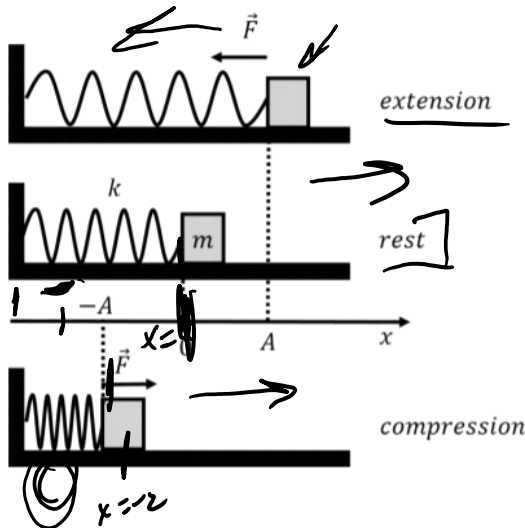


Sec 3.4 Mechanical Vibrations

Mechanical vibrations are everywhere in the world. Simple mass and spring models are typically good models for such things. These mass and spring models are also often written down as a homogeneous linear differential equation with constant coefficients.

Let's first examine the simplest case, a spring attached to a wall with a mass on the other end sliding on a frictionless surface.



First thing we need to know is hook's law. This law describes the force exerted on the mass by a spring. It goes as

$F = -kx$ where k is called the spring constant and x is the distance the spring is compressed or stretched.

$$k > 0 \quad m > 0$$

Note $x = 0$ corresponds to no compression or extension of the spring. Compression would be negative x making the force positive and stretching would be positive making the force point in the negative direction.

If we apply newton's second law to such a system $\sum F = ma$ we get:

$$ma = -kx \rightarrow mx'' = -kx \quad x(t)$$

Assuming k is positive what is general solution?

$$mx'' + kx = 0$$

$$mr^2 + k = 0$$

$$r^2 = -k/m$$

$$r = \pm \sqrt{-k/m}$$

$$r = \pm \sqrt{k/m} \sqrt{-1} = \pm \sqrt{k/m} i$$

$$r = 0 \pm \sqrt{\frac{k}{m}} i$$

$$x = e^{0t} (c_1 \cos(\sqrt{k/m} t) + c_2 \sin(\sqrt{k/m} t))$$

Now let's assume the spring was initially compressed as distance of 2 to start $x(0) = -2$ and that it began at rest $x'(0) = 0$ What is our particular solution?

$$x = c_1 \cos(\sqrt{k/m} t) + c_2 \sin(\sqrt{k/m} t) \quad x' = -c_1 \sqrt{k/m} \sin(\sqrt{k/m} t) + c_2 \sqrt{k/m} \cos(\sqrt{k/m} t)$$

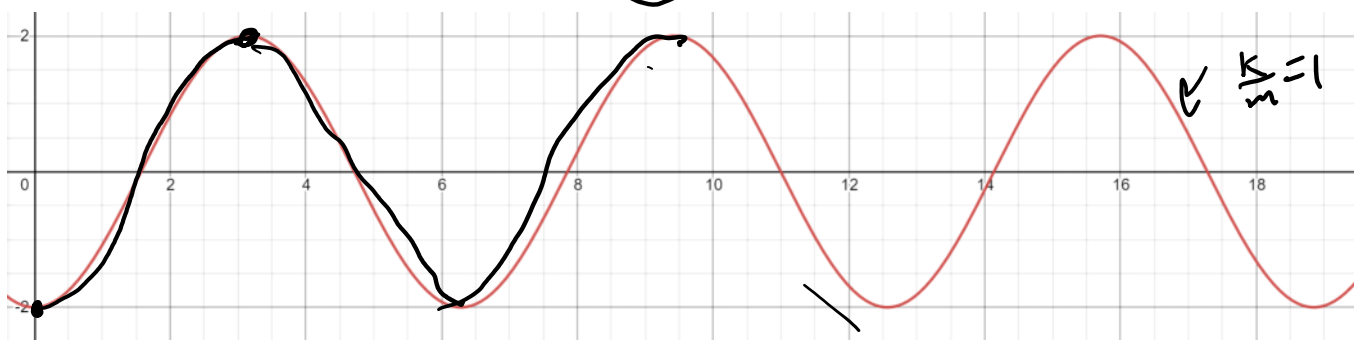
$$x(0) = -2 = c_1 \cos(0) + c_2 \sin(0) \rightarrow -2 = c_1 \Rightarrow c_1 = -2$$

$$x'(0) = 0 = -c_1 \sqrt{k/m} \sin(0) + c_2 \sqrt{k/m} \cos(0) \rightarrow 0 = c_2 \sqrt{k/m} \Rightarrow c_2 = 0$$

$$x(t) = -2 \cos(\sqrt{k/m} t)$$

Sec 3.4 Mechanical Vibrations

So we have the solution $x(t) = -2 \cos\left(\sqrt{\frac{k}{m}}t\right)$ Note the significance of the initial amount the spring was compressed. What is the significance of $\sqrt{\frac{k}{m}}$?



$$\sqrt{\frac{k}{m}} = \omega \quad -2 \cos(\omega t) \quad \frac{2\pi}{\omega}$$

$$\cos(b t) \quad T = \frac{2\pi}{b}$$

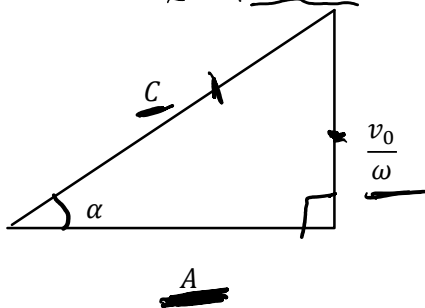
Let's look at another set of initial conditions, one where we give an initial push to the mass...

$x(0) = A, x'(0) = v_0$ Our general solution is $x(t) = c_1 \cos(\omega t) + c_2 \sin(\omega t)$ where $\omega = \sqrt{\frac{k}{m}}$
And $x'(t) = -\omega c_1 \sin(\omega t) + \omega c_2 \cos(\omega t)$

$\nearrow c_2 = v_0/\omega$
 $x(0) = A = c_1, x'(0) = v_0 = \omega c_2$ So $x(t) = A \cos(\omega t) + \frac{v_0}{\omega} \sin(\omega t)$

You might recall $\cos(x - y) = \cos(x) \cos(y) + \sin(x) \sin(y)$

Define $C = \sqrt{A^2 + \left(\frac{v_0}{\omega}\right)^2}$



Consider $x(t) = C \left(\frac{A}{C} \cos(\omega t) + \frac{v_0/\omega}{C} \sin(\omega t) \right)$
 $\frac{A}{C} = \cos(\alpha) \quad \frac{v_0/\omega}{C} = \sin(\alpha)$

$$x(t) = C (\cos(\alpha) \cos(\omega t) + \sin(\alpha) \sin(\omega t))$$

$$= C (\cos(\alpha - \omega t))$$

$\tan(\alpha) = \frac{v_0/\omega}{A}$
 $\alpha = \arctan\left(\frac{v_0/\omega}{A}\right)$

$$C (\cos(-(\omega t - \alpha))) = C \cos(\omega t - \alpha)$$

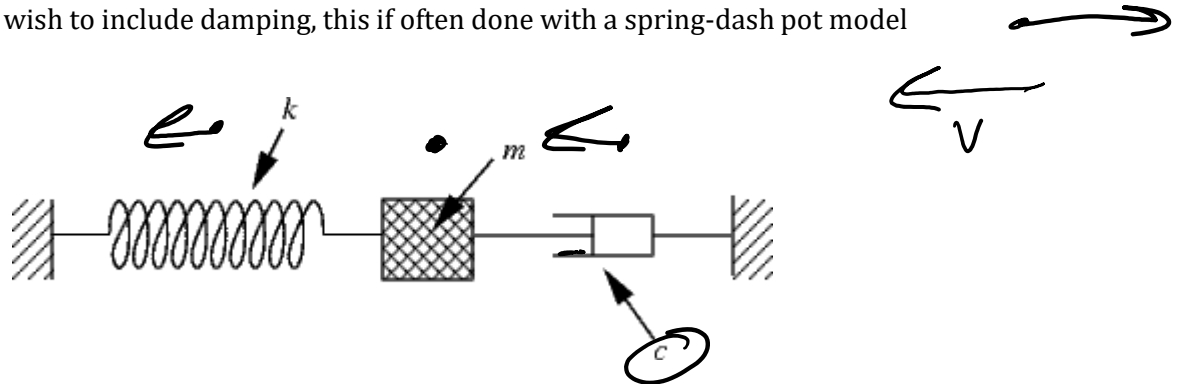
$\omega = \sqrt{\frac{k}{m}}$ is the frequency and α the phase angle

<https://www.desmos.com/calculator/usuwge1lwq>

Sec 3.4 Mechanical Vibrations

Of course not everything gets to vibrate freely. Often vibration is damped by friction or by design (like shock absorbers)

In some situations we wish to include damping, this is often done with a spring-dash pot model



Dashpots can be modeled a bit like air resistance, that is the force of resistance is proportional to the velocity.

$$F_d = -cv$$

Note that like with air resistance we need that minus sign so that the force points in the right direction based on the sign of the velocity.

Let's apply Newton's second law and see what kind of DE we get.

$$m x'' = -kx - cv = -kx - c x'$$

$$m x'' + c x' + k x = 0$$

Sec 3.4 Mechanical Vibrations

So we have: $mx'' + cx' + kx = 0$

We know how to solve this! We write down the characteristic equation, solve it, and based on the roots build our solution. Let's solve this one in terms of m, c, k and see what we can learn about such spring-dash pot systems in General

Our characteristic equation is:

$$mr^2 + cr + k = 0$$

$$ax^2 + bx + c = 0$$

$$r = \frac{-c}{2m} \pm \frac{\sqrt{c^2 - 4mk}}{2m}$$

If we use the quadratic formula we get roots of:

$$r = \frac{-c}{2m} \pm \frac{\sqrt{c^2 - 4mk}}{2m}$$

So we have some distinct possibilities...

If $c^2 > 4mk$ we will have two distinct real roots! Our solution will look like: $x(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$

Note that $\frac{\sqrt{c^2 - 4mk}}{2m} < \frac{\sqrt{c^2}}{2m} = \frac{c}{2m}$ so $-\frac{c}{2m} \pm \frac{\sqrt{c^2 - 4mk}}{2m} < 0$ for both roots!

This means that we have negative exponents in each exponential and so $x \rightarrow 0$ quite quickly.

We call this the **overdamped case**.

$$e^{im} \quad c_1 e^{r_1 t} + c_2 e^{-|r_2| t} = 0$$

If $c^2 = 4mk$ we will have a repeated root and our solution will be of the form: $x(t) = c_1 e^{r_1 t} + c_2 t e^{r_1 t}$

The root is for sure negative and we do go to zero but more slowly than the overdamped case.

We call this the **critically damped case**

If $c^2 < 4mk$ we have a negative in the square root and have complex roots in complex conjugate pairs $r_1 = a + bi, r_2 = a - bi$ which gives a solution of the form $y(x) = e^{ax}(c_1 \cos(bx) + c_2 \sin(bx))$

Note $a < 0$ and so we still go to zero but we get to wiggle inbetween.

We call this the **underdamped case**.

$$\underbrace{\frac{-c}{2m}}_a + \underbrace{\frac{\sqrt{4mk - c^2}}{2m} i}_b \quad \frac{-c}{2m} - \frac{\sqrt{4mk - c^2}}{2m} i$$

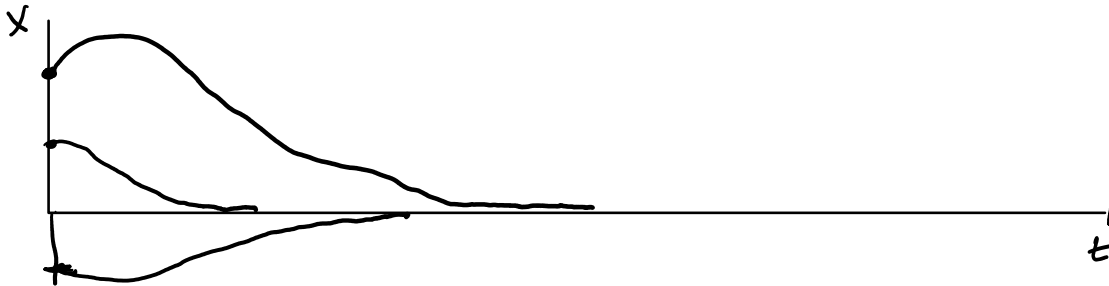
Sec 3.4 Mechanical Vibrations

$$-\frac{c}{2m} \pm \frac{\sqrt{c^2 - 4mk}}{2m}$$

Over damped $c^2 > 4mk$ $r_1 = -\frac{c}{2m} + \frac{\sqrt{c^2 - 4mk}}{2m}$, $r_2 = -\frac{c}{2m} - \frac{\sqrt{c^2 - 4mk}}{2m}$

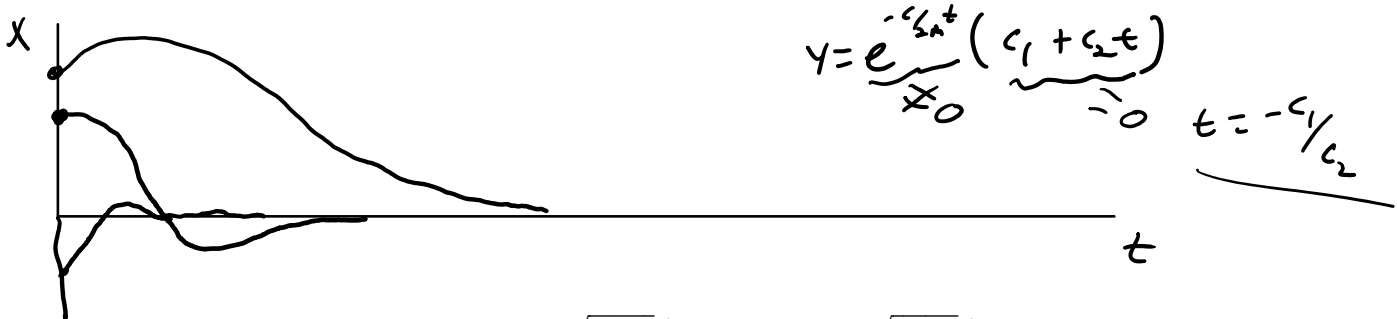
$$x(t) = c_1 e^{\left(-\frac{c}{2m} + \frac{\sqrt{c^2 - 4mk}}{2m}\right)t} + c_2 e^{\left(-\frac{c}{2m} - \frac{\sqrt{c^2 - 4mk}}{2m}\right)t}$$

Note that neither of the exponentials will ever truly be zero so we will never cross the equilibrium point again. We will just damp to it, no oscillation.



Critically damped $c^2 = 4mk$, $r_1 = -\frac{c}{2m}$ $x(t) = c_1 e^{-\frac{c}{2m}t} + c_2 t e^{-\frac{c}{2m}t} = e^{-\frac{c}{2m}t} (c_1 + c_2 t)$

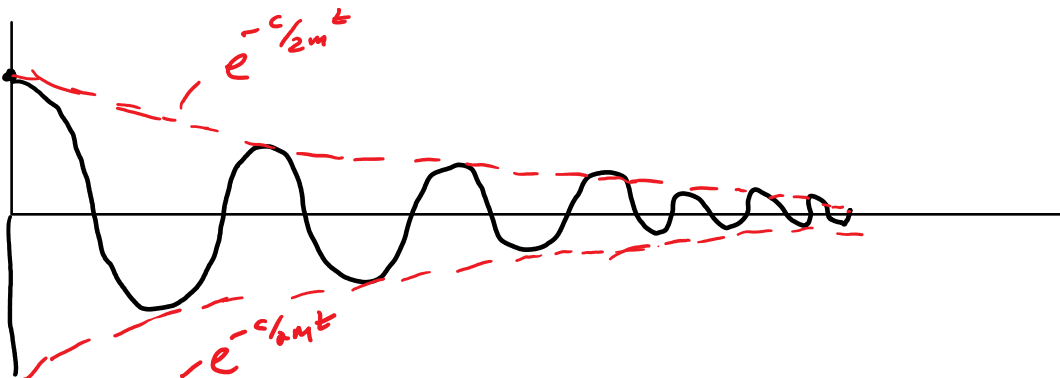
Note that in this case it might be possible to pass equilibrium at most once, if t is such that $c_1 + c_2 t = 0$ after that time we will just damp down to it. So we might get one and only one oscillation.



Underdamped case $c^2 < 4km$ $r_1 = -\frac{c}{2m} + \frac{\sqrt{4mk - c^2}}{2m}i$, $r_2 = -\frac{c}{2m} - \frac{\sqrt{4mk - c^2}}{2m}i$

$$x(t) = e^{-\left(\frac{c}{2m}t\right)} \left[c_1 \cos\left(\frac{\sqrt{4mk - c^2}}{2m}t\right) + c_2 \sin\left(\frac{\sqrt{4mk - c^2}}{2m}t\right) \right]$$

In this case we can cross zero lots of times but the exponential eventually pushes us to zero so we have a dying out series of oscillations



Sec 3.4 Mechanical Vibrations

Let's play with the solution to the initial value problem $mx'' + cx' + kx = 0$ $x(0) = A$, $x'(0) = v_0$

Head over to the following web address and play along if you like.

<https://www.desmos.com/calculator/zx2g8udgaz>

1. Pick values for c, k, m so that $c^2 - 4km > 0$ and turn on only the red plot. See what the solution looks like for different initial conditions A, v_0 . How does it change as c, k, m change?

2. Pick values for c, k, m so that $c^2 - 4km = 0$ and turn on only the blue plot. Play around with different initial conditions, can you get the solution to cross equilibrium once? What did you have to change?

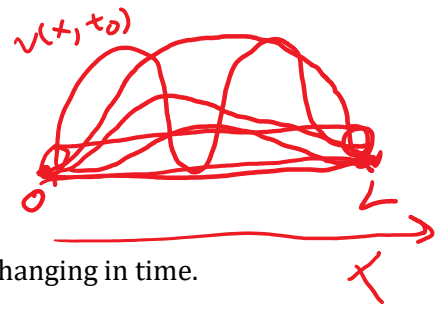
3. Pick values for c, k, m so that $c^2 - 4km < 0$ and turn on only the green plot. Play around with different initial conditions and values for c, k, m .

Sec 3.4 Mechanical Vibrations

In section 1.4 we took a look at the heat equation in 1-d

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}$$

Where $u(x, t)$ is the temperature is the temperature along a rod which is changing in time.



We assumed a solution of the form $u(x, t) = \phi(x)\theta(t)$ and substituted this into the PDE which told us

$$\frac{\theta'(t)}{\theta(t)} = \frac{k\phi''(x)}{\phi(x)}$$

Since θ and ϕ are functions of two different independent the only way that equality can hold is if each ratio was equal to a constant.

$$\frac{\theta'(t)}{\theta(t)} = \lambda$$

$$\frac{k\phi''(x)}{\phi(x)} = \lambda$$

$$k\phi''(x) = \lambda\phi(x)$$

$$\phi''(x) = \frac{\lambda}{k}\phi(x)$$

This gave us two ODE's

$$\theta' = \lambda\theta$$

$$\theta = Ce^{\lambda t}$$

$$\phi''(x) - \frac{\lambda}{k}\phi(x) = 0$$

And $\phi'' - \frac{\lambda}{k}\phi = 0$ which we did not know how to solve yet. However now we Do!

If $\lambda > 0$ $k > 0$

$$r^2 - \frac{\lambda}{k} = 0$$

$$r^2 = \frac{\lambda}{k}$$

$$r = \pm \sqrt{\frac{\lambda}{k}}$$

$$\phi = c_1 e^{\sqrt{\frac{\lambda}{k}}x} + c_2 e^{-\sqrt{\frac{\lambda}{k}}x}$$

If $\lambda < 0$ $k > 0$

$$r^2 - \frac{(-|\lambda|)}{k} = 0$$

$$r^2 = -\frac{\lambda}{k}$$

$$r = \pm \sqrt{\frac{\lambda}{k}} i$$

$$\phi = c_1 \cos(\sqrt{\frac{\lambda}{k}}x) + c_2 \sin(\sqrt{\frac{\lambda}{k}}x)$$