

Sec 3.2 General Solutions of Linear Equations

We are finally ready to broaden our solution methods to a general class of differential equations, n^{th} order linear differential equations.

An n^{th} order linear differential equation has the form:

$$p_0(x)y^{(n)} + p_1(x)y^{(n-1)} + p_2(x)y^{(n-2)} + \dots + p_n(x)y = F(x)$$

It is often convenient to write these in the form:

$$p_0 \neq 0$$

$$y^{(n)} + p_1(x)y^{(n-1)} + p_2(x)y^{(n-2)} + \dots + p_n(x)y = F(x)$$

Where under the assumption that $p_0(x) \neq 0$ on the intervals we wish to find solutions on, we have divided both sides by it.

An equation importantly tied to these linear differential equations is that of its **homogeneous equation**. This is when $F(x) = 0$

$$y^{(n)} + p_1(x)y^{(n-1)} + p_2(x)y^{(n-2)} + \dots + p_n(x)y = 0$$

As it will turn out, when we solve a non-homogeneous linear differential equation we will also have to solve its homogeneous relative. Before we get into that we have the following theorem.

Theorem Principle of superposition for homogeneous equations.

Let $y_1, y_2, \dots, y_n$ be solutions of the homogeneous linear differential equation,

$$y^{(n)} + p_1(x)y^{(n-1)} + p_2(x)y^{(n-2)} + \dots + p_n(x)y = 0,$$

on the interval I . If $c_1, c_2, \dots, c_n$ are constants then the linear combination

$$y = c_1y_1 + c_2y_2 + \dots + c_ny_n$$

is also a solution on the interval I

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Example: Show that $y = c_1 e^{-3x} + c_2 \sin(2x) + c_3 \cos(2x)$ is a solution to the third order linear differential equation:

$$y^{(3)} + 3y'' + 4y' + 12y = 0$$

Use your result to see that $y_1 = e^{-3x}$, $y_2 = \sin(2x)$, $y_3 = \cos(2x)$ are also individually solutions.

$$\begin{aligned} & (c_1 e^{-3x} + c_2 \sin(2x) + c_3 \cos(2x))''' + 3(c_1 e^{-3x} + c_2 \sin(2x) + c_3 \cos(2x))'' \\ & + 4(c_1 e^{-3x} + c_2 \sin(2x) + c_3 \cos(2x))' + 12(c_1 e^{-3x} + c_2 \sin(2x) + c_3 \cos(2x)) \end{aligned}$$

$$\begin{aligned} & (-27c_1 e^{-3x} - 8c_2 \cos(2x) + 8c_3 \sin(2x)) + 3(9c_1 e^{-3x} - 4c_2 \sin(2x) - 4c_3 \cos(2x)) \\ & + 4(-3c_1 e^{-3x} + 2c_2 \cos(2x) - 2c_3 \sin(2x)) + 12(c_1 e^{-3x} + c_2 \sin(2x) + c_3 \cos(2x)) \end{aligned}$$

$$= (-27c_1 e^{-3x} + 27c_1 e^{-3x} - 12c_1 e^{-3x} + 12c_1 e^{-3x}) = 0$$

$$+ (-8c_2 \cos(2x) - 12c_2 \sin(2x) + 8c_2 \cos(2x) + 12c_2 \sin(2x))$$

$$+ (8c_3 \sin(2x) - 12c_3 \cos(2x) - 8c_3 \sin(2x) + 12c_3 \cos(2x)) = 0 \checkmark$$

$$= c_1 [-27e^{-3x} + 3(9e^{-3x}) + 4(-3e^{-3x}) + 12e^{-3x}]$$

$$y_1 = e^{-3x} \quad y_1' = -3e^{-3x} \quad y_1'' = 9e^{-3x} \quad y_1^{(3)} = -27e^{-3x}$$

$$= c_1 [(e^{-3x})''' + 3(e^{-3x})'' + 4(e^{-3x})' + 12(e^{-3x})]$$

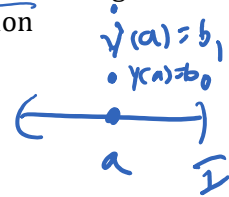
$$c_1 [y_1^{(3)} + 3y_1'' + 4y_1' + 12y_1] = 0$$

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Theorem: Existence and Uniqueness for Linear Equations

Suppose that the functions $p_1, p_1 \dots, p_n$ and f are continuous on the open interval I containing the point a . Then given n numbers $b_0, b_1 \dots, b_{n-1}$ the n^{th} order linear differential equation

$$y^{(n)} + p_1(x)y^{(n-1)} + p_2(x)y^{(n-2)} + \dots + p_n(x)y = F(x)$$



Has a unique solution on the entire interval I and that solution satisfies the n initial conditions

$$y(a) = b_0, y'(a) = b_1, \dots, y^{(n-1)}(a) = b_{n-1}$$

Let's some initial data to our problem $y^3 + 3y'' + 4y' + 12y = 0$ where $y = c_1 e^{-3x} + c_2 \sin(2x) + c_3 \cos(2x)$ was a solution. Let's say $y(0) = 0, y'(0) = 5$, and $y''(0) = -39$. What are the values of c_1, c_2, c_3 that give let y solve the initial value problem?

$$y(0) = 0$$

$$0 = c_1 e^0 + c_2 \sin(2(0)) + c_3 \cos(2(0))$$

$$0 = c_1 + c_2(0) + c_3 = c_1 + c_3$$

$$y' = -3c_1 e^{-3x} + 2c_2 \cos(2x) - 2c_3 \sin(2x)$$

$$y'(0) = -3c_1 + 2c_2 = 5$$

$$y'' = 9c_1 e^{-3x} - 4c_2 \sin(2x) - 4c_3 \cos(2x)$$

$$y''(0) = 9c_1 - 4c_3 = -39$$

$$\Rightarrow c_1 + c_3 = 0 \quad (1)$$

$$-3c_1 + 2c_2 = 5 \quad (2)$$

$$9c_1 - 4c_3 = -39 \quad (3)$$

$$\begin{aligned} -9c_1 - 9c_3 &= 0 \\ + 9c_1 - 4c_3 &= -39 \end{aligned}$$

$$-13c_3 = -39$$

$$c_3 = 39/13 = 3$$

$$c_1 + 3 = 0$$

$$c_1 = -3$$

$$-3(-3) + 2c_2 = 5$$

$$9 + 2c_2 = 5$$

$$2c_2 = -4$$

$$c_2 = -2$$

$$\Rightarrow y = -3e^{-3x} - 2\sin(2x) + 3\cos(2x)$$

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We saw and proved in Section 3.2 that for a **second** order homogeneous linear differential equation the solution was always a linear combination of **two linearly independent functions**.

Theorem: The General Solution

Let y_1, y_2 be two linearly independent solutions of the homogeneous equation

$$y'' + p(x)y + q(x)y = 0, \quad y(a) = b_0, y'(a) = b_1$$

with p, q continuous on the open interval I . If Y is any solution to the differential equation above there exist numbers c_1, c_2 such that

$$Y(x) = c_1 y_1(x) + c_2 y_2(x)$$

for all $x \in I$.

Something extremely similar is true for n th order equations... In fact a second order equation is just a special case of an n th order one.

Theorem: General solutions of homogeneous equations.

Let $y_1, y_2, \dots, y_n$ be n **linearly independent** solutions of the homogeneous equation

$$y^{(n)} + p_1(x)y^{(n-1)} + p_2(x)y^{(n-2)} + \dots + p_n(x)y = 0$$

on an open interval I where the p_i are continuous. If Y is any solution whatsoever of the differential equation above, then there exist numbers $c_1, c_2, \dots, c_n$ such that

$$Y(x) = c_1 y_1 + c_2 y_2 + \dots + c_n y_n$$

for all x in I .

Let's just think about what this says... It is telling us that if we find all n linearly independent functions that solve the differential equation we have found them **all**. This theorem allows to actually know when we have fully solved the differential equation.

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In section 3.1 we discussed when two functions f_1, f_2 were linearly independent. That what when one was **not** a multiple of the other, ie $f_1 \neq kf_2$

To extend this to n functions $f_1, f_2, \dots, f_n$ we can say they are linearly independent if none of them can be written as a linear combination of the others.

There are no $c_1, \dots, c_n$ s.t.

$$f_i = c_1 f_1 + c_2 f_2 + \dots + c_{n-1} f_{n-1}$$

Another way to say this is

The n functions $f_1, f_2, \dots, f_n$ are **linearly independent** on an interval I if the equation

$$c_1 f_1 + c_2 f_2 + \dots + c_n f_n = 0$$

Implies that $c_1 = c_2 = \dots = c_n = 0$

The n functions $f_1, f_2, \dots, f_n$ are **linearly dependent** on an interval I if the equation

$$c_1 f_1 + c_2 f_2 + \dots + c_n f_n = 0$$

Is true for some set of nonzero $c_1, c_2, \dots, c_n$

Just like before we can check for linear independence using the Wronskian.

$$W = \begin{vmatrix} f_1 & f_2 & \dots & f_n \\ f_1' & f_2' & \dots & f_n' \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \dots & f_n^{(n-1)} \end{vmatrix}$$

But before we do that let's look at some examples to gain intuition for this definition

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Example: Show that $y_1 = (x+2)^2$, $y_2 = 2x^2 + 8x$, $y_3 = 8$ are linearly dependent by directly finding $c_1, c_2, c_3 \neq 0$ such that $c_1 y_1 + c_2 y_2 + c_3 y_3 = 0$.

$$c_1(x+2)^2 + c_2(2x^2+8x) + c_3 8 = 0$$

$$c_1(x^2+4x+4) + c_2(2x^2+8x) + c_3 8 = 0$$

choose

$$c_1 = -2 \quad c_2 = 1 \quad c_3 = 1$$

$$-2(x^2+4x+4) + (2x^2+8x) + 8$$

$$-2x^2 - 8x - 8 + 2x^2 + 8x + 8 = 0$$

Example: Show that $y_1 = 1$, $y_2 = x$, $y_3 = x^2$ by assuming $c_1(1) + c_2x + c_3x^2 = 0$ means that $c_1 = c_2 = c_3 = 0$

$$\rightarrow c_1 + c_2x + c_3x^2 = 0 \rightarrow c_1 + 0x + 0(x)^2 = 0$$

$$\rightarrow c_2 + 2c_3x = 0$$

$$\rightarrow \frac{2c_3 = 0}{c_3 = 0}$$

$$c_2 + 2(0)(x) = 0$$

$$c_2 = 0$$

$$c_1 = 0$$

$$c_1 = c_2 = c_3 = 0$$

Linearly Independent

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Suppose that $y_1, y_2, \dots, y_n$ are n solutions of the homogeneous n^{th} order linear equation

$$y^{(n)} + p_1(x)y^{(n-1)} + p_2(x)y^{(n-2)} + \dots + p_n(x)y = 0$$

On an open interval I where p_i are continuous. Then the Wronskian of $y_1, y_2, \dots, y_n$, W is either zero or nonzero on I and

- (a) If $y_1, y_2, \dots, y_n$ are linearly dependent then $W = 0$ on I
 (b) If $y_1, y_2, \dots, y_n$ are linearly independent then $W \neq 0$ on I

Where $W = \begin{vmatrix} y_1 & y_2 & \dots & y_n \\ y_1' & y_2' & \dots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \dots & y_n^{(n-1)} \end{vmatrix}$

Example: Show that $y_1 = x, y_2 = x \ln(x), y_3 = x^2$ are linearly independent on the interval $(0, \infty)$

$$\begin{vmatrix} x & x \ln(x) & x^2 \\ 1 & \ln(x) + 1 & 2x \\ 0 & \frac{1}{x} & 2 \end{vmatrix} = x \begin{vmatrix} \ln(x) + 1 & 2x \\ \frac{1}{x} & 2 \end{vmatrix} - \begin{vmatrix} x \ln(x) & x^2 \\ \frac{1}{x} & 2 \end{vmatrix} + 0 \begin{vmatrix} x \ln(x) & x^2 \\ \ln(x) + 1 & 2x \end{vmatrix}$$

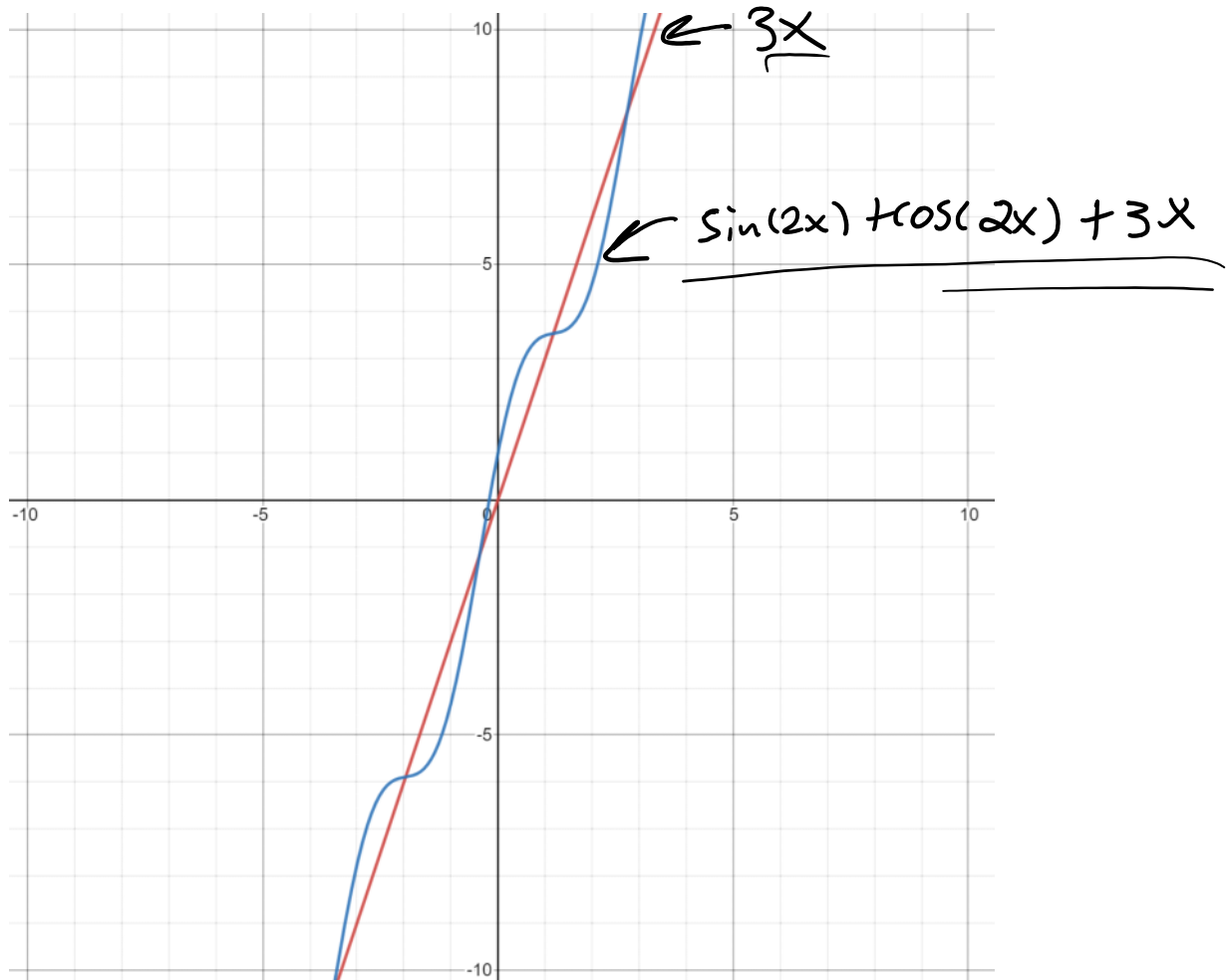
$$= x \left(2(\ln(x) + 1) - \frac{2x}{x} \right) - \left(2x \ln(x) - \frac{x^2}{x} \right)$$

$$= 2x \ln(x) + 2 - 2 - 2x \ln(x) + x$$

$$= x \neq 0 \text{ on } (0, \infty)$$

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If this equation is modeling something physical we had better make sure we actually have the full solution! We can see how different just the particular solution from the full solution in the plot below.



Theorem: Solutions of Nonhomogeneous Equations.

Let y_p be a particular solution of the nonhomogeneous linear differential equation

$$y^{(n)} + p_1(x)y^{(n-1)} + p_2(x)y^{(n-2)} + \cdots + p_n(x)y = f(x)$$

On an open interval I where the p_i and f are continuous. Let $y_1, y_2, \dots, y_n$ be linearly independent solutions of the associated homogeneous equation,

$$y^{(n)} + p_1(x)y^{(n-1)} + p_2(x)y^{(n-2)} + \cdots + p_n(x)y = 0.$$

If Y is any solution whatsoever of the non-homogeneous equation on I there exist $c_1, c_2, \dots, c_n$ such that

$$Y(x) = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n + y_p$$

for all $x \in I$

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Example: Find the General Solution to $y'' - 4y = 12$

$$\underline{y_p = 3}$$

$$\begin{aligned} y_p &= -3 \\ y_p' &= 0 \\ y_p'' &= 0 \end{aligned} \quad 0 - 4(-3) = \underline{12}$$

also solve

$$y'' - 4y = 0$$

$$ay'' + by' + cy = 0$$

C.E.

$$a=1 \quad b=0 \quad c=-4$$

$$ar^2 + br + c = 0$$

$$r^2 - 4 = 0$$

$$r^2 = 4$$

$$r = \pm\sqrt{4} = \pm 2$$

$$y_c = c_1 e^{2x} + c_2 e^{-2x}$$

full

$$y = y_c + y_p = c_1 e^{2x} + c_2 e^{-2x} - 3$$