

Section 3.3 Homogeneous Equations With Constant Coefficients

In this section we will study solution methods for n th order homogeneous linear differential equations with constant coefficients. That is equations of the form

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_2 y'' + a_1 y' + a_0 y = 0$$

Where the a_i 's are all constants.

From Section 3.2 we learned that n^{th} order homogeneous linear differential equations have n linearly independent solutions.

→ Theorem: General solutions of homogeneous equations.

Let $y_1, y_2, \dots, y_n$ be n **linearly independent** solutions of the homogeneous equation

$$y^{(n)} + p_1(x)y^{(n-1)} + p_2(x)y^{(n-2)} + \dots + p_n(x)y = 0$$

on an open interval I where the p_i are continuous. If Y is any solution whatsoever of the differential equation above, then there exist numbers $c_1, c_2, \dots, c_n$ such that

$$Y(x) = c_1 y_1 + c_2 y_2 + \dots + c_n y_n$$

for all x in I .



We will invoke this theorem to know when we have found all of the possible solutions to one of our n^{th} order equations as we develop solution methods.

You might recall from section 3.1 that we were able to solve second order equations like

$$ay'' + by' + cy = 0$$

Assuming a solution of the form $y = e^{rx}$ plugging this into our differential equation to get

$$ar^2 e^{rx} + bre^{rx} + ce^{rx} = 0$$

Factoring out the e^{rx} and dividing by it gave us the characteristic equation we used to find solutions.

$$(ar^2 + br + c)e^{rx} = 0 \rightarrow ar^2 + br + c = 0$$

↑ never zero

$$\begin{aligned} r_1, r_2 & \quad y_1 = e^{r_1 x} \quad y_2 = e^{r_2 x} \\ y &= c_1 e^{r_1 x} + c_2 e^{r_2 x} \end{aligned}$$

Section 3.3 Homogeneous Equations With Constant Coefficients

We then solved that equation and were able to build solutions using its roots.

The characteristic equation for the differential equation

$$ay'' + by' + cy = 0$$

is the polynomial

$$ar^2 + br + c = 0$$

If this polynomial has two distinct real roots r_1, r_2 the solution to the differential equation above is

$$y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x}$$

If it has a real repeated root r_1 , the solution is

$$y(x) = c_1 e^{r_1 x} + c_2 x e^{r_1 x}$$

$$(r - r_1)^2$$

Today we will play the same game. For an n^{th} order equation we have

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_2 y'' + a_1 y' + a_0 y = 0$$

Again we see a mix of multiples of derivatives of our unknown function and itself all adding up to zero. Just as before this might lead us to try a solution like

$$\begin{aligned} \rightarrow y &= e^{rx} \\ \rightarrow y' &= r e^{rx} \\ \rightarrow y'' &= r^2 e^{rx} \\ \rightarrow y^{(3)} &= r^3 e^{rx} \\ &\vdots \\ \rightarrow y^{(n)} &= r^n e^{rx} \end{aligned}$$

Since derivatives of it are just multiples of it and if each derivative of the equation is multiplied by the right constants we might get it all to add up to zero.

$$a_n r^n e^{rx} + a_{n-1} r^{n-1} e^{rx} + \dots + a_2 r^2 e^{rx} + a_1 r e^{rx} + a_0 e^{rx} = 0$$

To find values for r that might work we again factor out the e^{rx} and divide it away

Section 3.3 Homogeneous Equations With Constant Coefficients

$$(a_n r^n + a_{n-1} r^{n-1} + \dots + a_2 r^2 + a_1 r + a_0) e^{rx} = 0 \rightarrow a_n r^n + a_{n-1} r^{n-1} + \dots + a_2 r^2 + a_1 r + a_0 = 0$$

So we again have what we call the characteristic equation, sometimes also called the auxiliary equation.

$$a_n r^n + a_{n-1} r^{n-1} + \dots + a_2 r^2 + a_1 r + a_0 = 0$$

Note that for the unknown number r that this is just an n^{th} degree polynomial. For an equation like this we have 3 possibilities of solutions for r .

1. n distinct roots in which case the characteristic equation factors as:

$$a_n (r - r_1)(r - r_2)(r - r_3) \dots (r - r_n) = 0$$

2. $k < n$ repeated roots and $n - k$ distinct roots where $k = m$ having all repeated roots. In which case the characteristic equation factors as:

$$(r - r_1)(r - r_2) \dots (r - r_{n-k})(r - p)^k$$

3. Some or all complex roots ($r = a \pm bi$) in which case the characteristic equation would factor like:

$$(r - r_1)(r - r_2)(r - (a + bi))(r - (a - bi))$$

The simplest case is that of n distinct roots $r_1, r_2, \dots, r_n$ since each $y_i = e^{r_i x}$ is a solution to the differential equation since,

$$a_n r_i^n e^{r_i x} + a_{n-1} r_i^{n-1} e^{r_i x} + \dots + a_1 r_i e^{r_i x} + a_0 e^{r_i x} = 0$$

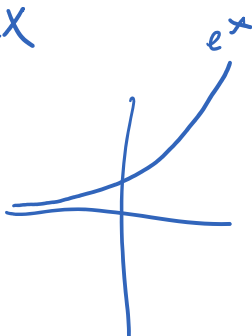
$$(a_n r_i^n + a_{n-1} r_i^{n-1} + \dots + a_1 r_i + a_0) e^{r_i x} = 0 \quad \checkmark \text{ Since } r_i \text{ is a root}$$

$$= 0$$

Since there are n of them and since if $r_i \neq r_j$ $y_i = e^{r_i x}$, $y_j = e^{r_j x}$ will be linearly independent. Easy to check with the wronskian...

$$y_1 = e^{r_1 x} \quad y_2 = e^{r_2 x} \quad r_1 \neq r_2$$

$$\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{r_1 x} & e^{r_2 x} \\ r_1 e^{r_1 x} & r_2 e^{r_2 x} \end{vmatrix} = r_2 e^{r_1 x} e^{r_2 x} - r_1 e^{r_1 x} e^{r_2 x}$$

$$= (r_2 - r_1) e^{r_1 x} e^{r_2 x} \neq 0 \neq 0$$


Section 3.3 Homogeneous Equations With Constant Coefficients

We then know that if we have n linearly independent solutions of our n^{th} order linear homogeneous equation that the full solution is a linear combination of all of them. This gives us the following theorem:

For the n^{th} order linear homogeneous differential equation with constant coefficients given by:

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_2 y'' + a_1 y' + a_0 y = 0$$

Which has characteristic equation

$$a_n r^n + a_{n-1} r^{n-1} + \dots + a_2 r^2 + a_1 r + a_0 = 0$$

That has n distinct roots $r_1, r_2, \dots, r_n$ has the general solution,

$$Y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x} + \dots + c_{n-1} e^{r_{n-1} x} + c_n e^{r_n x}$$

Example: Find the general solution the differential equation

$$y^{(4)} + 6y^{(3)} + 7y'' - 6y' - 8y = 0$$

Our Characteristic equation is:

$$r^4 + 6r^3 + 7r^2 - 6r - 8 = 0$$

$$(1) r^4 + 6r^3 + 7r^2 - 6r - 8 = 0$$

The hardest part of all this is factoring higher order polynomials. One trick to finding some roots comes from the rational root theorem from Algebra.

If the polynomial $a_n r^n + a_{n-1} r^{n-1} + \dots + a_2 r^2 + a_1 r + a_0 = 0$ has integer coefficients then any rational roots of this equation will have the form $\pm p/q$ where p is an integer factor of a_0 and q an integer factor of a_n .

$$a_0 = 8$$

$$P: 8, 4, 2, 1$$

easy to check 1, -1

$$a_n = (1)$$

$$Q: 1$$

Possible roots: $\pm 8, \pm 4, \pm 2, \pm 1$

$$\begin{aligned} r=1 \\ 1^4 + 6(1)^3 + 7(1)^2 - 6(1) - 8 &= 0 \\ 1 + 6 + 7 - 6 - 8 &= 0 \quad (x-1) \text{ factor} \end{aligned}$$

$$\begin{aligned} r=-1 \\ (-1)^4 + 6(-1)^3 + 7(-1)^2 - 6(-1) - 8 &= 0 \\ 1 - 6 + 7 + 6 - 8 &= 0 \quad (x+1) \text{ factor} \end{aligned}$$

Section 3.3 Homogeneous Equations With Constant Coefficients

So we found that $r = 1, r = -1$ are roots this means that $(r+1)(r-1) = r^2 - 1$ are both factors of the polynomial

So if we want to simplify our polynomial with this knowledge we can divide by them! This is the same as dividing by $r^2 - 1$. Polynomial long division here we go!

$$\begin{array}{r}
 \begin{array}{r}
 r^2 - 1 \overline{) r^4 + 6r^3 + 7r^2 - 6r - 8} \\
 \underline{-r^4 + r^2} \\
 6r^3 + 8r^2 - 6r - 8 \\
 \underline{-6r^3 + 6r} \\
 8r^2 - 8 \\
 \underline{-8r^2 + 8} \\
 0
 \end{array}
 \end{array}
 \quad
 \begin{array}{l}
 (r^2 - 1) r^2 = r^4 - r^2 \\
 6r(r^2 - 1) \\
 6r^3 - 6r \\
 8(r^2 - 1) \\
 8r^2 - 8
 \end{array}$$

$$\begin{aligned}
 r^4 + 6r^3 + 7r^2 - 6r - 8 &= (r+1)(r-1)(r^2 + 6r + 8) \\
 (r+1)(r-1)(r+2)(r+4) &= 0 \\
 r_1 = 1, r_2 = -1, r_3 = -2, r_4 = -4
 \end{aligned}$$

So we have the roots $r_1 = 1, r_2 = -1, r_3 = -2, r_4 = -4$ so our general solution is:

$$y = c_1 e^{1x} + c_2 e^{-1x} + c_3 e^{-2x} + c_4 e^{-4x}$$

$$\begin{aligned}
 y(a) &= b_1 \\
 y'(a) &= b_2 \\
 y''(a) &= b_3 \\
 y'''(a) &= b_4
 \end{aligned}
 \quad \text{for particular soln.}$$

Section 3.3 Homogeneous Equations With Constant Coefficients

Now let's deal with what happens with repeated roots. Since we know that we need n linearly independent solutions to form the full general solution simply repeating e^{rx} won't cut it.

To figure out what the general solution should look like consider again our basic equation:

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_2 y'' + a_1 y' + a_0 y = 0$$

Which could also be written as

$$a_n D^n y + a_{n-1} D^{n-1} y + \dots + a_2 D^2 y + a_1 D y + a_0 y = 0$$

Where $D = \frac{d}{dx}$, $D^2 = \frac{d^2}{dx^2}$, ..., $D^n = \frac{d^n}{dx^n}$

If we divide by a_n and assume we appropriately change the other a 's for something nice we have

$$D^n y + a_{n-1} D^{n-1} y + \dots + a_2 D^2 y + a_1 D y + a_0 y = 0$$

Note that in this form we can factor out y

$$(D^n + a_{n-1} D^{n-1} + \dots + a_2 D^2 + a_1 D + a_0) y = 0$$

$Ly = 0$

Then we can factor the differential operators!

$$(D^n + a_{n-1} D^{n-1} + \dots + a_2 D^2 + a_1 D + a_0) y = (D - r_1)(D - r_2) \dots (D - r_n) y$$

Or suppose it factors with a repeated root of multiplicity k with only one other root it might look like

$$D^n + a_{n-1} D^{n-1} + \dots + a_2 D^2 + a_1 D + a_0 = (D - r_1)(D - r_2)^k$$

Before we keep going let's just make this concrete with an example....

$$\text{Show that } y'' + 4y' + 3y = D^2 y + 4Dy + 3y = (D+3)(D+1)y$$

$$(D+3)(D+1)y = (D+3)(Dy + y) = D^2 y + Dy + 3Dy + 3y$$

$$= D^2 y + 4Dy + 3y = y'' + 4y' + 3y = (D+1)(D+3)y$$

Section 3.3 Homogeneous Equations With Constant Coefficients

So if

$$D^n y + a_{n-1} D^{n-1} y + \dots + a_2 D^2 y + a_1 D y + a_0 y = 0$$

Has characteristic equation $r^n + a_{n-1} r^{n-1} + \dots + a_1 r + a_0$ with roots $r_1, r_2 \dots r_n$ it can be written as

$$(D - r_1)(D - r_2) \dots (D - r_n) y = 0$$

Also suppose we have a repeated root ρ of multiplicity $k < n$ then it could be written as

$$(D - r_1)(D - r_2) \dots (D - r_{n-k})(D - \rho)^k y = 0$$

With this out of the way we can figure out what the general solution in the case of repeated roots looks like.

Let's suppose we have a repeated root ρ of our characteristic equation with multiplicity k then our DE has the form $(D - r_1)(D - r_2) \dots (D - r_{n-k})(D - \rho)^k y = 0$ note that if y is such that $(D - \rho)^k y = 0$ that y would indeed be a solution. So let's just look at

$$(D - \rho)^k y = 0$$

We already know $y = e^{\rho x}$ is a solution but there need to be $k - 1$ more. Let's try to find another one by assuming it has the form $y = u(x)e^{\rho x}$

What happens when we calculate $(D - \rho)u(x)e^{\rho x}$?

$$(D - \rho)u(x)e^{\rho x} = D(u(x)e^{\rho x}) - \rho u(x)e^{\rho x} = u'(x)e^{\rho x} + \cancel{u(x)\rho e^{\rho x}} - \cancel{e^{\rho x}\rho u(x)} = u'(x)e^{\rho x}$$

So if we apply this k times $(D - \rho)^k u(x)e^{\rho x}$ we would just get....

$$\begin{aligned} (D - \rho)^2 u(x)e^{\rho x} &= (D - \rho)[(D - \rho)u(x)e^{\rho x}] = (D - \rho)[u'(x)e^{\rho x}] \\ &= D(u'(x)e^{\rho x}) - \rho u'(x)e^{\rho x} \\ &= u''(x)e^{\rho x} + \cancel{u'(x)\rho e^{\rho x}} - \cancel{\rho u'(x)e^{\rho x}} \\ \Rightarrow (D - \rho)^k u(x)e^{\rho x} &= u^{(k)}(x)e^{\rho x} = [D^k u(x)]e^{\rho x} \end{aligned}$$

Section 3.3 Homogeneous Equations With Constant Coefficients

So $(D - \rho)^k u(x) e^{\rho x} = e^{\rho x} [D^k u(x)]$ this will be zero as long as $D^k u(x) = 0$ This will happen exactly when $u(x)$ is a polynomial of degree $k - 1$! That is

$$u(x) = (1 + x + x^2 + x^3 + \dots + x^{k-1})$$

$$\begin{aligned} (D - \rho)^k u(x) e^{\rho x} &= 0 \\ &= e^{\rho x} [D^k u(x)] = 0 \\ &= 0 \end{aligned}$$

What's more is that the set of functions $\{1, x, x^2, \dots, x^{k-1}\}$ are linearly independent! This means we now have the other k functions we need for the full solution.... Namely

$$y = (1 + x + x^2 + \dots + x^{k-1}) e^{\rho x} = \underline{e^{\rho x}} + \underline{x e^{\rho x}} + \underline{x^2 e^{\rho x}} + \dots + \underline{x^{k-1} e^{\rho x}}$$

$$y_1 = e^{\rho x}, y_2 = x e^{\rho x}, y_3 = x^2 e^{\rho x}, \dots, y_k = x^{k-1} e^{\rho x}$$

This gives us the following theorem:

For the n^{th} order linear homogeneous differential equation with constant coefficients given by:

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_2 y'' + a_1 y' + a_0 y = 0$$

Which has characteristic equation

$$a_n r^n + a_{n-1} r^{n-1} + \dots + a_2 r^2 + a_1 r + a_0 = 0$$

That has $n - k$ distinct roots and a root ρ with multiplicity k has the general solution,

$$Y(x) = \underline{c_1 e^{r_1 x} + c_2 e^{r_2 x} + \dots + c_{n-k} e^{r_{n-k} x}} + \underline{c_{n-(k+1)} e^{\rho x} + c_{n-(k+2)} x e^{\rho x} + c_{n-(k+3)} x^2 e^{\rho x} + \dots + c_n x^{k-1} e^{\rho x}}$$

Example: Find the general solution to the DE $y^{(3)} + 3y'' - 4y = 0$

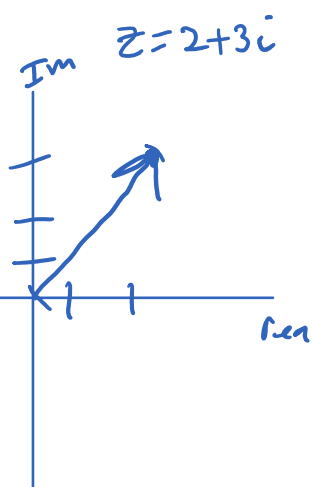
Note: $r^3 + 3r^2 - 4 = (r - 1)(r + 2)^2$

$$\begin{aligned} &\downarrow \\ &r^3 + 3r^2 - 4 = 0 \\ &(r - 1)(r + 2)^2 \leftarrow \text{multiplicity} \quad r_1 = 1 \quad r_2 = 2 \quad \text{with multiplicity } k = 2 \end{aligned}$$

$$y(x) = c_1 e^x + c_2 e^{-2x} + c_3 x e^{-2x}$$

Section 3.3 Homogeneous Equations With Constant Coefficients

Now we are ready to deal with complex roots! Recall what a complex number is:

$$\begin{aligned} & \frac{a+bi}{i^2 = -1} \quad \sqrt{-1} = i \\ & \frac{1}{i} = -i \\ & \frac{1}{i} \cdot \frac{i}{i} = \frac{i}{i^2} = \frac{i}{-1} = -i \\ & (a_1 + b_1 i) + (a_2 + b_2 i) = (a_1 + a_2) + (b_1 + b_2)i \\ & (a_1 + b_1 i)(a_2 + b_2 i) = (a_1 a_2 - b_1 b_2 + (a_1 b_2 + a_2 b_1)i) \text{ (foil it if you like.)} \\ & \quad \quad \quad i^2 = -1 \end{aligned}$$


With that memory jog lets now talk about the wonderful Euler formula!

$$e^{ix} = \cos(x) + i\sin(x)$$

If you recall from calculus the taylor series expansions for e^x , $\sin(x)$, and $\cos(x)$ are:

$$\begin{aligned} e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \dots = \sum_{n=0}^{\infty} \frac{x^n}{n!} \\ \sin(x) &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \\ \cos(x) &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \end{aligned}$$

$$e^{i\pi} = \cos(\pi) + i\sin(\pi) = -1$$

Let's plug in ix to e^x and see what happens....

$$\begin{aligned} e^{ix} &= 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} + \frac{(ix)^6}{6!} \\ &= 1 + ix + \frac{-x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!} + \frac{ix^5}{5!} - \frac{x^6}{6!} \\ &= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \right) + i \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \right) \\ &= \cos(x) + i\sin(x) \end{aligned}$$

Section 3.3 Homogeneous Equations With Constant Coefficients

So $e^{ix} = \cos(x) + i\sin(x)$ Let's keep this in mind while we solve $y'' - 2y' + 5y = 0$ Let's solve the characteristic equation

$$r^2 - 2r + 5 = 0$$

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\frac{2 \pm \sqrt{4 - 4(1)(5)}}{2(1)} = 1 \pm \frac{\sqrt{-16}}{2} = 1 \pm \frac{4i}{2}$$

$$= 1 \pm 2i \quad 1 + 2i \quad 1 - 2i$$

$$(x - (1 + 2i))(x - (1 - 2i))$$

$\sqrt{(-1)(16)}$
 $\sqrt{-1}\sqrt{16}$
 $i4$

So we have two roots of the characteristic equation $r_1 = 1 + 2i$, $r_2 = 1 - 2i$ recall that our characteristic equation was based on the assumption that $y = e^{rx}$ in this case what does our solutions look like?

$$y = c_1 e^{(1+2i)x} + c_2 e^{(1-2i)x} = c_1 e^x e^{2ix} + c_2 e^x e^{-2ix}$$

$$= e^x (c_1 e^{2ix} + c_2 e^{-2ix}) = e^x [c_1 (\cos(2x) + i\sin(2x)) + c_2 (\cos(-2x) + i\sin(-2x))]$$

$$= e^x ((c_1 + c_2)\cos(2x) + (c_1 - c_2)\sin(2x))$$

cos even so
 $\cos(-2x) = \cos(2x)$
 sin odd
 $\sin(-2x) = -\sin(2x)$

So our general solution when we have complex roots $a + bi$, $a - bi$ is $y(x) = e^{ax} ((c_1 + c_2)\cos(bx) + (c_1 - c_2)\sin(bx))$
 Let's suppose we have real initial data though for our previous problem through namely $y(0) = 1, y'(0) = 0$ what are c_1, c_2 ?

We have that

$$\rightarrow y = e^x ((c_1 + c_2)\cos(2x) + (c_1 - c_2)\sin(2x))$$

and

$$\rightarrow y' = e^x ((c_1 + c_2)\cos(2x) + (c_1 - c_2)\sin(2x)) + e^x (-2(c_1 + c_2)\sin(2x) + 2(c_1 - c_2)\cos(2x))$$

So our initial conditions give us two equations

$$\begin{aligned} \rightarrow 1 &= c_1 + c_2 \\ 0 &= (c_1 + c_2) + 2i(c_1 - c_2) \end{aligned}$$

Lets find c_1, c_2

Section 3.3 Homogeneous Equations With Constant Coefficients

$$1 = c_1 + c_2$$

$$0 = \underbrace{(c_1 + c_2)}_{=1} + 2i(c_1 - c_2)$$

$$0 = 1 + 2i(c_1 - c_2)$$

$$-1 = 2i(c_1 - c_2)$$

$$\frac{-1}{2i} = c_1 - c_2$$

$$-\frac{1}{2}(-i) = c_1 - c_2$$

$$\frac{1}{2}i = c_1 - c_2$$

$$y = e^x \left(\underbrace{(c_1 + c_2)}_{=1} \cos(2x) + \underbrace{(c_1 - c_2)}_{=\frac{1}{2}i} i \sin(2x) \right)$$

$$y = e^x \left(\cos(2x) + \frac{1}{2} i \cdot i \sin(2x) \right)$$

$$y = e^x \left(\cos(2x) - \frac{1}{2} \sin(2x) \right)$$

$$\frac{1}{i} \cdot \frac{i}{i} = \frac{i}{i^2} = \frac{i}{-1} = -i$$

So our particular solution would be $y = e^x \left(\cos(2x) - \frac{1}{2} \sin(2x) \right)$

What if we had just guessed the solution $y = e^x(k_1 \cos(2x) + k_2 \sin(2x))$ and applied the same initial data?

We would have $1 = k_1$, $0 = k_1 + 2k_2$

So $0 = 1 + 2k_2$

Then $k_2 = -\frac{1}{2}$

And we recover the same solution!

$$y = e^x \left(\cos(2x) - \frac{1}{2} \sin(2x) \right)$$

The point is that if we use real initial data what multiplies our sines and cosines will end up real so we may as well guess the solution

$$y(x) = e^{ax}(c_1 \cos(bx) + c_2 \sin(bx))$$

Section 3.3 Homogeneous Equations With Constant Coefficients

This gives us the following theorem:

For the n^{th} order linear homogeneous differential equation with constant coefficients given by:

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_2 y'' + a_1 y' + a_0 y = 0$$

Which has characteristic equation

$$a_n r^n + a_{n-1} r^{n-1} + \dots + a_2 r^2 + a_1 r + a_0 = 0$$

That $k < n$ complex roots will have the general solution

$$a_1 + b_1 i \dots a_n + b_n i$$

$$Y(x) = y_{\text{real roots}} + e^{a_1 x} (c_1^1 \cos(b_1 x) + c_2^1 \sin(b_1 x)) + \dots + e^{a_k x} (c_1^k \cos(b_k x) + c_2^k \sin(b_k x))$$

Where $y_{\text{real roots}}$ will follow the rules of the previous two theorems depending on having distinct real or repeating roots.

Example Find the general solution to the differential equation

$$y^{(5)} + 6y^{(4)} + 10y^{(3)} + 16y'' + 69y' + 90y = 0$$

$$\text{Note } r^5 + 6r^4 + 10r^3 + 16r^2 + 69r + 90 = (r+2)(r+3)^2(r-(1+2i))(r-(1-2i))$$

$$r = -2$$

$$y_1 = c_1 e^{-2x}$$

$$r = -3 \text{ multiplicity } 2$$

$$y_2 = c_2 e^{-3x}$$

$$y_3 = c_3 x e^{-3x}$$

$$\begin{matrix} a+bi \\ a-bi \end{matrix}$$

$$\begin{matrix} a+bi & a=1 & b=2 \\ a-bi \end{matrix}$$

$$y_4 = e^x (c_4 \cos(2x) + c_5 \sin(2x))$$

$$Y(x) = c_1 e^{-2x} + c_2 e^{-3x} + c_3 x e^{-3x} + e^x [c_4 \cos(2x) + c_5 \sin(2x)]$$

Section 3.3 Homogeneous Equations With Constant Coefficients

What happens if our complex roots are repeated?

This repeated root theorem still applies.

For the n^{th} order linear homogeneous differential equation with constant coefficients given by:

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_2 y'' + a_1 y' + a_0 y = 0$$

Which has characteristic equation

$$a_n r^n + a_{n-1} r^{n-1} + \dots + a_2 r^2 + a_1 r + a_0 = 0$$

That has $n - k$ other roots and a complex roots $\rho = a \pm bi$ with multiplicity k has the general solution,

$$Y(x) = y_{\text{other roots}} + \sum_{p=0}^{k-1} x^p e^{ax} (c_p \cos(bx) + d_p \sin(bx))$$

Where $y_{\text{other roots}}$ follows the rules for those types of roots.

Example find the general solution to $y^{(4)} - 4x^{(3)} + 14y'' - 20y' + 25 = 0$

$$\text{note: } r^4 - 4r^3 + 14r^2 - 20r + 25 = (r^2 - 2r + 5)^2 = ((x - (1 + 2i))(x - (1 - 2i)))^2$$

$$\frac{2 \pm \sqrt{4 - 4(1)(5)}}{2} = \frac{2 \pm \sqrt{4 - 20}}{2} = \frac{2 \pm \sqrt{-16}}{2} = \frac{2 \pm 4i}{2} = 1 \pm 2i$$

$$y_1 = e^x (c_1 \cos(2x) + c_2 \sin(2x)) \quad , \quad y_2 = e^x x (c_3 \cos(2x) + c_4 \sin(2x))$$

$$y = e^x (c_1 \cos(2x) + c_2 \sin(2x)) + x e^x (c_3 \cos(2x) + c_4 \sin(2x))$$

Section 3.3 Homogeneous Equations With Constant Coefficients

So in summary if you have a linear homogeneous differential equation with constant coefficients you will solve the characteristic equation and based on the types of roots you will build your solution.

DE: $a_n y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_2 y'' + a_1 y' + a_0 y = 0$ ↙

Characteristic Equation: $a_n r^n + a_{n-1} r^{n-1} + \dots + a_2 r^2 + a_1 r + a_0 = 0$ ↙

Real distinct roots give parts of the general solution of the form,

$$y_i = \underline{c_i e^{r_i x}}$$

Real repeated roots of multiplicity k give parts of the general solution of the form,

$$\underline{y_i = c_i e^{\rho x}}, \underline{y_{i+1} = c_{i+1} x e^{\rho x}}, \underline{y_{i+2} = c_{i+2} x^2 e^{\rho x}}, \dots, \underline{y_{i+(k-1)} = c_{i+(k-1)} x^{k-1} e^{\rho x}}$$

Complex roots $a + bi, a - bi$ will come in complex conjugate pairs give parts of the general solution of the form

$$\underline{y_i = e^{ax} (c_i \cos(bx) + d_i \sin(bx))}$$

Complex repeated roots with multiplicity k will give parts of the general solution of the form

↙

$$y_i = e^{ax} (c_i \cos(bx) + d_i \sin(bx)), y_{i+1} = x e^{ax} (c_{i+1} \cos(bx) + d_{i+1} \sin(bx)),$$
$$y_{i+2} = x^2 e^{ax} (c_{i+2} \cos(bx) + d_{i+2} \sin(bx)), \dots, y_{i+(k-1)} = x^{k-1} e^{ax} (c_{i+(k-1)} \cos(bx) + d_{i+(k-1)} \sin(bx))$$

The full solution will be a linear combination of these pieces!