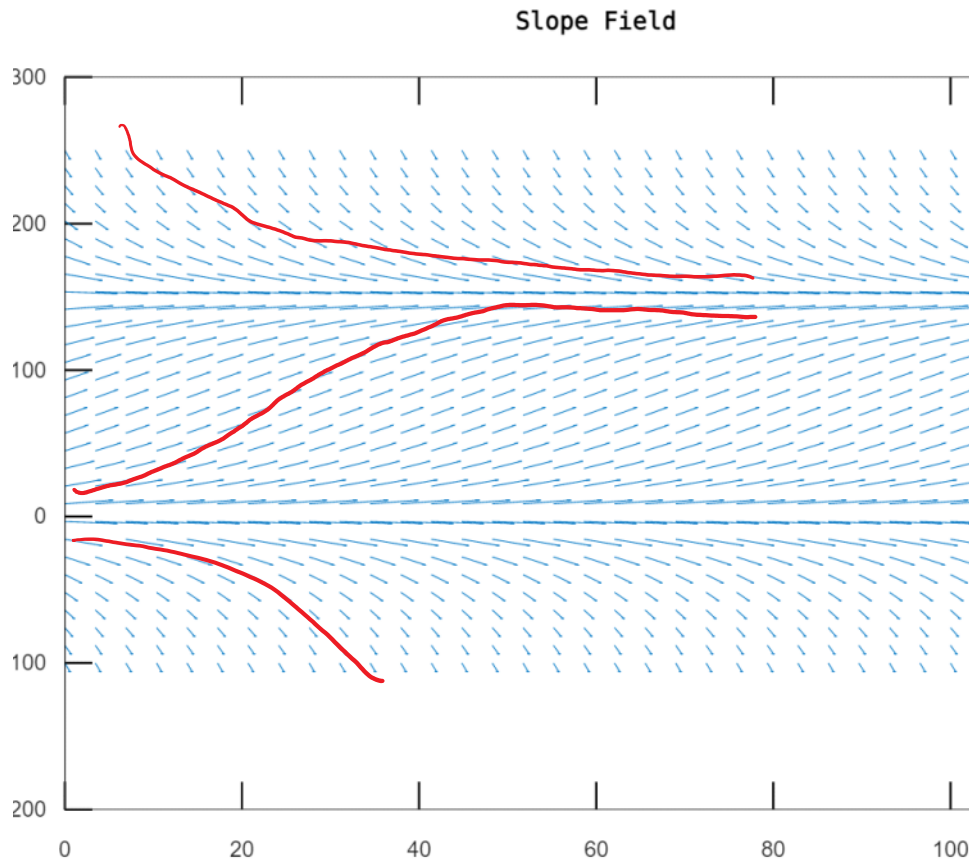


Sec 2.4 Numerical Approximation: Euler's Method

We have used slope fields to help us visualize the solutions to DE's this is especially useful when we can't easily solve the them. We have also used them to help us sketch specific solutions.



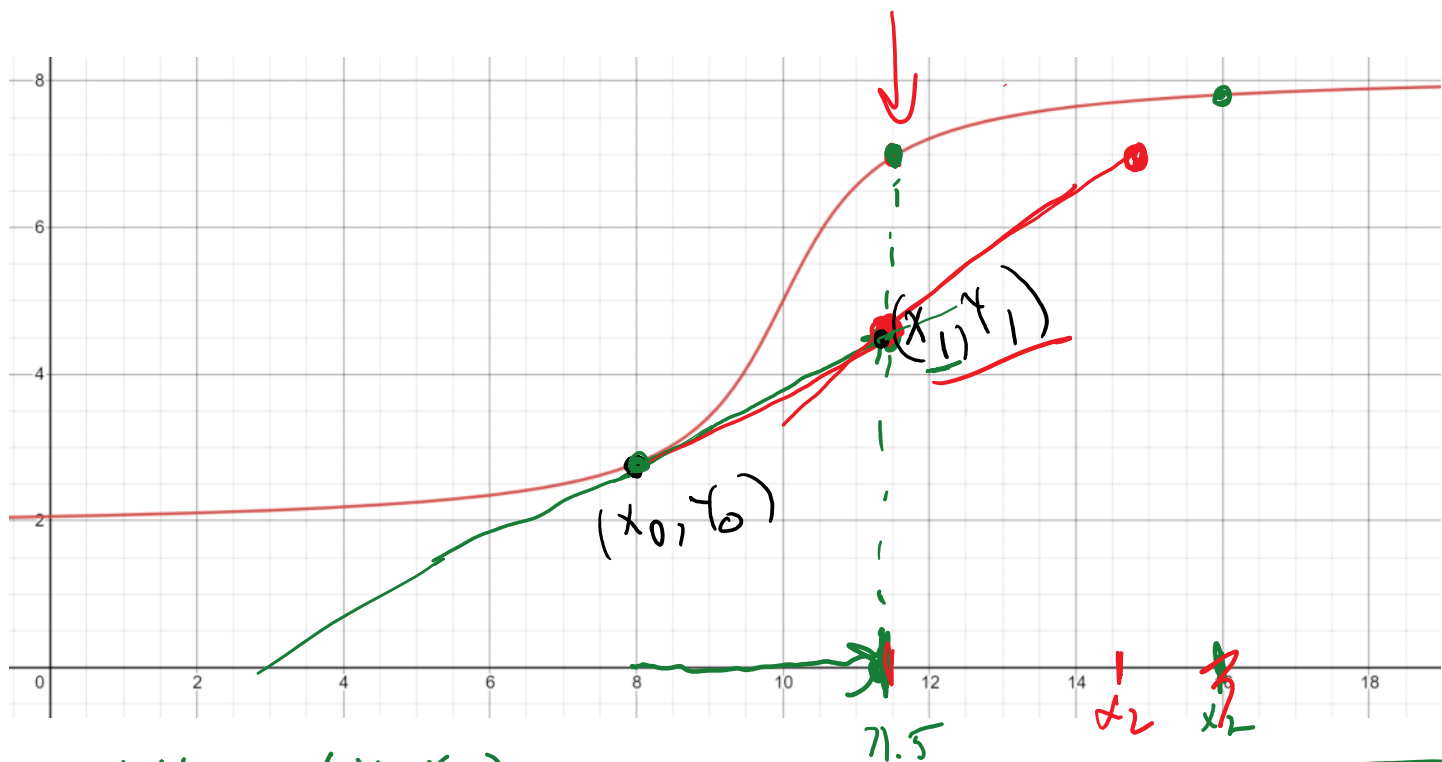
In doing this sketching of solutions we were simply following the tangent lines from one point to the next. This concept is the basis of Euler's Method which helps us approximate the solutions to differential equations that might otherwise be difficult to solve.

$$\frac{dy}{dx} = f(x, y) \leftarrow \text{Slopes of Tangent lines}$$



Sec 2.4 Numerical Approximation: Euler's Method

Let $\frac{dy}{dx} = f(x, y)$ suppose we are at a point (x_0, y_0) and want to follow the tangent line there to the next point close to the true solution at (x_1, y_1) . What formula can we use to get there?



$$y - y_0 = m(x - x_0)$$

$$\downarrow \quad \downarrow$$

$$y_1 - y_0 = f(x_0, y_0)(x_1 - x_0)$$

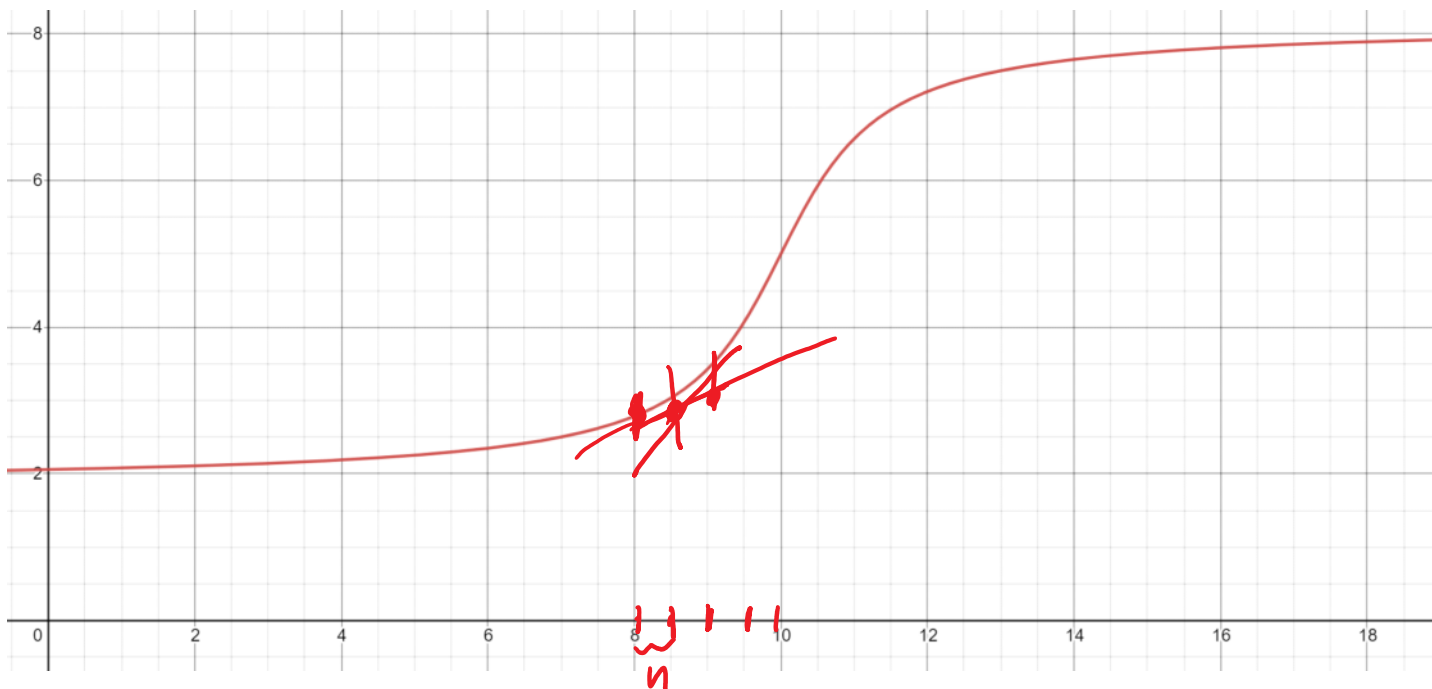
$$\boxed{y_1 = y_0 + f(x_0, y_0)(x_1 - x_0)}$$

So $y_1 = y_0 + f(x_0, y_0)(x_1 - x_0)$ What might we do if we wanted to go from (x_1, y_1) to another approximate point?

$$y_2 = y_1 + f(x_1, y_1)(x_2 - x_1)$$

Sec 2.4 Numerical Approximation: Euler's Method

Clearly this first approximation is not so great, how can we improve it?



Let's take the same distances between x-values for each of our approximate points, let's call it h . In this case our first approximation becomes...

$$x_1 - x_0 = h \quad x_2 - x_1 = h$$

$$y_1 = y_0 + f(x_0, y_0)h$$

The second is:

$$y_2 = y_1 + f(x_1, y_1)h$$

Indeed the $n + 1^{th}$ approximation would be defined in terms of the n^{th} approximation right before it.

$$y_{n+1} = y_n + f(x_n, y_n)h$$

We can also see that the smaller we take h the more accurate the approximation can be.

Sec 2.4 Numerical Approximation: Euler's Method

Given the initial value problem:

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0$$

Euler's method with step size h consists of applying the iterative formula below with $n \geq 0$.

$$y_{n+1} = y_n + h \cdot f(x_n, y_n)$$

To calculate successive approximations y_1, y_2, y_3 to the true values $y(x_1), y(x_2), y(x_3)$ of the function $y(x)$ that is a solution to the initial value problem at the points x_1, x_2, x_3 .

Example: Apply Euler's method with a step size of $h = 1$ to find y_1, y_2, y_3 to the initial value problem

$$y_{n+1} = y_n + f(x_n, y_n)h$$

$$\frac{dy}{dx} = \overbrace{x + y}^{f(x, y)}, \quad \overbrace{y(0) = 1}^{x_0 \quad y_0}$$

$$x_0 = 0 \\ y_0 = 1$$

$$x_1 = x_0 + h \\ 0 + 1$$

$$x_1 = 1$$

$$x_2 = 2$$

$$x_3 = 3$$

$$y_1 = 1 + (0 + 1)(1)$$

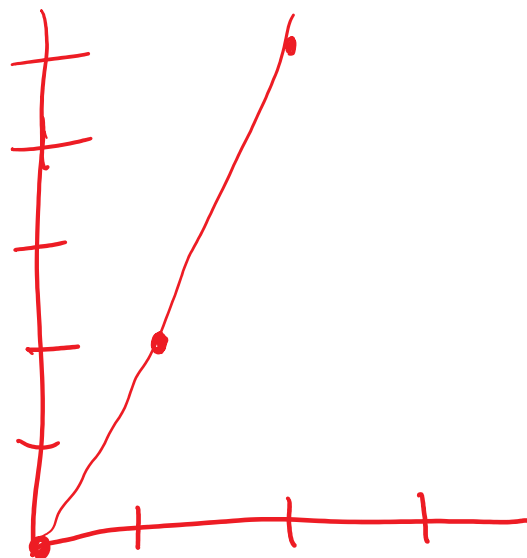
$$y_1 = 2$$

$$y_2 = 2 + \overbrace{(1 + 2)}^{f(x_1, y_1)}(1)$$

$$y_2 = 2 + 3 = 5$$

$$y_3 = 5 + (2 + 5)(1)$$

$$y_3 = 5 + 7 = 12$$



Sec 2.4 Numerical Approximation: Euler's Method

Find the true solution to

$$\frac{dy}{dx} = x + y, \quad y(0) = 1$$

$$\frac{dy}{dx} + \underset{-1}{P(x)}y = \underset{x}{Q(x)}$$

$$\frac{dy}{dx} - y = x$$

$$I(x) = e^{\int P(x)} = e^{\int -1 dx} = e^{-x}$$

$$\frac{dy}{dx} e^{-x} - y e^{-x} = x e^{-x}$$

$$\int \frac{d}{dx}(e^{-x}y) = \int x e^{-x}$$

$$\begin{aligned} u &= x & dv &= e^{-x} \\ du &= dx & v &= -e^{-x} \end{aligned}$$

$$e^{-x}y = -x e^{-x} + \int e^{-x} dx$$

$$e^{-x}y = -x e^{-x} - e^{-x} + C$$

$$y = \frac{-x e^{-x}}{e^{-x}} - \frac{e^{-x}}{e^{-x}} + \frac{C}{e^{-x}}$$

$$y = -x - 1 + C e^x$$

$$y(0) = 1 = -0 - 1 + C e^0$$

$$1 = -1 + C$$

$$C = 2$$

$$y(x) = -x - 1 + 2e^x$$

Sec 2.4 Numerical Approximation: Euler's Method

Obviously in the days before the computer this method would be very tedious, but these days we can get very good solutions using numerical methods.

$$\frac{dy}{dx} = x + y, \quad y(0) = 1$$

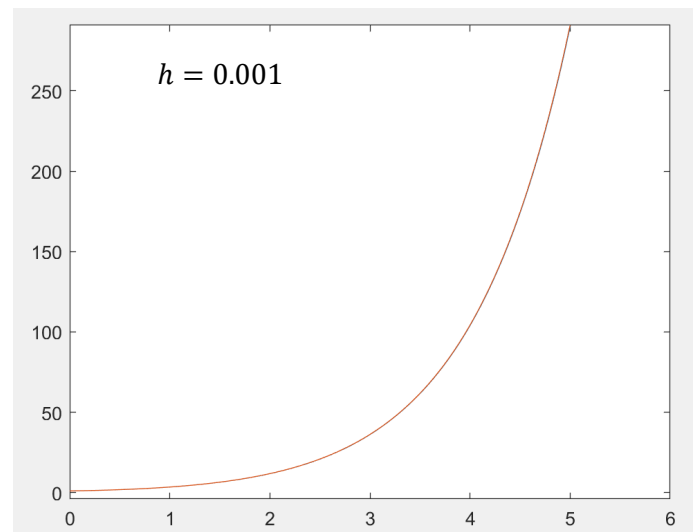
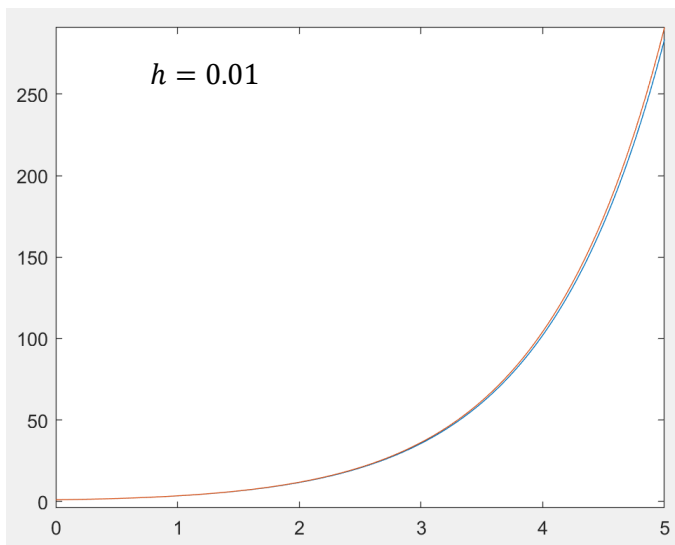
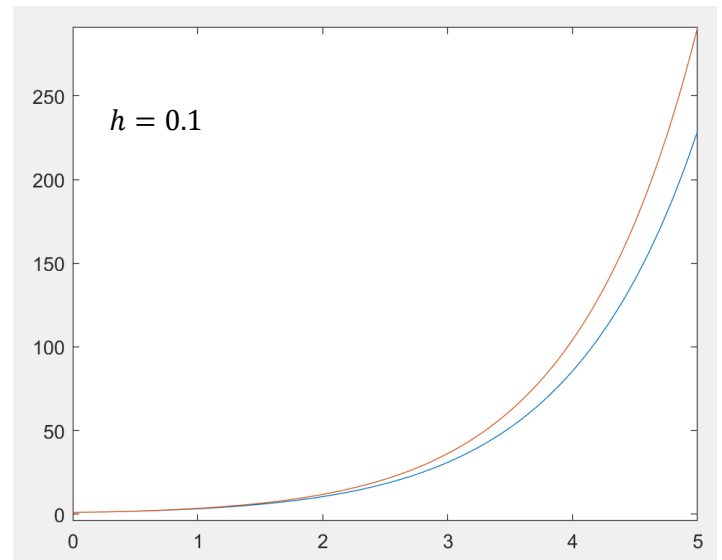
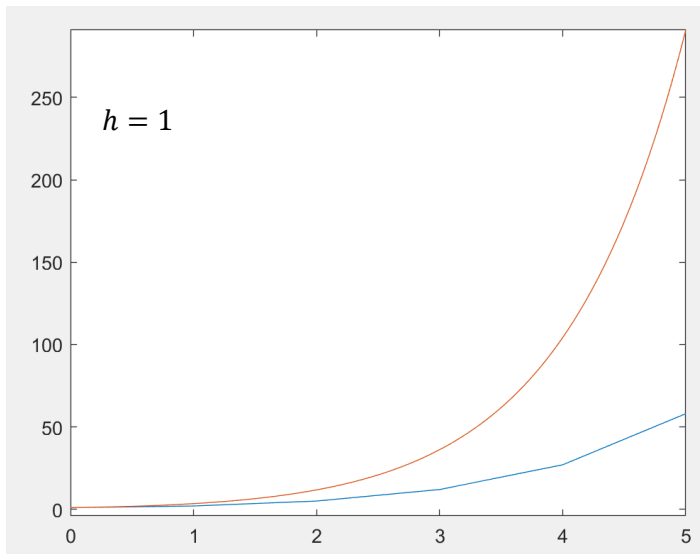
In this case our true solution is $y(x) = -x - 1 + 2e^x$

Let's go through some code that can make our approximation for us.

```
clear all
close all
f = @(x,y)x+y; % function dy/dx=f(x, y)
x0=0; %initial x value
y0=1; %initial y value
xfinal=5; %end of interval over which we want solutions
h=1; %step size
N=(xfinal-x0)/h; %number of approximations to cover interval
x=zeros(1,N+1); %initialize vector to store x values
y=zeros(1,N+1); %initialize vector to store y values
x(1)=x0; %set first spot in vector to initial data
y(1)=y0; %set first spot in vector to initial data
xtrue=linspace(x0,xfinal,100); % x points for plotting true solution
ytrue=-xtrue-1+2*exp(xtrue); % y points for plotting true solution
for i=1:N %set up loop to iteratively get each approximation
    y(i+1)=y(i)+h*f(x(i),y(i)); % get each new y approximation
    x(i+1)=x(i)+h; % get each new x
end
plot(x,y) % plot approximations
hold on % tells octave to hold on and put next plot on top of first one
plot(xtrue,ytrue) %plot truth
ylim([ytrue(1)-5 ytrue(end)]); %set y limits for better picture with this problem
```

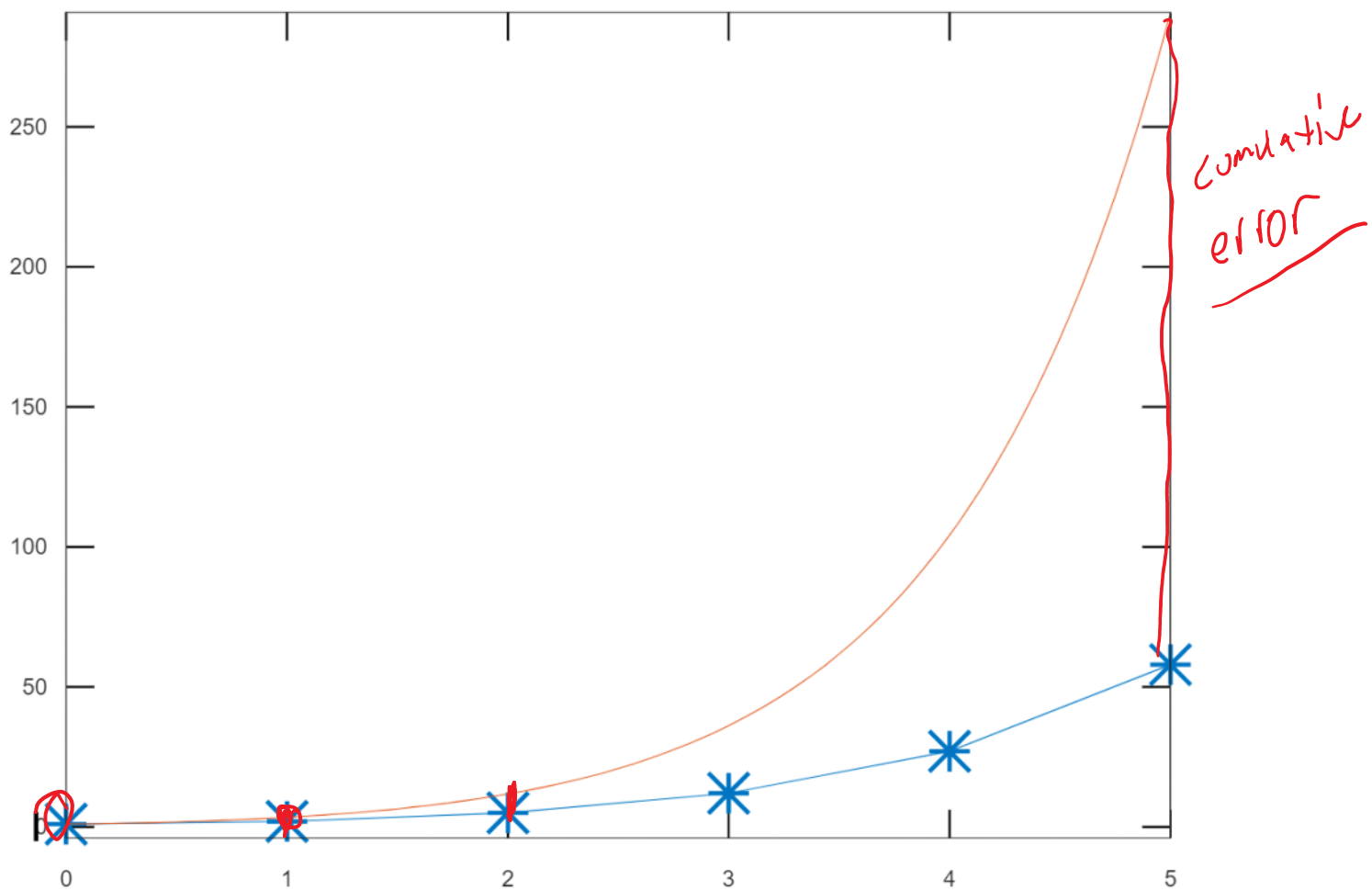
Let's head over to octave and see how shrinking h improves our solution.

Sec 2.4 Numerical Approximation: Euler's Method



Sec 2.4 Numerical Approximation: Euler's Method

Some terminology, local vs cumulative error

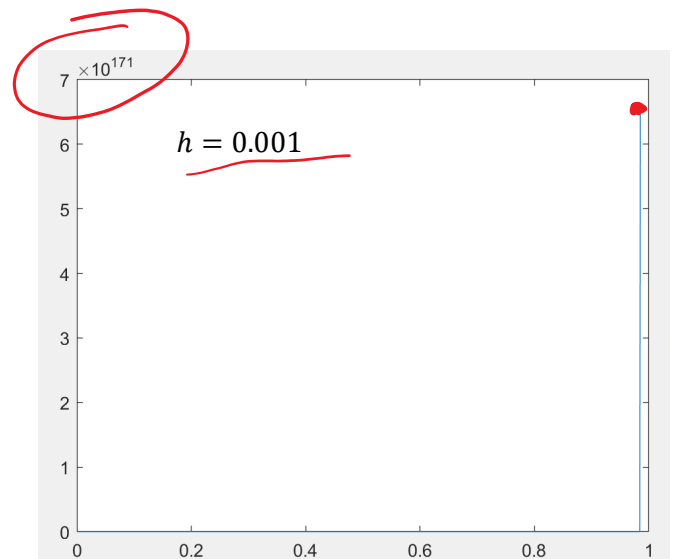
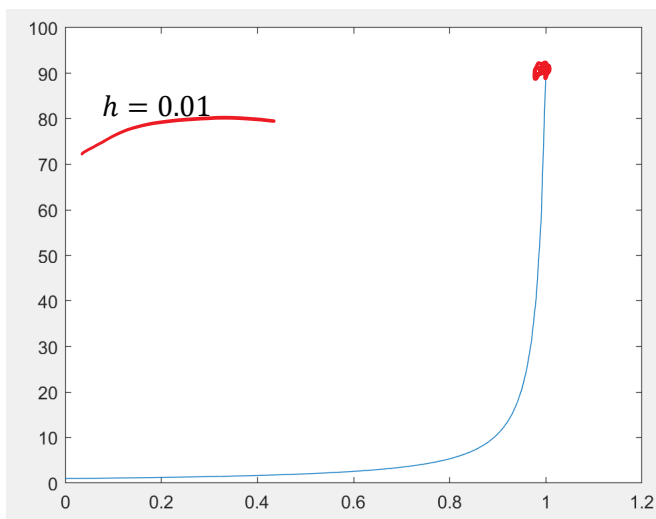
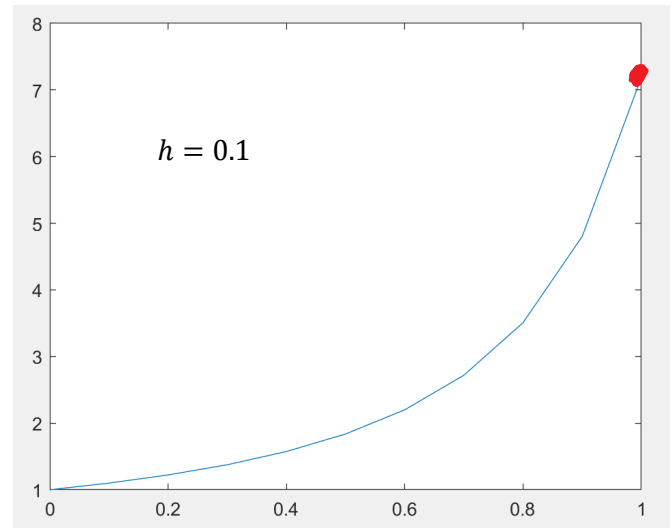
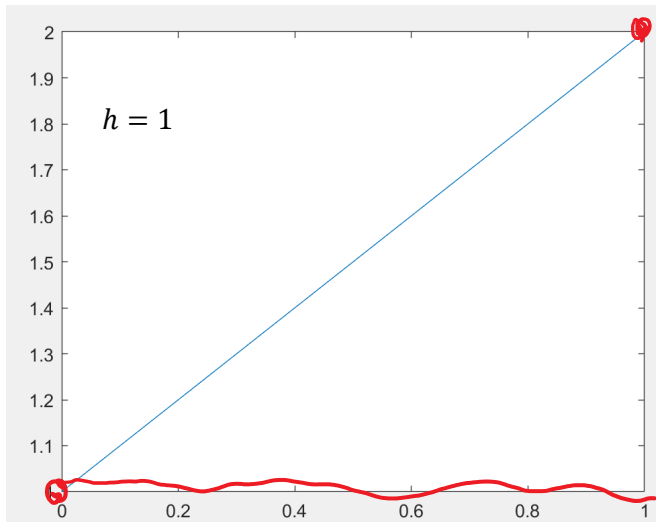


Sec 2.4 Numerical Approximation: Euler's Method

So far it seems that Euler's method can work extremely well as long as we have a small enough step size h . But things can explode when the slopes of the tangent lines explode!

Let's look at the Euler approximation on $[0,1]$ for $h = 1, 0.1, 0.01, 0.001$

$$\frac{dy}{dx} = x^2 + y^2 \quad y(0) = 1$$



It turns out that the solution to this DE has an asymptote at $x = 0.97$. In Euler's method remember we are stepping by a multiple of the slope of the tangent line. Near an asymptote that slope can get really big really fast ruining our approximations.

$$y_{n+1} = y_n + h f(x, y)$$

Sec 2.4 Numerical Approximation: Euler's Method

