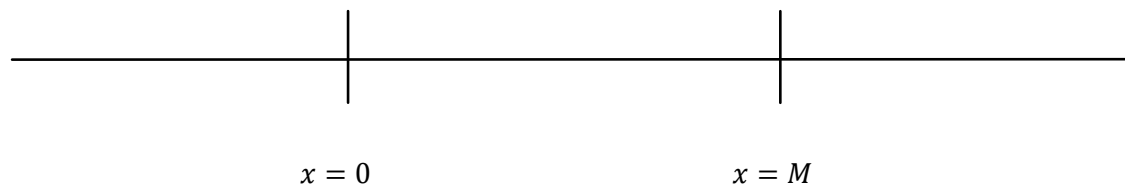
Let c be a critical point (equilibrium solution) of the differential equation $\frac{dx}{dt} = f(x)$. We say that c is stable provided that: for every $\epsilon > 0$ there exists ($\exists$) a $\delta > 0$ such that whenever the initial point x_0 is within distance δ of c ($|x_0 - c| < \delta$) we have that our solution $x(t)$ is within ϵ of c ($|x(t) - c| < \epsilon$). We say that c is unstable when it is not stable.



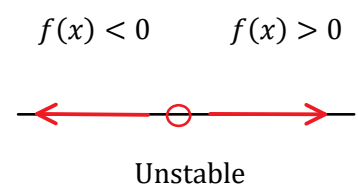
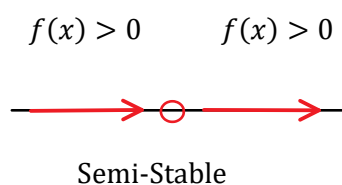
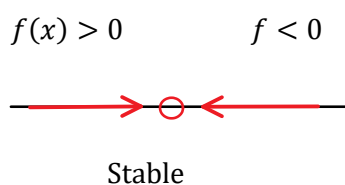
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The Phase Diagram. Let's keep working with $\frac{dx}{dt} = kx(M - x)$

How can we tell whether the equilibrium solutions are stable or unstable without plotting or solving the DE?



So if on either side of a critical point the derivative indicates that $x(t)$ is moving to it this means it is stable. If the derivative indicates that $x(t)$ is moving away from the critical point on both sides we say it is unstable. If the derivative indicates $x(t)$ moves toward the critical point on one side but away on the other we call it semistable.



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Harvesting of a population.

Let's return to the logistic equation once again but now add in a harvesting rate h this could foreexample be the rate of fish that are caught and not released in a lake.

$$\frac{dx}{dt} = kx(M - x) - h$$

What do our equilibrium solutions look like now? To understand them let's complete the square for the right hand side and make a sketch.

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Let's suppose $k = 1$, $M = 4$ and let's consider 3 cases for h . $h = 1$, $h = 4$, $h = 5$

$$\frac{dx}{dt} = x(4 - x) - h$$

Let's complete the square again for ease of analysis.

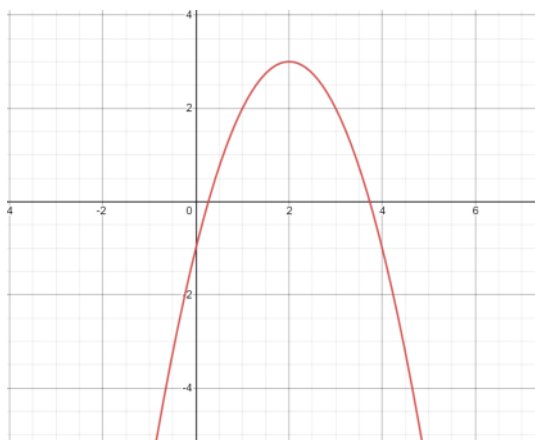
So when $h = 1$,

$h = 4$,

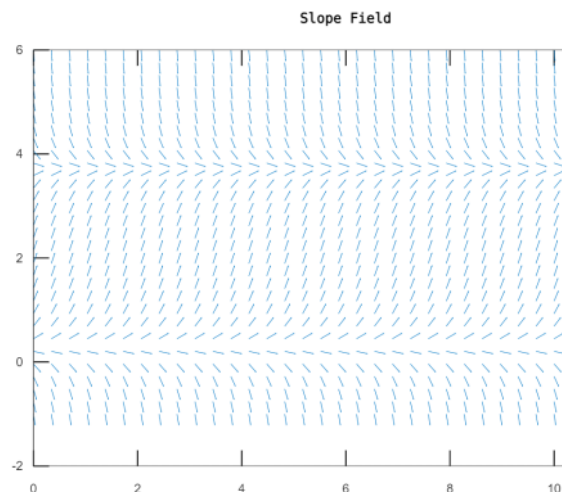
and $h = 5$,

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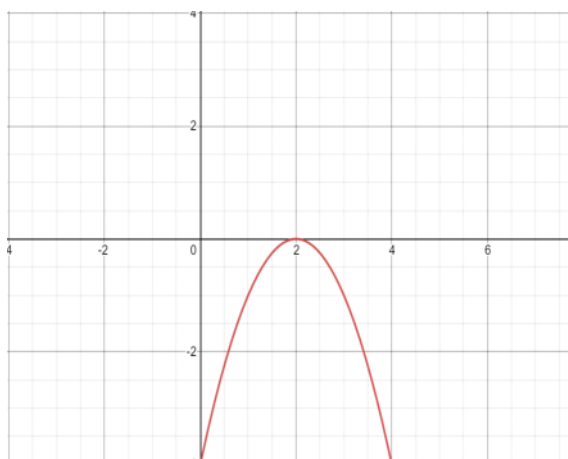
$$\frac{dx}{dt} = x(4 - x) - h$$



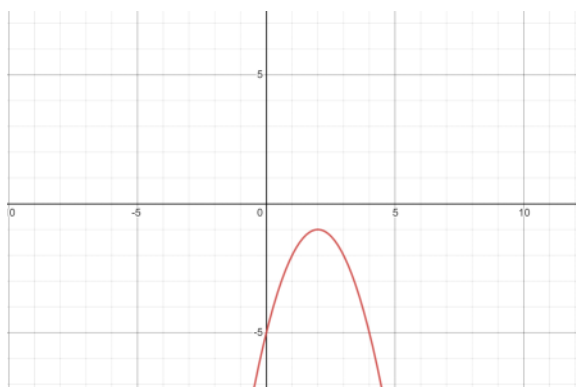
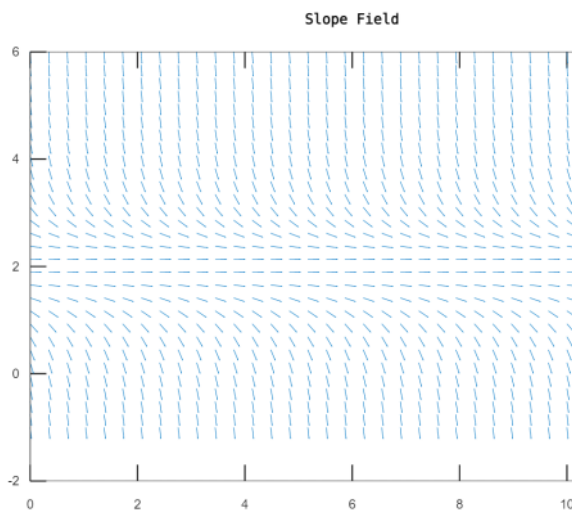
$h = 1$



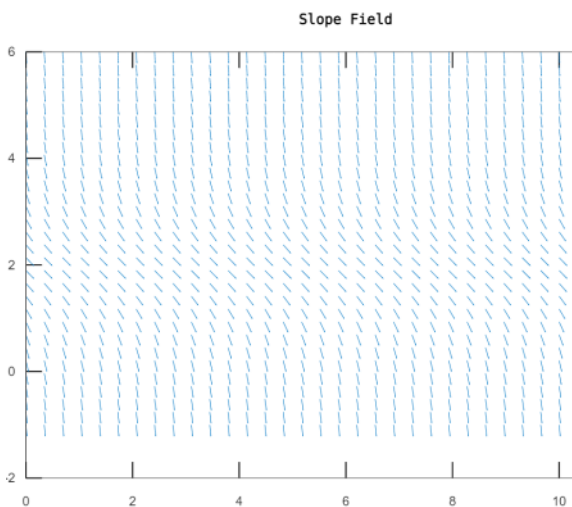
We call $h = 4$ a bifurcation point as the qualitative behavior of the system changes when we get to this value.



$h = 4$



$h = 5$



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We can diagram how the equilibrium solutions change with a given parameter in with what is called a bifurcation diagram. Let's make one for the equation we just studied.

$$\frac{dx}{dt} = x(4 - x) - h$$

Here we will plot the h on the x-axis and the equilibrium solutions c on the y.

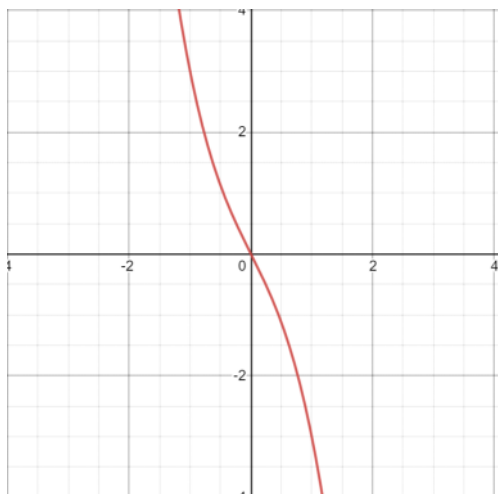
The equilibrium solutions come about when $\frac{dx}{dt} = 0$ which means $h = x(4 - x)$

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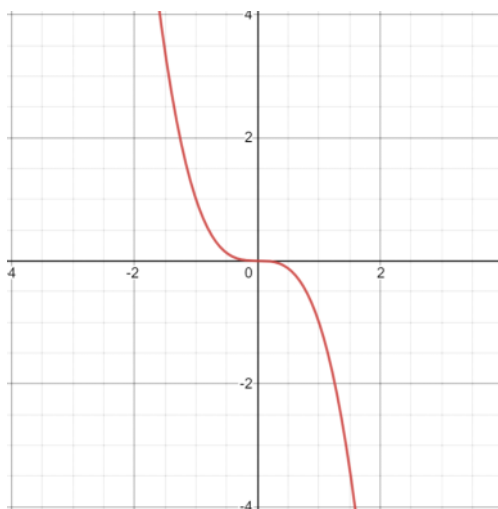
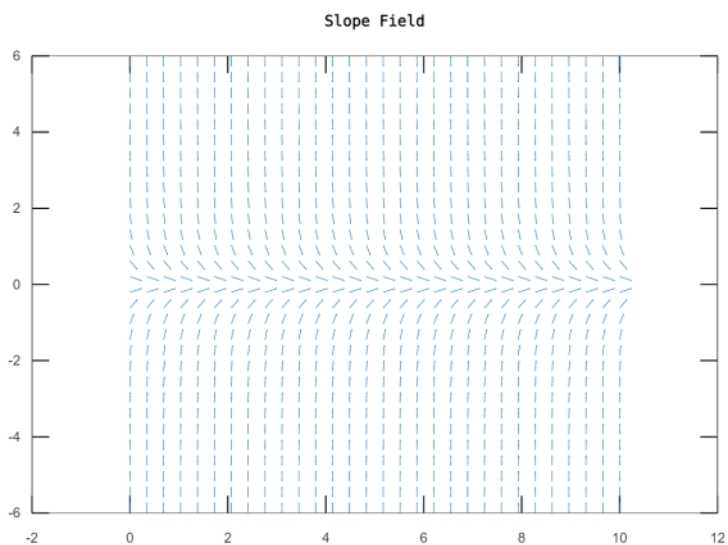
Let's examine the ODE

$$\frac{dx}{dt} = kx - x^3$$

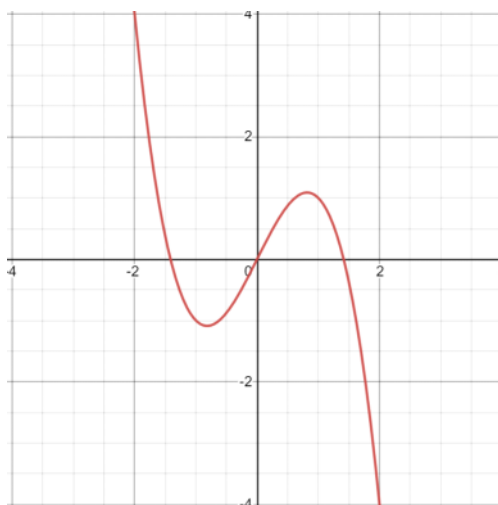
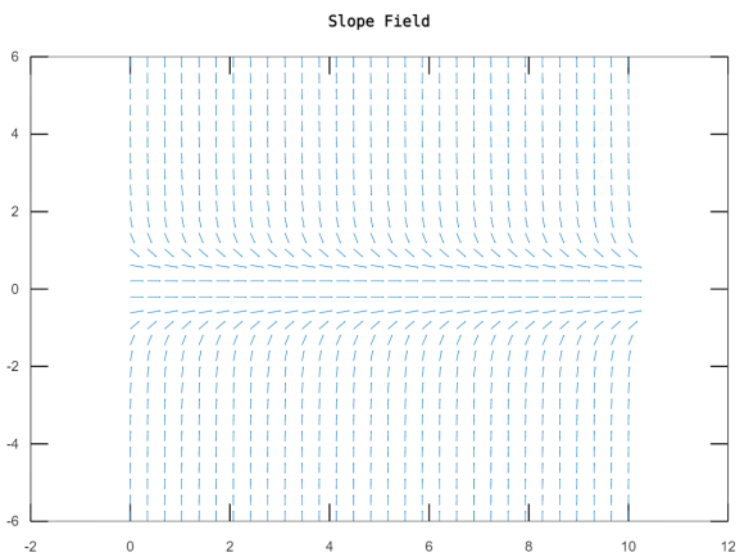
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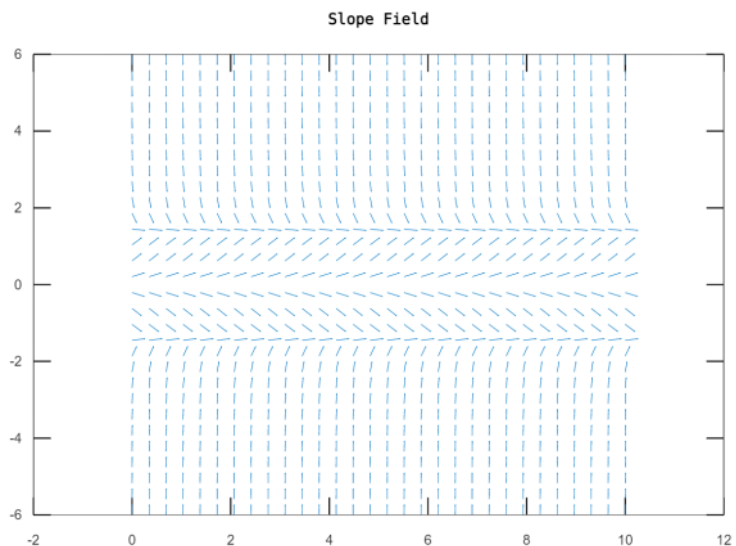
$k = -2$



$k = 0$
Bifurcation!



$k = 2$
Now 3
Eq Solns.



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An Interesting Application of Bifurcations and ODE's

There is a very nice paper by Ian Eisenmann in 2010 where an ODE model of the volume of Arctic sea ice is developed with a parameter that represents the amount of CO2 in the atmosphere.

<http://eisenman.ucsd.edu/papers/Eisenman-Wettlaufer-2009.pdf>

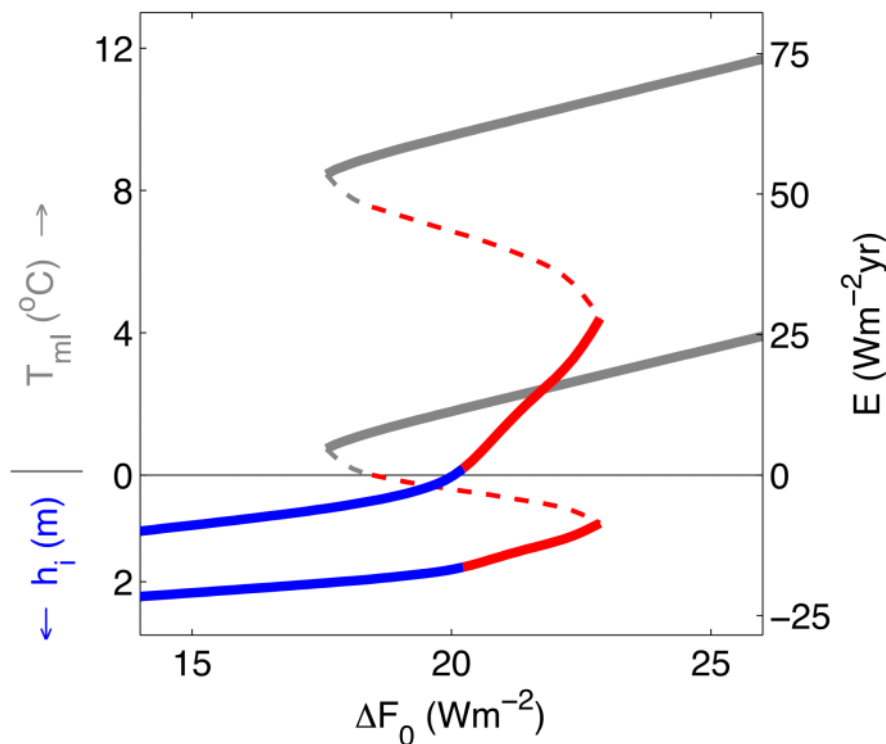
I. Eisenman and J.S. Wettlaufer (2009). **Nonlinear threshold behavior during the loss of Arctic sea ice.** *Proceedings of the National Academy of Sciences* 106, 28-32.

The ice is modeled with the amount of energy it contains..... With lots of parameters all coming from atmospheric physics.

$$\frac{dE}{dt} = [1 - \alpha(E)]F_S(t) - F_0(t) + \Delta F_0 - F_T(t)T(t, E) + F_B + v_0 \mathcal{R}(-E).$$

Here ΔF_0 is a stand in for the amount of CO2 in the atmosphere. If $E < 0$ there is ice, if $E > 0$ there is not.

The bifurcation diagram suggests dire consequences if we do not get our CO2 emissions under control...



The top curve is the summer ice and the bottom the winter, note that there are stable ice covered all year solutions which then become unstable with a stable ice free all year solution. Note that once you reach the ice free all year solution you need to cool the earth to a point beyond where you lost the ice to recover it. This is called hysteresis.