

Sec 3.1 Introduction: Second Order Linear Equations

The most general form of an n-th order differential equation with independent variable x and unknown function $y(x)$ can be written as:

$$F(x, y, y', y'', \dots, y^n) = 0$$

where F is a real valued function of $n + 2$ variables.

So for a second order differential equation the form is

$$F(x, y, y', y'') = 0$$

Really it is just that the highest derivative we see is the second derivative. Here though we will also focus on second order **linear** equations.

Recall that a linear function is one that satisfies the following conditions.

$f(x)$ is a linear function in x if

—

$$f(cx) = cf(x)$$

$$f(x_1 + x_2) = f(x_1) + f(x_2)$$

Just to see this definition lets show that $f(x) = 2x$ is linear

$$f(cx) = cf(x)$$

$$2(cx) = c(2x)$$

$$f(x_1 + x_2) = f(x_1) + f(x_2)$$

$$2(x_1 + x_2) = 2x_1 + 2x_2$$

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For our second order linear equations they will have the form:

$$\underline{A(x)y'' + B(x)y' + C(x)y = F(x)}$$

To see how this is linear we can look at the left hand side and define

$$\underline{G(x, y, y', y'') = A(x)y'' + B(x)y' + C(x)y}$$

And we can verify that G is linear in y, y', y'' by checking that

$$G(x, cy, cy', cy'') = cG(x, y, y', y'')$$

$$(cy)' = \frac{d}{dx}(cy) = c \frac{d}{dx}y$$

$$G(x, y_1 + y_2, (y_1 + y_2)', (y_1 + y_2)'') = G(x, y_1, y_1', y_1'') + G(x, y_2, y_2', y_2'')$$

$$\begin{aligned} G(x, cy, cy', cy'') &= A(x)(cy)'' + B(x)(cy)' + C(x)(cy) \\ &= cA(x)y'' + cB(x)y' + cC(x)y \\ &= c(A(x)y'' + B(x)y' + C(x)y) \\ &= cG(x, y, y', y'') \end{aligned}$$

$$\begin{aligned} &A(x)(y_1 + y_2)'' + B(x)(y_1 + y_2)' + C(x)(y_1 + y_2) \\ &\quad \overbrace{A(x)(y_1'' + y_2'')} + B(x)(y_1' + y_2') + C(x)(y_1 + y_2) \\ &A(x)y_1'' + B(x)y_1' + C(x)y_1 + A(x)y_2'' + B(x)y_2' + C(x)y_2 \\ &\quad \underline{G(x, y_1, y_1', y_1'')} \quad \underline{G(x, y_2, y_2', y_2'')} \end{aligned}$$

$\left\{ \begin{aligned} \frac{d}{dx} y_1 + y_2 \\ = \frac{d}{dx} y_1 + \frac{d}{dx} y_2 \end{aligned} \right.$

The main thing to note is that there are no powers of y, y' or y'' other than 1. Also there are no products of y and its derivatives.

$$yy''$$

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One very important class of linear differential equations is that of the **homogeneous** one. A homogeneous second order differential equation is one of the form

$$A(x)y'' + B(x)y' + C(x)y = 0$$

Theorem: The superposition principle of linear homogeneous differential equations.

Let y_1, y_2 be two solutions to the differential equation

$$A(x)y'' + B(x)y' + C(x)y = 0$$

Then $y_3 = c_1y_1 + c_2y_2$ is also a solution.

Proof:

Assume y_1, y_2 solve $A(x)y'' + B(x)y' + C(x)y = 0$

$$\Rightarrow A(x)y_1'' + B(x)y_1' + C(x)y_1 = 0 \text{ and } A(x)y_2'' + B(x)y_2' + C(x)y_2 = 0$$

check $c_1y_1 + c_2y_2$

$$A(x)(c_1y_1 + c_2y_2)'' + B(x)(c_1y_1 + c_2y_2)' + C(x)(c_1y_1 + c_2y_2)$$

$$= \underbrace{A(x)c_1y_1''}_{=0} + \underbrace{A(x)c_2y_2''}_{=0} + \underbrace{B(x)c_1y_1'}_{=0} + \underbrace{B(x)c_2y_2'}_{=0} + \underbrace{C(x)c_1y_1}_{=0} + \underbrace{C(x)c_2y_2}_{=0}$$

$$= c_1(\underbrace{A(x)y_1'' + B(x)y_1' + C(x)y_1}_{=0}) + c_2(\underbrace{A(x)y_2'' + B(x)y_2' + C(x)y_2}_{=0})$$

$$= c_1(0) + c_2(0) = 0 \quad \checkmark \quad \Rightarrow y_3 = c_1y_1 + c_2y_2$$

solves the D.E.

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Theorem: Existence and uniqueness of solutions to linear DE's

Suppose that the functions $p, q,$ and f are continuous on the open interval I containing the point a .
Then given any two numbers b_0, b_1 the equation

$$y'' + p(x)y' + q(x)y = f(x)$$

Has a unique solution on the entire interval I that satisfies the initial data

$$y(a) = b_0, \quad y'(a) = b_1$$

For the form above we have divided by $A(x)$ assuming it is never zero on our interval I

The key thing to note here is that we will need two pieces of initial data to get a particular solution. This is precisely because the solutions to these second order equations will be a linear combination of two linearly independent functions with two constants. Let's just see this in an example.

Example: Show that $y_1 = e^x$ and $y_2 = xe^x$ are solutions to the differential equation $y'' - 2y' + y = 0$

Then find a solution that satisfies $y(0) = 3, y'(0) = 1$

$$(e^x)'' - 2(e^x)' + e^x = e^x - 2e^x + e^x = 0 \quad \checkmark$$

$$(xe^x)'' - 2(xe^x)' + xe^x$$

$$(xe^x)' = 1e^x + xe^x$$

$$(xe^x)'' = (e^x + xe^x)'$$

$$= e^x + e^x + xe^x$$

$$(e^x + e^x + xe^x) - 2(e^x + xe^x) + xe^x$$

$$2e^x + xe^x - 2e^x - 2xe^x + xe^x = 0$$

$$\Rightarrow y = c_1 e^x + c_2 x e^x$$

$$y(0) = 3 \quad y'(0) = 1 \rightarrow y' = c_1 e^x + c_2 (e^x + x e^x)$$

$$\downarrow$$

$$3 = c_1 e^0 + c_2 (0 e^0) \quad 1 = c_1 e^0 + c_2 (e^0 + 0 e^0)$$

$$3 = c_1 \rightarrow 1 = c_1 + c_2 \rightarrow 1 = 3 + c_2 \quad c_2 = -2$$

$$y = 3e^x - 2xe^x$$

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What exactly do we mean when we say two functions are linearly independent.

Definition: Two functions f_1 and f_2 are linearly independent if neither can be written as a multiple of the other.

(1)
Example: Show that $f_1(x) = \cos(x)$ and $f_2(x) = \sin(x)$ are **linearly independent**. Then show that
(2) $f_1(x) = (x+2)^2 + 1$ and $f_2(x) = 2x^2 + 8x + 10$ are **linear dependent**. What about $f_1(x) = \sin(2x)$, (3)
 $f_2(x) = \sin(x)\cos(x)$?

Let's suppose f_1, f_2 are LD $\Rightarrow f_1 = k f_2$

$$\frac{f_1}{f_2} = k$$

$$\frac{\sin(x)}{\cos(x)} = \text{const?} \quad \text{No} \quad \frac{\sin(x)}{\cos(x)} = \tan(x)$$

(2) $f_1 = (x+2)^2 + 1$ $f_2 = 2x^2 + 8x + 10$

↓

$$x^2 + 4x + 4 + 1$$

↓

$$2(x^2 + 4x + 5)$$

$$x^2 + 4x + 5$$

$$f_2 = 2(f_1)$$

$$\frac{f_1}{f_2} = \frac{x^2 + 4x + 5}{2(x^2 + 4x + 5)} = \frac{1}{2}$$

(3) $f_1 = \sin(2x)$ $f_2 = \sin(x)\cos(x)$

$$\sin(2x) = 2\sin(x)\cos(x) \quad \text{Trig}$$

$$f_1 = 2(f_2)$$

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Why do we care if two functions that solve a second order linear homogeneous DE are linearly independent?

We'll remember that if y_1 and y_2 are solutions of the DE then so is $c_1 y_1 + c_2 y_2$ but if y_1, y_2 are linearly dependent then there is some constant such that $y_1 = c y_2$ and our sum of solutions would then just be

$$y = c_1 y_1 + c_2 (c y_1) = (c_1 + c_2 c) y_1 = C y_1$$

$y_2 = \frac{1}{c} y_1$

That is our sum is actually just one solution. So if we are worrying about finding all of the solutions we only want to add the linearly independent ones. It might be nice if we had a way to check if two functions were linearly independent.

If y_1, y_2 are solutions to the differential equation $y'' + p(x)y + q(x)y = 0$ on an interval I then the Wronskian defined as,

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_1' y_2,$$

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Will be nonzero for all points in I if y_1, y_2 are linearly independent and will be equal to zero if they are linearly dependent.

To see why the Wronskian will be zero when they are linearly dependent we only have to assume this and calculate it. If y_1 and y_2 are linearly dependent then $y_1 = c y_2$

$$\begin{aligned} \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} &= y_1 y_2' - y_1' y_2 \quad \text{but } y_1 = c y_2 \Rightarrow y_1' = c y_2' \\ & \quad y_2 = \frac{1}{c} y_1 \quad y_2' = \frac{1}{c} y_1' \\ &= y_1 \left(\frac{1}{c} y_1' \right) - y_1' \left(\frac{1}{c} y_1 \right) \\ &= \frac{1}{c} (y_1 y_1' - y_1' y_1) = 0 \end{aligned}$$

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Theorem: The General Solution

Let y_1, y_2 be two linearly independent solutions of the homogeneous equation

$$y'' + p(x)y + q(x)y = 0$$

$$y(a) = b_0, y'(a) = b_1$$

with p, q continuous on the open interval I . If Y is any solution to the differential equation above there exist numbers c_1, c_2 such that

$$Y(x) = c_1 y_1(x) + c_2 y_2(x)$$

for all $x \in I$.

To prove this we first show that we can find c_1, c_2 such that for $a \in I$

$$Y(a) = c_1 y_1(a) + c_2 y_2(a)$$

Since y_1, y_2 solve the ODE $\Rightarrow Y = c_1 y_1 + c_2 y_2$ solves ODE

$$Y(a) = b_0 = c_1 y_1(a) + c_2 y_2(a)$$

$$Y'(a) = b_1 = c_1 y_1'(a) + c_2 y_2'(a)$$

$$\begin{bmatrix} y_1(a) & y_2(a) \\ y_1'(a) & y_2'(a) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} b_0 \\ b_1 \end{bmatrix}$$

we'll have soln iff $\begin{vmatrix} y_1(a) & y_2(a) \\ y_1'(a) & y_2'(a) \end{vmatrix} \neq 0$

$$\begin{vmatrix} y_1(a) & y_2(a) \\ y_1'(a) & y_2'(a) \end{vmatrix} = y_1(a) y_2'(a) - y_1'(a) y_2(a)$$

= Wronskian
Since y_1, y_2 LI.
 $\neq 0$

Then we define the solution $G(x) = c_1 y_1(x) + c_2 y_2(x)$ which we know is a solution because y_1 and y_2 are.

Finally we look at the initial data at $x = a$. Because of the constants we found $G(a) = Y(a)$ and $G'(a) = Y'(a)$. Since both G, Y are solutions on I with the same initial data by our existence and uniqueness theorem they must be the same function. Thus if we find any solution to the DE above it must be that it can be written as a linear combination of two linearly independent functions.

$$G(x) = Y(x) \quad \text{By prev th.}$$

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A special case of second order homogeneous linear differential equations is that of the one with constant coefficients.

$$\underline{ay'' + by' + cy = 0}$$

These equations pop up often and we have some nice ways to solve them! We will cover more of them in later sections but for now we will look at one method called the characteristic equation.

Let's look at our equation and think about what kinds of solutions might make sense....

const (derivative) + const (derivative) + const (function) = 0

$$\begin{array}{l|l} \frac{d}{dx} e^{rx} = r e^{rx} & ar^2 e^{rx} + br e^{rx} + c e^{rx} = 0 \\ \frac{d^2}{dx^2} e^{rx} = r^2 e^{rx} & \underline{(ar^2 + br + c) e^{rx} = 0} \\ & \quad \quad \quad = 0 \end{array}$$

So e^{rx} is the kind of solution we are looking for and we could find r using what we call **the characteristic equation** $\rightarrow \underline{ar^2 + br + c = 0}$. Let's suppose r_1, r_2 are the two roots of the characteristic equation. What would the solution to the DE be? How do we know there are not any other solutions?

$y_1 = e^{r_1 x}$ is a soln if I plug it in $(ar_1^2 + br_1 + c) e^{r_1 x} = 0$
 r_1 is a root

$y_2 = e^{r_2 x}$ is a soln $(ar_2^2 + br_2 + c) e^{r_2 x} = 0$
 r_2 is a root

The full soln

$$\boxed{y = c_1 e^{r_1 x} + c_2 e^{r_2 x}}$$

$$\frac{e^{r_1 x}}{e^{r_2 x}} = e^{(r_1 - r_2)x} \quad \text{Not const}$$

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What if we have a repeated root r_1 ? Well clearly $e^{r_1 x}$ solves the DE from the same arguments as before but we know there should be two linearly independent functions for the full solution.

$$ay'' + by' + cy = 0 \quad ar^2 + br + c = 0 \quad \text{If only } r_1 \text{ as a root}$$

$$y_1 = e^{r_1 x} \text{ is a soln} \quad (ar_1^2 + br_1 + c)e^{r_1 x} = 0$$

We know there is another

$$y_2 = xe^{r_1 x}$$

$$y_2 = xe^{r_1 x}$$

$$y_2' = e^{r_1 x} + x r_1 e^{r_1 x}$$

$$y_2'' = r_1 e^{r_1 x} + r_1 e^{r_1 x} + x r_1^2 e^{r_1 x}$$

$$= 2r_1 e^{r_1 x} + x r_1^2 e^{r_1 x}$$

$$a(2r_1 e^{r_1 x} + x r_1^2 e^{r_1 x}) + b(e^{r_1 x} + x r_1 e^{r_1 x}) + c x e^{r_1 x}$$

$$x e^{r_1 x} (ar_1^2 + br_1 + c) + e^{r_1 x} (2ar_1 + b) \stackrel{?}{=} 0$$

$$\stackrel{!}{=} 0 \quad \stackrel{!}{=} 0 \quad \frac{d}{dx}(ar^2 + br + c)|_{r_1}$$

The characteristic equation for the differential equation

$$ay'' + by' + cy = 0$$

is the polynomial

$$ar^2 + br + c = 0$$



If this polynomial has two distinct real roots r_1, r_2 the solution to the differential equation above is

$$y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x}$$

If it has a real repeated root r_1 , the solution is

$$y(x) = c_1 e^{r_1 x} + c_2 x e^{r_1 x}$$

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Example: Find the general solution to the following DE $y'' - 2y' - 3y = 0$

$$ay'' + by' + cy = 0$$

$$y'' - 2y' - 3y = 0$$

$$a=1 \quad b=-2 \quad c=-3$$

characteristic eqn

$$ar^2 + br + c = 0$$

$$r^2 - 2r - 3 = 0$$

$$(r-3)(r+1) = 0$$

$$\Rightarrow r_1 = 3, r_2 = -1$$

$$y = c_1 e^{3x} + c_2 e^{-x}$$

Example: Find the general solution to the following DE $y'' + 4y' + 4y = 0$

$$ay'' + by' + cy = 0$$

$$y'' + 4y' + 4y = 0$$

$$a=1 \quad b=4 \quad c=4$$

characteristic eqn.

$$r^2 + 4r + 4 = 0$$

$$(r+2)^2 = 0$$

$$r = -2 \quad \text{repeated root}$$

$$y = c_1 e^{-2x} + c_2 x e^{-2x}$$