

Section 2.3 Acceleration and Velocity Models

We have studied some acceleration and velocity models already....

$$\frac{dv}{dt} = -g$$

Falling body

$$\frac{dv}{dt} = -g - kv$$

Falling body with air resistance proportional to velocity

$$\frac{dv}{dt} = 30(1 - e^{-2t})$$

A gas car acceleration as torque increases with engine rpm.

Of course in general because the derivative of velocity is the acceleration we are talking about a general class of models

$$\frac{dv}{dt} = a(t, v, x \dots)$$

Really this is just Newton's second law $F = ma$

$$m \frac{dv}{dt} = F$$

$$F = ma$$

KVP $1 < P < 2$

Let's revisit the case of air resistance but a bit more generally. It turns out that for low speeds air resistance often does go as kv , but for higher speeds it goes more like kv^2 . Incidentally being proportional to the square of the velocity is an assumption made by Newton in his original Principia book. The first real book of calculus.

If you recall the falling egg quiz problem when the air resistance was proportional to the velocity ($F_r = kv$) our ODE had the form:

$$m \frac{dv}{dt} = -mg - kv \rightarrow \frac{dv}{dt} = -g - \rho v, \text{ with } \rho = \frac{k}{m}$$

$-kv(-)$ $v < 0$ $\uparrow F_r = kv$ $\downarrow mg$

The minus sign on the air resistance term was somewhat counterintuitive since we assume it will point in the opposite direction of gravity. Upon closer examination though we realized that for a falling object the velocity itself would be negative so we needed the minus sign to ensure the acceleration caused by the air resistance was indeed in the positive direction.

However if the acceleration due to air resistance is ρv^2 this is no longer a problem for downward motion as we will square the velocity, so we would instead use a +.

$$\frac{dv}{dt} = -g + \rho v^2$$

$$\uparrow \rho v^2$$

$$\downarrow -g$$

However if we are moving upward we would need a minus sign since then the air resistance would be acting in the same direction as gravity.

$$\frac{dv}{dt} = -g - \rho v^2$$

$$\uparrow +$$

$$\downarrow -g$$

$$\downarrow \rho v^2$$

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We can summarize this in a more general way for both upward and downward motion by writing:

$$\frac{dv}{dt} = -g - \underbrace{pv|v|}_{\substack{\text{down} \\ \text{up}}}$$

$-g - p(-)|-| = -g + pv^2$
 $-g - p(+)|+| = -g - pv^2$

Let's solve the case of upward motion.

$$\begin{aligned} \frac{dv}{dt} &= -g - pv^2 \\ \frac{dv}{dt} &= -(g + pv^2) \\ \frac{dv}{g + pv^2} &= -dt \\ \int \frac{dv}{g + pv^2} &= \int -dt \\ \int \frac{1}{1 + u^2} du &= \arctan(u) + C \\ \frac{1}{g} \int \frac{dv}{1 + \frac{p}{g}v^2} &= \int -dt \end{aligned}$$

$$\begin{aligned} \frac{1}{g} \int \frac{dv}{1 + (\sqrt{\frac{p}{g}}v)^2} &= \int -dt \\ u = \sqrt{\frac{p}{g}}v \quad du = \sqrt{\frac{p}{g}}dv \\ \frac{1}{g} \sqrt{\frac{g}{p}} \int \frac{du}{1 + u^2} &= \int -dt \\ \frac{1}{\sqrt{gp}} \arctan(u) &= -t + C \\ \arctan(u) &= -\sqrt{gp}t + C \\ u &= \tan(-\sqrt{gp}t + C) \\ \sqrt{\frac{p}{g}}v &= \tan(-\sqrt{gp}t + C) \\ \boxed{v = \sqrt{\frac{g}{p}} \tan(-\sqrt{gp}t + C)} \end{aligned}$$

$v(0) = v_0$
 $v_0 = \sqrt{\frac{g}{p}} \tan(0 + C)$
 $v_0 = \sqrt{\frac{g}{p}} \tan(C)$
 $\tan(C) = v_0 \sqrt{\frac{p}{g}}$
 $\boxed{C = \arctan(v_0 \sqrt{\frac{p}{g}})}$

Let's solve the case of downward motion.

$$\begin{aligned} \frac{dv}{dt} &= -g + pv^2 = -(g - pv^2) \\ \frac{dv}{g - pv^2} &= -dt \\ \int \frac{dv}{g - pv^2} &= \int -dt \\ \int \frac{1}{1 - u^2} du &= \operatorname{arctanh}(u) + C \\ \frac{1}{g} \int \frac{dv}{1 - \frac{p}{g}v^2} &= \int -dt \end{aligned}$$

$$\begin{aligned} \frac{1}{g} \int \frac{dv}{1 - (\sqrt{\frac{p}{g}}v)^2} &= \int -dt \\ u = \sqrt{\frac{p}{g}}v \\ \frac{1}{\sqrt{gp}} \operatorname{arctanh}(u) &= -t + C \\ u &= \tanh(\sqrt{gp}t + C) \\ \sqrt{\frac{p}{g}}v &= \tanh(\sqrt{gp}t + C) \\ v &= \sqrt{\frac{g}{p}} \tanh(\sqrt{gp}t + C) \\ \lim_{t \rightarrow \infty} \sqrt{\frac{g}{p}} \left(\frac{e^{-\sqrt{gp}t + C} - e^{\sqrt{gp}t - C}}{e^{-\sqrt{gp}t + C} + e^{\sqrt{gp}t - C}} \right) &= -\sqrt{\frac{g}{p}} \left(\frac{e^C}{e^C} \right) \end{aligned}$$

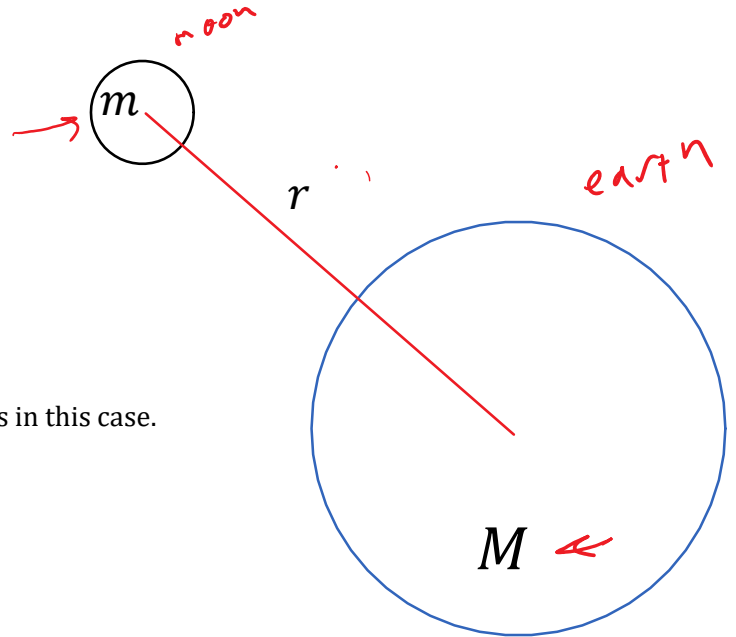
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Gravity is not the same everywhere. For most purposes the objects we throw around are close to the surface of the Earth that $g = -9.8 \frac{m}{s^2}$ or $-32 \frac{ft}{s^2}$ works just fine, however if we get far enough the force of gravity exerted on us by the earth is less.

According to Newton the force of gravity between two objects a distance r apart is described by:

$$F_G = G \frac{Mm}{r^2}$$

Where $G = 6.6726 \times 10^{-11}$



The distance r is measured from the center of the spheres in this case.

How much does a 80Kg person weigh on the top of mount Everest?

Mass of the earth $5.972 \times 10^{24} \text{ kg}$

Radius of the Earth: $6.3781 \times 10^6 \text{ m}$

Height of Mount Everest $8,848.86 \text{ m}$

$$M = 5.972 \times 10^{24}$$

$$m = 80;$$

$$r = 6.371 \times 10^6 + 8848.86$$

$$G = 6.6726 \times 10^{-11}$$

$$F = G \cdot M \cdot m / r^2$$

$$F_{\text{surf}} = F = G \cdot M \cdot m / (6.37 \times 10^6)^2$$

$$M = 5.9720 \times 10^{24}$$

$$r = 6.3798 \times 10^6$$

$$G = 6.6726 \times 10^{-11}$$

$$F = 783.22$$

$$F_{\text{surf}} = 785.64$$



Section 2.3 Acceleration and Velocity Models

Let's explore a relevant problem in modern times using this equation and what we have learned about DE's. As you may know Elon Musk wants to go live on Mars and the space ship SpaceX is designing is meant to land vertically on the Martian surface. It would be good to know when exactly the ships thrusters should be fired to provide for a soft touch down on the surface! Let's see if we can help Elon out.

$$F_g = \frac{GMm}{r^2}$$

Let's suppose the ship is first in free fall, and since mars atmosphere is thin we will neglect air resistance, this would be totally valid on the moon. After falling some distance the ship will ignite its thrusters to produce a constant acceleration from the thrust T . From Newtons second law

$$\underbrace{m \frac{d^2 r}{dt^2}}_{\text{Free fall}} = - \underbrace{\frac{GMm}{r^2}}_F$$

$$\underbrace{m \frac{d^2 r}{dt^2}}_{\text{Rocket on.}} = \underbrace{mT}_{F_T} - \underbrace{\frac{GMm}{r^2}}_{F_g}$$

M mass starship
 M mass mars.



Free Fall

Where m is the mass of the starship and M the mass of Mars. Note we can divide by m to obtain.

$$\frac{d^2 r}{dt^2} = - \frac{GM}{r^2}$$

$$\frac{d^2 r}{dt^2} = T - \frac{GM}{r^2}$$

Here we are interested in the velocity however, for a soft touch down we want the velocity to be zero ($v = 0$) when we make touchdown.

The chain rule saves the day again!

$$\frac{d^2 r}{dt^2} = \frac{dv}{dt} = \frac{dv}{dr} \frac{dr}{dt} = \frac{dv}{dr} v$$

So now have new DE's in terms of v and a new independent variable r namely,

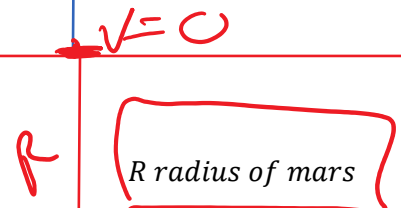
$$v \frac{dv}{dr} = - \frac{GM}{r^2}$$

$$v \frac{dv}{dr} = T - \frac{GM}{r^2}$$



Height to fire thrusters h

$r - R$ distance between the starship and the center.



R radius of mars

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So we have:

$$v \frac{dv}{dr} = -\frac{GM}{r^2} dr$$

$$v \frac{dv}{dr} = T - \frac{GM}{r^2}$$

Solving gives:

$$\int v dv = \int -\frac{GM}{r^2} dr$$

$$\frac{v^2}{2} = GM r^{-1} + C_1$$

$$\frac{v^2}{2} = \frac{GM}{r} + C_1$$

$$v^2 = \frac{2GM}{r} + C_1$$

$$v dv = T - \frac{GM}{r^2} dr$$

$$\frac{v^2}{2} = Tr + \frac{GM}{r} + C_2$$

$$v^2 = 2Tr + \frac{2GM}{r} + C_2$$

If the starship had a velocity of v_0 at the top of the atmosphere a height H we can find the constant of integration as:

$$v_0^2 = \frac{2GM}{R+H} + C_1$$

$$C_1 = v_0^2 - \frac{2GM}{R+H}$$

$$v_f^2 = \frac{2GM}{r} + v_0^2 - \frac{2GM}{R+H}$$

Then we know we want $v = 0$ when $r = R$ we can use this as a condition to find the constant of integration for our equation involving thrust.

$$0^2 = 2TR + \frac{2GM}{R} + C_2$$

$$C_2 = -2TR - \frac{2GM}{R}$$

$$v^2 = 2Tr + \frac{2GM}{r} - 2TR - \frac{2GM}{R}$$

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Finally we know that the velocity at the instant of ignition had to be the same for both solutions. If we set them equal to each other we can solve for the distance r from the center of mars the rockets had to fire to ensure the soft touchdown. Then the height above the ground is just $r - R$.

$$V_F^2 = \frac{2GM}{r} + V^0 - \frac{2GM}{R+H}$$

$$v^2 = 2Tr + \frac{2GM}{r} - 2Tr - \frac{2GM}{R}$$

$$\frac{2GM}{r-R} + \sqrt{\frac{-2GM}{R+H}}$$

$$\rightarrow 2T(r-R) + \frac{2GM}{r-R} - 2TR - \frac{2GM}{R}$$

Maybe Elon should give us dome doge coin for the help!

Another interesting question one might ask using Newtons law of gravitation is that of the velocity needed to escape the earth in general. We might consider any velocity needed to be an initial velocity that ensures the solution to our DE is always positive, that it never reverses and falls back to the Earth.

Neglecting air resistance we have that

$$v \frac{dv}{dr} = -\frac{GM}{r^2}$$

With an initial velocity v_0 at the surface of the Earth a distance R from the center of the Earth.

$$v \frac{dv}{dr} = -\frac{GM}{r^2}$$

$$\int v dv = - \int \frac{GM}{r^2} dr$$

$$\frac{v^2}{2} = \frac{5m}{r} + C$$

$$\frac{v_0^2}{2} = \frac{GM}{R} + C$$

$$C = \frac{v_1^2}{2} - \frac{GM}{R}$$

$$\text{So } \frac{v^2}{2} - \frac{GM}{r} + \frac{v_0^2}{2} - \frac{GM}{R}$$

Better argument.

3) Better argument:
suppose we do escape the earth's gravity. This means that $r \rightarrow \infty$
that there was never a time where it stopped increasing. If so

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$$\frac{v^2}{2} = \frac{GM}{r} + \frac{v_0^2}{2} - \frac{GM}{R}$$

$\frac{Gm}{r} \rightarrow 0$

and our implicit equation becomes

$$\frac{v^2}{2} - \frac{v_0^2}{2} - \frac{Gm}{R}$$

Then since $\frac{2}{\sqrt{2}} > 0$ since it is

$$\text{Squared} \Rightarrow \frac{2}{V_{2/2}} - \frac{5m}{R} > 0$$

$$\Rightarrow \sqrt{V_0} > \sqrt{25 \text{ m} / R}$$

$$\frac{\left(\frac{\sqrt{26}m}{r}\right)^2}{2} - \frac{Gm}{R}$$

$$\frac{2Gm}{2R} - \frac{Gm}{R} = 0$$

$$\frac{2Gm}{4R} - \frac{Gm}{R} = 0 \quad \left\{ \quad 2 \right.$$

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$$\Rightarrow \boxed{V_0 > \sqrt{2Gm/R}}$$

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$$\rightarrow \sinh(x) = \frac{e^x - e^{-x}}{2} \leftarrow$$

$$\rightarrow \cosh(x) = \frac{e^x + e^{-x}}{2} \leftarrow$$

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

$$\frac{d}{dx} \sinh(x) = \frac{e^x - (-e^{-x})}{2} = \frac{e^x + e^{-x}}{2} = \cosh(x)$$

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$$-g + kv^2$$

Show that if you solve the differential equation $\frac{dx}{dt} = -1 + x^2$ with $x(0) = 0$ using separation of variables and a partial fraction decomposition that the solution is $x(t) = \tanh(-t)$. Note DO NOT use the integral formula $\int \frac{1}{1-u^2} du = \operatorname{arctanh}(u) + C$. This problem is designed to show how the hyperbolic tangent function arises from some differential equations. Some hints: (1) Solve for C right after you integrate both sides using the initial condition (2) Remember your rules of logs. (3) After you solve for $x(t)$ multiply the top and bottom of your solution by e^t . Compare your result to $\tanh(-t)$ using its definition in terms of exponential functions

$$\frac{dx}{dt} = -1 + x^2$$

$$\int \frac{dx}{1-x^2} = \int dt$$

↓

$$= -t + C$$

$$\int \frac{1}{(1+x)(1-x)} = \int -dt$$

$$\frac{1}{(1+x)(1-x)} = \frac{A}{1+x} + \frac{B}{1-x}$$

$$1 = A(1-x) + B(1+x)$$

$$x=1$$

$$1 = A(0) + B(2) \Rightarrow B = \frac{1}{2}$$

$$x=-1 \quad 1 = A(1-(-1)) + B(0) \Rightarrow A = \frac{1}{2}$$

$$\int \frac{1}{2(1+x)} + \int \frac{1}{2(1-x)} = -t + C$$

$$u=1+x$$

$$du=dx$$

$$u=1-x$$

$$du=-dx$$

$$\frac{1}{2} \ln(1+x) - \frac{1}{2} \ln(1-x) = -t + C$$

$$\frac{1}{2} (\ln(1+x) - \ln(1-x)) = -t + C$$

$$\frac{1}{2} \ln\left(\frac{1+x}{1-x}\right) = -t + C$$

$$\tanh(t) = \frac{\sinh(t)}{\cosh(t)} = \frac{e^t - e^{-t}}{e^t + e^{-t}}$$

$$\tanh(-t) = \frac{e^{-t} - e^{-(-t)}}{e^{-t} + e^{-(-t)}} = \frac{e^{-t} - e^t}{e^{-t} + e^t}$$

$$\ln\left(\frac{1+x}{1-x}\right) = -2t + C$$

$$\ln\left(\frac{1+0}{1-0}\right) = -2(0) + C$$

$$\ln(1) = C \quad C=0$$

$$\ln\left(\frac{1+x}{1-x}\right) = -2t$$

$$\frac{1+x}{1-x} = e^{-2t}$$

$$1+x = (1-x)e^{-2t}$$

$$1+x = e^{-2t} - xe^{-2t}$$

$$x + xe^{-2t} = e^{-2t} - 1$$

$$x(1+e^{-2t}) = e^{-2t} - 1$$

$$x = \frac{(e^{-2t} - 1)(e^t)}{(1+e^{-2t})(e^t)}$$

$$x = \frac{e^{-t} - e^t}{e^t + e^{-t}} = \tanh(-t)$$

Section 2.3 Acceleration and Velocity Models

A more general way to write Newton's second law is using momentum (P). In Newtonian physics $P = mv$ where m is mass and v velocity. With momentum Newton's second law takes the form:

$$\frac{dP}{dt} = F$$

$$ma = \bar{F}$$

$$\frac{d}{dt}P = \frac{d}{dt}mv = m \frac{dv}{dt} = ma = \bar{F}$$

Where F represents the sum of all forces acting on the moving object. Suppose there is a rocket which produces a constant thrust F_T as it burns through its fuel. We know that in this situation the mass of the rocket is not constant as the fuel is burned, indeed as more fuel is burned it becomes less massive. Neglecting air resistance, what differential equation involving $\frac{dv}{dt}$ and $\frac{dm}{dt}$ models the motion of the rocket?

$$P(t) = m(t)v(t)$$

$$\frac{dP}{dt} = \frac{d}{dt}m(t)v(t) = \frac{dm}{dt}v + m \frac{dv}{dt}$$

$$\frac{dm}{dt}v + m \frac{dv}{dt} = F_T - mg$$

