

Section 3.5

Nonhomogeneous Equations and Undetermined Coefficients

We have spent a lot of time studying homogeneous linear equations but now we will turn our attention to Non-Homogeneous equations. That is linear differential equations of the form:

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_2 y'' + a_1 y' + a_0 y = f(x)$$

We saw in section 3.2 that the full solution to the non-homogeneous equation was actually a sum of two solutions.

$$y = y_c + y_p$$

Where y_c is called the complementary solution and solves the homogeneous problem:

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_2 y'' + a_1 y' + a_0 y = 0$$

While y_p is called the particular solution and solves directly the nonhomogeneous problem

$$G(y) = a_n y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_2 y'' + a_1 y' + a_0 y = f(x)$$

To see why we need the complementary solution recall that linear in y gives us

$$G(y_1 + y_2) = G(y_1) + G(y_2)$$

$$\text{Here } G(y) = a_n y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_2 y'' + a_1 y' + a_0 y = f(x)$$

So then if y_c solves the homogeneous problem $G(y) = 0$ and y_p solves the nonhomogeneous problem $G(y) = f(x)$ then $Y = y_c + y_p$ into $G(y)$ would give....

$$\begin{aligned} G(y_c + y_p) &= a_n \frac{d^n}{dx^n} (y_c + y_p) + a_{n-1} \frac{d^{n-1}}{dx^{n-1}} (y_c + y_p) + \dots + a_0 (y_c + y_p) \\ &= a_n \left(\frac{d^n}{dx^n} y_c + \frac{d^n}{dx^n} y_p \right) + a_{n-1} \left(\frac{d^{n-1}}{dx^{n-1}} y_c + \frac{d^{n-1}}{dx^{n-1}} y_p \right) + \dots + a_0 (y_c + y_p) \\ &= \left(a_n \frac{d^n}{dx^n} y_c + a_{n-1} \frac{d^{n-1}}{dx^{n-1}} y_c + \dots + a_0 y_c \right) + \left(a_n \frac{d^n}{dx^n} y_p + a_{n-1} \frac{d^{n-1}}{dx^{n-1}} y_p + \dots + a_0 y_p \right) \\ &= G(y_c) + G(y_p) \\ &= 0 + f(x) \\ &= f(x) \end{aligned}$$

$y_c + y_p$ solves the N.H. equation.

Section 3.5

Nonhomogeneous Equations and Undetermined Coefficients

In section 3.3 we studied the methods to solve the complementary solution y_c

DE: $a_n y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_2 y'' + a_1 y' + a_0 y = 0$

Characteristic Equation: $a_n r^n + a_{n-1} r^{n-1} + \dots + a_2 r^2 + a_1 r + a_0 = 0$

Real distinct roots give parts of the general solution of the form,

$y_i = c_i e^{r_i x}$

Real repeated roots of multiplicity k give parts of the general solution of the form,

$y_i = c_i e^{\rho x}, y_{i+1} = c_{i+1} x e^{\rho x}, y_{i+2} = c_{i+2} x^2 e^{\rho x}, \dots, y_{i+(k-1)} = c_{i+(k-1)} x^{k-1} e^{\rho x}$

Complex roots $a + bi, a - bi$ with will come in complex conjugate pairs give parts of the general solution of the form

$y_i = e^{ax} (c_i \cos(bx) + d_i \sin(bx))$

Complex repeated roots with multiplicity k will give parts of the general solution of the form

$y_i = e^{ax} (c_i \cos(bx) + d_i \sin(bx)), y_{i+1} = x e^{ax} (c_{i+1} \cos(bx) + d_{i+1} \sin(bx)),$
 $y_{i+2} = x^2 e^{ax} (c_{i+2} \cos(bx) + d_{i+2} \sin(bx)), \dots, y_{i+(k-1)} = x^{k-1} e^{ax} (c_{i+(k-1)} \cos(bx) + d_{i+(k-1)} \sin(bx))$

The full solution will be a linear combination of these pieces!

Now we will learn how to find y_p for the non-homogeneous equations. The method we will study is typically called the **Method of Undetermined Coefficients**.

For a non-homogeneous equation $a_n y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_2 y'' + a_1 y' + a_0 y = f(x)$

The general strategy will be to look at the form of $f(x)$ and guess a linear combination of functions ($y_p = c_1 g_1(x) + c_2 g_2(x) + \dots$) for which when derivatives are taken, multiplied by the coefficients in the DE, and added up might produce $f(x)$. We will find the coefficients of the linear combination of guess functions so that our guess indeed does produce $f(x)$ when it is plugged into the differential equation.

Section 3.5

Nonhomogeneous Equations and Undetermined Coefficients

Let's start with an example. Find the solution to the differential equation

$$y'' + 2y' + y = 2x^2 + 1$$

What function might make a good guess for y_p ?

$$y_p = Ax^2 + Bx + C$$

Let's plug this into the differential equation and find the coefficients.

~~$y'' + 2y' + y = 2x^2 + 1$~~

$y_p = Ax^2 + Bx + C$
 $y_p' = 2Ax + B$
 $y_p'' = 2A$

$$2A + 2(2Ax + B) + Ax^2 + Bx + C = 2x^2 + 1$$
$$2A + 4Ax + 2B + Ax^2 + Bx + C = 2x^2 + 1$$
$$Ax^2 + 4Ax + Bx + 2A + 2B + C = 2x^2 + 1$$
$$Ax^2 + (4A + B)x + (2A + 2B + C) = 2x^2 + 1$$
$$\begin{cases} A = 2 \\ 4A + B = 0 \\ 2A + 2B + C = 1 \end{cases}$$
$$\begin{cases} 4(2) + B = 0 \Rightarrow B = -8 \\ 2(2) + 2(-8) + C = 1 \\ 4 - 16 + C = 1 \\ -12 + C = 1 \Rightarrow C = 13 \end{cases}$$
$$y_p = 2x^2 - 8x + 13$$

Also we can't forget to solve the complementary equation: $y'' + 2y' + y = 0$

$$r^2 + 2r + 1 = 0$$

$$(r+1)^2 = 0$$

$$r = -1 \text{ multiplicity 2}$$

$$y_c = c_1 e^{-x} + c_2 x e^{-x}$$

Full

$$y = y_c + y_p = c_1 e^{-x} + c_2 x e^{-x} + 2x^2 - 8x + 13$$

Section 3.5

Nonhomogeneous Equations and Undetermined Coefficients

For nonhomogeneous equations

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_2 y'' + a_1 y' + a_0 y = f(x)$$

Where $f(x) = A_n x^n + A_{n-1} x^{n-1} + \dots + A_0$ where some of the coefficients may be zero the guess you want to make for your y_p is

$$y_p = C_n x^n + C_{n-1} x^{n-1} + \dots + C_0$$

(Provided none of the terms are duplicates of any part of the complementary solution)

You then find the C 's by plugging y_p in to the differential equation equating both sides and matching coefficients.

Let's look at another example. Find the particular solution to the differential equation

$$3y'' + y' - 2y = 2\cos(x)$$

What might be a good guess for y_p ?

$$y_p = A\cos(x) + B\sin(x)$$

Let's find the undetermined coefficients.

$$y_p' = -A\sin(x) + B\cos(x)$$

$$y_p'' = -A\cos(x) - B\sin(x)$$

$$3(-A\cos(x) - B\sin(x)) + (-A\sin(x) + B\cos(x)) - 2(A\cos(x) + B\sin(x)) = 2\cos(x)$$

$$(-3A + B - 2A)\cos(x) + (-3B - A - 2B)\sin(x) = 2\cos(x)$$

$$-5A + B = 2$$

$$-5B - A = 0$$

$$A = -5B$$

$$-5(-5B) + B = 2$$

$$26B = 2$$

$$B = \frac{1}{13}$$

$$-5\left(\frac{1}{13}\right) - A = 0$$

$$A = -\frac{5}{13}$$

$$y_p = -\frac{5}{13}\cos(x) + \frac{1}{13}\sin(x)$$

Section 3.5

Nonhomogeneous Equations and Undetermined Coefficients

In general for nonhomogeneous equations

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_2 y'' + a_1 y' + a_0 y = f(x)$$

Where $f(x) = A_1 \cos(x) + B_1 \sin(x) + A_2 \cos(2x) + B_2 \sin(2x) + \dots$ where some of the coefficients may be zero the guess you want to make for your y_p is

$$y_p = C_1 \cos(x) + C_2 \sin(x) + C_3 \cos(2x) + C_4 \sin(2x) + \dots$$

(Provided none of the terms are duplicates of any part of the complementary solution)

You then find the C 's by plugging y_p in to the differential equation equating both sides and matching coefficients.

Another example: Find the particular solution to the differential equation:

$$y'' + 16y = e^{3x}$$

What might be a good choice for y_p ?

$$y_p = Ae^{3x}$$

Let's find the undetermined coefficients.

$$y_p' = 3Ae^{3x}$$

$$y_p'' = 9Ae^{3x}$$

$$9Ae^{3x} + 16Ae^{3x} = e^{3x}$$

$$(9A + 16A)e^{3x} = e^{3x}$$

$$25A = 1$$

$$A = 1/25$$
$$y_p = \frac{1}{25}e^{3x}$$

Section 3.5

Nonhomogeneous Equations and Undetermined Coefficients

In general for nonhomogeneous equations

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_2 y'' + a_1 y' + a_0 y = f(x)$$

Where $f(x) = A_1 e^{r_1 x} + A_2 e^{r_2 x} + \dots$ where some of the coefficients may be zero the guess you want to make for your y_p is

$$y_p = C_1 e^{r_1 x} + C_2 e^{r_2 x} + \dots$$

(Provided none of the terms are duplicates of any part of the complementary solution)

You then find the C 's by plugging y_p in to the differential equation equating both sides and matching coefficients.

Based on the above what might we try?

$y'' - 4y = 2e^{2x}$

What happens when we do?

$y_p = Ae^{2x}$

$y_p' = 2Ae^{2x}$

$y_p'' = 4Ae^{2x}$

$4He^{2x} - 4Ae^{2x} = 0$

$\Rightarrow Ae^{2x}$ solves $y'' - 4y = 0$

Let's check the complementary solution, the one that solves $y'' - 4y = 0$

$$r^2 - 4 = 0 \quad r^2 = 4 \quad r = \pm 2$$

$$y_c = C_1 e^{2x} + C_2 e^{-2x}$$

Any guess as to what we might try to do to fix this?

$$y_p = x e^{2x}$$

Section 3.5

Nonhomogeneous Equations and Undetermined Coefficients

Let's try $y_p = Axe^{2x}$

$$y'' - 4y = 2e^{2x}$$

$$y'_p = Ae^{2x} + 2Axe^{2x}$$

$$y''_p = 2Ae^{2x} + 2Ae^{2x} + 4Axe^{2x}$$

$$4Ae^{2x} + 4Axe^{2x} - 4(Axe^{2x}) = 2e^{2x}$$

$$4Ae^{2x} = 2e^{2x}$$

$$4A = 2$$

$$A = 1/2$$

$$y_p = \frac{1}{2}xe^{2x}$$

So what do we do when parts of our guesses for our particular solution end up also satisfying the homogeneous equation?

Consider the differential equation: $(D - 2)^3 y = e^{2x}$, what is the complementary solution?

$$(r - 2)^3 = 0 \quad (D - 2)^3 y = 0$$

$$r = 2 \text{ multiplicity } 3$$

$$y_c = c_1 e^{2x} + c_2 x e^{2x} + c_3 x^2 e^{2x}$$

Recall that $(D - r)^k u(x) e^{rx} = e^{rx} [D^k u(x)]$ before we chose polynomial terms of order $k - 1$ for $u(x)$ so that k derivatives of them would give us zero, which we want for the homogeneous solution. However now we want something that won't go away for our particular solution. In this case we want something that will just go to a constant. If we have k derivatives perhaps a power of x that will reduce to a constant after that. Like x^k

Since if $u(x) = x^k$ then $(D - r)^k u(x) e^{rx} = e^{rx} [D^k u(x)] = e^{rx} [D^k x^k] = e^{rx}$

So if our complementary solution is $y_c = c_1 e^{2x} + c_2 x e^{2x} + c_3 x^2 e^{2x}$ and our particular solution also needs a e^{2x} what might we try?

$$y_p = Ax^3 e^{2x}$$

What happens when we do?

$$- y'_p = 3Ax^2 e^{2x} + 2Ax^3 e^{2x}$$

$$- y''_p = 6Ax e^{2x} + 3Ax^2 e^{2x} + 6Ax^2 e^{2x} + 2Ax^3 e^{2x}$$

$$- y'''_p = 6Ae^{2x} + 12Ax e^{2x} + \dots$$

Section 3.5

Nonhomogeneous Equations and Undetermined Coefficients

So what we do when we have a term in our guess that is in the complementary solution is multiply it by the lowest power of x that we can so that it is no longer a repeated term.

→ Suppose $y_c = c_3 \cos(x) + c_4 \sin(x) + c_4 \cos(4x) + c_5 \sin(4x) + c_6 x \cos(4x) + c_7 x \sin(4x)$ is the complementary solution to the differential equation:

$$G(y) = 4 \cos(x) + 2 \sin(4x) + \sin(2x)$$

(1) (2) (3)

What guess would we make for our particular solution?

$$y_p = A_1 x \cos(x) + B_1 x \sin(x) + A_2 x^2 \cos(4x) + B_2 x^2 \sin(4x) + A_3 \cos(2x) + B_3 \sin(2x)$$

(1) (2) (3)

repeated once in y_c repeated twice in y_c not repeated

So let's summarize the types of guesses we might make for different $f(x)$'s

$$a_n y^{(n)} + \dots + a_0 = f(x)$$

$f(x)$	$y_p(x)$
$P_m(x) = b_0 + b_1 x + b_2 x^2 + \dots + b_n x^n$	$x^s (A_0 + A_1 x + A_2 x^2 + \dots + A_n x^n)$
$a \cos(kx) + b \sin(kx)$	$x^s (A \cos(kx) + B \sin(kx))$
$e^{rx} (a \cos(kx) + b \sin(kx))$	$x^s e^{rx} (A \cos(kx) + B \sin(kx))$
$(b_0 + b_1 x + b_2 x^2 + \dots + b_n x^n) e^{rx}$	$x^s (A_0 + A_1 x + A_2 x^2 + \dots + A_n x^n) e^{rx}$
$(b_0 + b_1 x + b_2 x^2 + \dots + b_n x^n) (a \cos(kx) + b \sin(kx))$	$x^s [(A_0 + A_1 x + A_2 x^2 + \dots + A_n x^n) \cos(kx) + (B_0 + B_1 x + B_2 x^2 + \dots + B_n x^n) \sin(kx)]$

Where s is the smallest integer such that no term in the particular solution is an exact repeat of any term in the complementary solution.

What do we do when multiple of these types of terms show up in a sum on the other? Well we sum our guesses.

Section 3.5

Nonhomogeneous Equations and Undetermined Coefficients

Find the general form of the particular solution of: $y^{(3)} + 9y' = \cos(3x) + x\sin(x) + x^2e^{2x}$

Let's make use of our table....

So let's summarize the types of guesses we might make for different $f(x)$'s

$f(x)$	$y_p(x)$
$P_m(x) = b_0 + b_1x + b_2x^2 + \dots + b_nx^n$	$x^s(A_0 + A_1x + A_2x^2 + \dots + A_nx^n)$
$a\cos(kx) + b\sin(kx)$	$x^s(A\cos(kx) + B\sin(kx))$ (1)
$e^{rx}(a\cos(kx) + b\sin(kx))$	$x^s e^{rx}(a\cos(kx) + b\sin(kx))$
$(b_0 + b_1x + b_2x^2 + \dots + b_nx^n)e^{rx}$ (2)	$x^s(A_0 + A_1x + A_2x^2 + \dots + A_nx^n)e^{rx}$ (3)
$(b_0 + b_1x + b_2x^2 + \dots + b_nx^n)(a\cos(kx) + b\sin(kx))$	$x^s[(A_0 + A_1x + A_2x^2 + \dots + A_nx^n)\cos(kx) + (B_0 + B_1x + B_2x^2 + \dots + B_nx^n)\sin(kx)]$ (2)

Where s is the smallest integer such that no term in the particular solution is an exact repeat of any term in the complementary solution.

First find that complementary solution to check for possible repeats...

$$y''' + 9y' = 0$$

$$r^3 + 9r = 0$$

$$r(r^2 + 9) = 0$$

$$r=0 \quad r^2 + 9 = 0 \quad r^2 = -9 \quad r = \pm 3i$$

$$c_1 e^{0x}$$

$$e^{0x}(c_2 \cos(3x) + c_3 \sin(3x))$$

$$y_c = c_1 + c_2 \cos(3x) + c_3 \sin(3x)$$

$$y_p = \begin{aligned} & \text{(1)} \quad x^s(A_1 \cos(3x) + B_1 \sin(3x)) \\ & \text{(2)} \quad x^s(d_1 + d_2)(A_2 \cos(x) + B_2 \sin(x)) + \dots \\ & \quad + x^s(d_3 x^2 + d_4 x + d_5)e^{2x} \\ & \text{(3)} \quad x^s(A_0 + A_1x + A_2x^2 + \dots + A_nx^n)e^{2x} \end{aligned}$$

Section 3.5

Nonhomogeneous Equations and Undetermined Coefficients

$$D = \frac{d}{dx}$$

Determine the appropriate form of the particular solution for the differential equation

$$(D - 2)^3(D^2 + 9)y = x^2 e^{2x} + x \sin(3x)$$

(1) (2)

So let's summarize the types of guesses we might make for different $f(x)$'s

$f(x)$	$y_p(x)$
$P_m(x) = b_0 + b_1x + b_2x^2 + \dots + b_nx^n$	$x^s(A_0 + A_1x + A_2x^2 + \dots + A_nx^n)$
$\cos(kx) + b\sin(kx)$	$x^s(A\cos(kx) + B\sin(kx))$
$e^{rx}(\cos(kx) + b\sin(kx))$	$x^s e^{rx}(\cos(kx) + b\sin(kx))$
$(b_0 + b_1x + b_2x^2 + \dots + b_nx^n)e^{rx}$ (1) $\rightarrow$	$x^s(A_0 + A_1x + A_2x^2 + \dots + A_nx^n)e^{rx}$ $\leftarrow$
$(b_0 + b_1x + b_2x^2 + \dots + b_nx^n)(\cos(kx) + b\sin(kx))$ (2) $\leftarrow$	$x^s[(A_0 + A_1x + A_2x^2 + \dots + A_nx^n)\cos(kx) + (B_0 + B_1x + B_2x^2 + \dots + B_nx^n)\sin(kx)]$ $\leftarrow$

Where s is the smallest integer such that no term in the particular solution is an exact repeat of any term in the complementary solution.

Complementary solution check.....

$$\rightarrow (D - 2)^3(D^2 + 9)y = 0$$

$$(r - 2)^3(r^2 + 9) = 0$$

$r = 2$ multiplicity 3
 $r^2 + 9 = 0 \quad r^2 = -9$
 $r = \pm 3i$

$$y_c = \underline{c_1 e^{2x}} + \underline{c_2 x e^{2x}} + \underline{c_3 x^2 e^{2x}} + \underline{c_4 \cos(3x)} + \underline{c_5 \sin(3x)}$$

$$\underline{x^2 e^{2x}} + \underline{x \sin(3x)}$$

$$x^{\underline{s}}(\underline{d_1} + \underline{d_2 x} + \underline{d_3 x^2})e^{2x} + x^{\underline{s}}[(\underline{d_4} + \underline{d_5 x})\sin(3x) + (\underline{d_6} + \underline{d_7 x})\cos(3x)]$$

$s=3$ $s=1$

$$y_p = x^3(d_1 + d_2x + d_3x^2)e^{2x} + x[(d_4 + d_5x)\sin(3x) + (d_6 + d_7x)\cos(3x)]$$

Section 3.5

Nonhomogeneous Equations and Undetermined Coefficients

Example: Solve the initial value problem

$$y'' - 3y' + 2y = 3e^{-x} - 10\cos(3x), y(0) = 1, y'(0) = 2$$

(1) complementary soln.

$$y'' - 3y' + 2y = 0 \quad r^2 - 3r + 2 = 0 \rightarrow (r-2)(r-1) = 0$$

$$r=1, r=2 \rightarrow y_c = c_1 e^{2x} + c_2 e^x$$

(2) particular U.C.

$$y_p = A e^{-x} + B \sin(3x) + C \cos(3x)$$

(3) plug y_p into the DE equate coeff.

$$y_p' = -A e^{-x} + 3B \cos(3x) - 3C \sin(3x)$$

$$y_p'' = A e^{-x} - 9B \sin(3x) - 9C \cos(3x)$$

$$\begin{aligned} & \underline{A e^{-x} - 9B \sin(3x) - 9C \cos(3x)} - 3(-A e^{-x} + 3B \cos(3x) - 3C \sin(3x)) \\ & + 2(A e^{-x} + B \sin(3x) + C \cos(3x)) \\ & = 3e^{-x} - 10\cos(3x) \end{aligned}$$

$$6A e^{-x} + (-9B + 9C + 2B) \sin(3x) + (-9C - 9B + 2C) \cos(3x) = 3e^{-x} - 10\cos(3x)$$

$$6A = 3 \rightarrow \boxed{A = \frac{1}{2}}$$

$$-9B + 9C + 2B = 0 \rightarrow -7B + 9C = 0 \rightarrow B = \frac{9}{7}C$$

$$-9C - 9B + 2C = -10 \rightarrow -7C - 9B = -10$$

$$-7C - 9\left(\frac{9}{7}C\right) = -10$$

$$-7C - 8\frac{1}{7}C = -10$$

$$-\frac{49C}{7} - \frac{81C}{7} = -10$$

$$\rightarrow -\frac{130}{7}C = -10$$

$$\boxed{C = \frac{70}{130} = \frac{7}{13}}$$

$$\boxed{B = \frac{9}{7} \left(\frac{7}{13}\right) = \frac{9}{13}}$$

$$y_p = \frac{1}{2} e^{-x} + \frac{9}{13} \sin(3x) + \frac{7}{13} \cos(3x)$$

Section 3.5

Nonhomogeneous Equations and Undetermined Coefficients

$$y_c = c_1 e^x + c_2 e^{2x}$$

$$y_p = \frac{1}{2} e^{-x} + \frac{9}{13} \sin(3x) + \frac{7}{13} \cos(3x)$$

$$y = c_1 e^x + c_2 e^{2x} + \frac{1}{2} e^{-x} + \frac{9}{13} \sin(3x) + \frac{7}{13} \cos(3x)$$

$$y(0) = 1 \quad y'(0) = 2$$

$$y(0) = c_1 e^0 + c_2 e^0 + \frac{1}{2} e^0 + \frac{9}{13} \sin(0) + \frac{7}{13} \cos(0) = 1$$

$$c_1 + c_2 + \frac{1}{2} + \frac{7}{13} = 1 \rightarrow c_1 + c_2 + \frac{27}{26} = 1 \rightarrow c_1 + c_2 = -\frac{1}{26}$$

$$y'(x) = c_1 e^x + 2c_2 e^{2x} - \frac{1}{2} e^{-x} + \frac{27}{13} \cos(3x) - \frac{21}{13} \sin(3x)$$

$$y'(0) = c_1 e^0 + 2c_2 e^0 - \frac{1}{2} e^0 + \frac{27}{13} \cos(0) - \frac{21}{13} \sin(0) = 2$$

$$c_1 + 2c_2 - \frac{1}{2} + \frac{27}{13} = 2 \rightarrow c_1 + 2c_2 + \frac{41}{26} = 2$$

$$c_1 + 2c_2 = \frac{52}{26} - \frac{41}{26} = \frac{11}{26}$$

$$c_1 + c_2 = -\frac{1}{26}$$

$$\rightarrow c_1 + 2c_2 = \frac{11}{26}$$

$$c_1 + 2c_2 = \frac{11}{26}$$

$$-c_1 - c_2 = +\frac{1}{26}$$

$$c_2 = \frac{12}{26} \rightarrow \frac{6}{13}$$

$$c_1 + \frac{12}{26} = -\frac{1}{26} \rightarrow c_1 = \frac{-13}{26} = -\frac{1}{2}$$

$$y = -\frac{1}{2} e^x + \frac{6}{13} e^{2x} + \frac{1}{2} e^{-x} + \frac{9}{13} \sin(3x) + \frac{7}{13} \cos(3x)$$

Section 3.5

Nonhomogeneous Equations and Undetermined Coefficients

Another method to finding particular solutions to differential equations is that of variation of parameters.

Let's consider the differential equation $G(y) = y'' + p(x)y' + q(x)y = f(x)$ and suppose it has complementary solution $y_c = c_1 y_1 + c_2 y_2$. Armed with this let's derive a formula for the particular solution y_p .

$$G(y) = 0$$

We will begin by assuming that the particular solution can be written involving the complementary solution functions and has the form:

$$y_p = u_1(x)y_1 + u_2(x)y_2$$

If we take the needed derivatives to plug into our differential equation we get.

$$y_p' = \underbrace{u_1' y_1 + u_1 y_1'} + \underbrace{u_2' y_2 + u_2 y_2'} = \cancel{(u_1' y_1 + u_2' y_2)} + (u_1 y_1' + u_2 y_2')$$

$$y_p'' = u_1'' y_1 + u_1' y_1' + u_1' y_1' + u_1 y_1'' + u_2'' y_2 + u_2' y_2' + u_2' y_2' + u_2 y_2''$$

One requirement we have on y_p is that $G(y) = f(x)$ but here we have two unknown functions u_1 and u_2 so we need more requirements to find them. Having nothing else we are free to seek the simplest solution we can. It would be nice if in trying to find our unknown functions we did not have to think about second derivatives so let us impose the requirement that

$$u_1' y_1 + u_2' y_2 = 0$$

Then our derivatives become

$$y_p' = \underbrace{u_1 y_1'} + \underbrace{u_2 y_2'}$$

$$y_p'' = u_1' y_1' + u_1 y_1'' + u_2' y_2' + u_2 y_2''$$

Section 3.5

Nonhomogeneous Equations and Undetermined Coefficients

Considering differential operator $G(y) = y'' + p(x)y' + q(x)y$ let's plug in $y_p = u_1(x)y_1 + u_2(x)y_2$

$$G(y) = y_p'' + p(x)y_p' + q(x)y_p$$

$$G(y_p) = u_1y_1'' + u_2y_2'' + u_1'y_1' + u_2'y_2' + p(x)(u_1y_1' + u_2y_2') + q(x)(u_1y_1 + u_2y_2)$$

Let's collect all of the terms with no derivatives of u

$$G(y_p) = (u_1y_1'' + u_2y_2'' + p(x)(u_1y_1' + u_2y_2') + q(x)(u_1y_1 + u_2y_2)) + u_1'y_1' + u_2'y_2'$$

$$G(y_p) = (u_1y_1'' + p(x)(u_1y_1') + q(x)(u_1y_1)) + (u_2y_2'' + p(x)(u_2y_2') + q(x)(u_2y_2)) + u_1'y_1' + u_2'y_2'$$

$$G(y_p) = (u_1(y_1'' + p(x)y_1' + q(x)y_1)) + (u_2(y_2'' + p(x)y_2' + q(x)y_2)) + u_1'y_1' + u_2'y_2'$$

$$G(y_1) = 0$$

$$G(y_2) = 0$$

Now recall that y_1, y_2 solve the homogeneous problem $y'' + p(x)y' + q(x)y = 0$ so $G(y_p)$ is

$$G(y_p) = u_1'y_1' + u_2'y_2' = f(x)$$

What we want of our particular solution is for $G(y_p) = f(x)$ so we can say we require that

$$u_1'y_1' + u_2'y_2' = f(x)$$

We can combine this with the previous requirement that

$$u_1'y_1 + u_2'y_2 = 0$$

To get two equations and two unknowns (the derivatives of u_1, u_2). Then if we solve for those derivatives we simply integrate them to get u_1, u_2 and then we build $y_p = u_1y_1 + u_2y_2$

$$u_1 = \int u_1'$$

$$u_2 = \int u_2'$$

Section 3.5

Nonhomogeneous Equations and Undetermined Coefficients

$$\begin{aligned} u_1' y_1 + u_2' y_2 &= 0 \\ u_1' y_1' + u_2' y_2' &= f(x) \end{aligned}$$

Let's solve the system.

Written in matrix form this system is

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ f(x) \end{bmatrix}$$

$y_1 u_1' + y_2 u_2' = 0$
 $y_1' u_1' + y_2' u_2' = f(x)$

Recall that if $A\vec{x} = \vec{b}$ then $\vec{x} = A^{-1}\vec{b}$ and for a two by two matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

So assuming we have a solution for all x we have

$$\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \frac{1}{y_1 y_2' - y_1' y_2} \begin{bmatrix} y_2' & -y_2 \\ y_1 & y_1' \end{bmatrix} \begin{bmatrix} 0 \\ f(x) \end{bmatrix} = \frac{1}{W(y_1, y_2)} \begin{bmatrix} -y_2 f(x) \\ y_1 f(x) \end{bmatrix}$$

Where $W(y_1, y_2)$ is the wronskian of y_1, y_2

Therefore

$$u_1' = -\frac{y_2 f(x)}{W(y_1, y_2)}$$

$$u_2' = \frac{y_1 f(x)}{W(y_1, y_2)}$$

So that ~~Type equation here.~~

$$u_1 = -\int \frac{y_2 f(x)}{W(y_1, y_2)} dx$$

$$u_2 = \int \frac{y_1 f(x)}{W(y_1, y_2)} dx$$

Thus the particular solution $y_p = y_1 u_1 + y_2 u_2$ becomes

$$y_p = -y_1 \underbrace{\int \frac{y_2 f(x)}{W(y_1, y_2)} dx}_{u_1} + y_2 \underbrace{\int \frac{y_1 f(x)}{W(y_1, y_2)} dx}_{u_2}$$

Section 3.5

Nonhomogeneous Equations and Undetermined Coefficients

If the non-homogeneous equation $y'' + p(x)y' + q(x)y = f(x)$ has complementary solution $y_c = c_1y_1 + c_2y_2$, then the particular solution is given by

$$y_p = -y_1 \int \frac{y_2 f(x)}{W(y_1, y_2)} dx + y_2 \int \frac{y_1 f(x)}{W(y_1, y_2)} dx$$

Where $W(y_1, y_2)$ is the wronskian of the two linearly independent functions y_1, y_2

Right here we see how important the complementary solution is and why we can't just ignore it. It in fact defines the particular solution.

We did this for second order equation for ease, but this certainly applies to higher order linear nonhomogeneous equations.

For something like

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_2 y'' + a_1 y' + a_0 y = f(x)$$

With complementary solution $y_c = c_1y_1 + c_2y_2 + \dots + c_ny_n$

We would take the form of y_p to be $y_p = u_1y_1 + u_2y_2 + \dots + u_ny_n$ and play the same game we did with our second order equation and you would obtain

$$y_p = y_1 \int \frac{W_1 f(x)}{W(y_1, \dots, y_n)} dx + y_2 \int \frac{W_2 f(x)}{W(y_1, \dots, y_n)} dx + \dots + y_n \int \frac{W_n f(x)}{W(y_1, \dots, y_n)} dx$$

Where $W_1, W_2, \dots, W_i, \dots, W_n$ are the determinants of the matrix we get when replacing the i^{th} column of the Wronskian by $(0, 0, 0, \dots, 1)$. Sometimes for higher order equations, it is just easier to make our guesses!

Section 3.5

Nonhomogeneous Equations and Undetermined Coefficients

Use variation of parameters to solve the differential equation $y'' + 4y = 5\sec(2x)$

If the non-homogeneous equation $y'' + p(x)y' + q(x)y = f(x)$ has complementary solution $y_c = c_1y_1 + c_2y_2$, then the particular solution is given by

$$\longrightarrow y_p = -y_1 \int \frac{y_2 f(x)}{W(y_1, y_2)} dx + y_2 \int \frac{y_1 f(x)}{W(y_1, y_2)} dx$$

Where $W(y_1, y_2)$ is the wronskian of the two linearly independent functions y_1, y_2

$$y'' + 4y = 0 \quad r^2 + 4 = 0 \quad r = \pm 2i \quad y_c = c_1 \underbrace{\cos(2x)}_{y_1} + c_2 \underbrace{\sin(2x)}_{y_2}$$

$$W = \begin{vmatrix} \cos(2x) & \sin(2x) \\ -2\sin(2x) & 2\cos(2x) \end{vmatrix} = 2\cos^2(2x) + 2\sin^2(2x) = 2(\underbrace{\cos^2(2x) + \sin^2(2x)}_1)$$

$$W = 2$$

$$y_p = -\cos(2x) \int \frac{\sin(2x) 5\sec(2x)}{2} dx + \sin(2x) \int \frac{\cos(2x) 5\sec(2x)}{2} dx$$

$$y_p = -\frac{5}{2}\cos(2x) \left(\int \tan(2x) dx \right) + \frac{5}{2}\sin(2x) \int (1) dx$$

$$y_p = -\frac{5}{2}\cos(2x) \left(-\frac{1}{2}\ln(|\cos(2x)|) \right) + \frac{5}{2}x\sin(2x)$$

$$= \frac{5}{4}\cos(2x) \ln(|\cos(2x)|) + \frac{5}{2}x\sin(2x) \quad \leftarrow$$

$$y = y_c + y_p$$